Rational Functions With Nonnegative Integer Coefficients

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Domaine Saint-Jacques

1. When are the coefficients of a rational function nonnegative?

2. When (if they are integers) do they have a combinatorial interpretation?

How can we prove that numbers are nonnegative?

1. explicit formula

2. subtract a smaller number from a larger

3. square or sum of squares

1. $\dfrac{(a-b+c)!\,(a+b-c-1)!}{(a-b-c-1)!\,a!\,b!\,c!}$ is nonnegative, where $a > b + c$.

2. $|1+2i| = \sqrt{5}$, so $|(1+2i)^{2n}| \leq 5^n$ and similarly, $|(1-2i)^{2n}| \leq 5^n$. Therefore

$$2 \cdot 5^n - (1+2i)^{2n} - (1-2i)^{2n}$$

is a nonnegative integer. It follows (dividing these numbers by 16) that the coefficients of

$$\frac{x + 5x^2}{1 + x - 5x^2 - 125x^3}$$

are nonnegative integers.

3. The coefficient of $x^p y^q z^n$ in

$$\frac{1}{1 - (1+x)(1+y)z + 4xyz^2}$$

is

$$\frac{p!\,q!}{(n-p)!\,(n-q)!}\left[\sum_i \frac{(n-i)!}{i!\,(p-i)!\,(q-i)!}(-2)^i\right]^2$$

How to get a combinatorial interpretation for a rational generating function?

The transfer matrix method.

Let M be a matrix and let a_n be the (i,j) entry of M^n. Then $\sum_{n=0}^{\infty} a_n x^n$ is rational. If the entries of M are nonnegative integers, it is reasonable to say that the a_n have a combinatorial interpretation.

More generally, if M is a matrix whose entries are polynomials with nonnegative coefficients and with no constant term then the entries of $(I-M)^{-1}$ are rational functions with combinatorial interpretations. Such rational functions are called $\mathbb{N}$-recognizable.

$\mathbb{N}$-rational functions.

The class of $\mathbb{N}$-rational functions in a set of variables is the smallest set of rational functions containing 1 and all the variables and closed under addition, multiplication, and the operation $f \mapsto f/(1-f)$ whenever f has constant term 0. $\mathbb{N}$-rational functions also have combinatorial interpretations.

Theorem (Schützenberger) A series is $\mathbb{N}$-recognizable if and only if it is $\mathbb{N}$-rational.

Are there other ways to get rational functions whose coefficients have nonnegative coefficients?

The Cartier-Foata theory of free partially commutative monoids

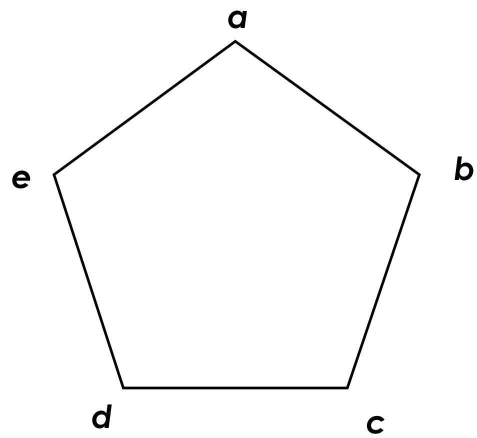
Take a graph whose vertices are variables. If S is a set of vertices, we denote by $\Pi(S)$ the product of the the elements of S.

Then

$$\Bigg(\sum_{S \text{ independent}} (-1)^{|S|} \prod(S)\Bigg)^{-1}$$

has nonnegative coefficients.

Example



$$\frac{1}{1-a-b-c-d-e+ac+ce+eb+ed+bd+da}$$

has nonnegative coefficients.

However, these rational generating functions are always $\mathbb{N}$-rational (Diekert), so we don't get anything new.

I don't know of any rational functions with combinatorial interpretations that aren't $\mathbb{N}$-rational. Are there any?

Problem: $\mathbb{N}$-rationality gives a combinatorial interpretation. But it does not necessarily give a *nice* combinatorial interpretation.

Are there rational functions with nonnegative integer coefficients that are known not to be $\mathbb{N}$-rational?

Yes. Let

$$\begin{aligned} a_n &= \frac{1}{16}\left(2 \cdot 5^n - (1+2i)^{2n} - (1-2i)^{2n}\right) \\ &= \frac{1}{4}\left(\mathrm{Im}(1+2i)^n\right)^2 = \frac{5^n}{4}\sin^2 n\theta \geq 0, \end{aligned}$$

where $\theta = \tan^{-1} 2$. Then

$$f(x) := \sum_{n=0}^{\infty} a_n x^n = \frac{x + 5x^2}{1 + x - 5x^2 - 125x^3}.$$

However, $f(x)$ is not $\mathbb{N}$-rational by a theorem of Berstel (1971): If $u(x)$ is $\mathbb{N}$-rational with radius of convergence r then r is a pole of $u(x)$, and if s is a pole of $u(x)$ with $|s| = r$ then s/r is a root of unity.

In other words, if $u(x)$ is $\mathbb{N}$-rational, then $u(x)$ can be expressed, by multisection, as a sum of rational functions with a single (necessarily positive) pole on the circle of convergence.

A converse of Berstel's theorem was given by Soittola (1976) and rediscovered by Katayama, Okamoto, and Enomoto (1978).

It implies, for example, that

$$\sum_{n=0}^{\infty} \left(2 \cdot 6^n - (1+2i)^{2n} - (1-2i)^{2n}\right) x^n$$

is $\mathbb{N}$-rational.

However, this is obvious because the series is equal to

$$\frac{18x + 86x^2}{1 - 11x^2 - 150x^3}$$

But Soittola's theorem also implies that

$$\frac{1+x}{1+x-2x^2-3x^3}$$

is $\mathbb{N}$-rational, which is not so obvious. Can we prove it directly?

By multisection, we have

$$\frac{1+x}{1+x-2x^2-3x^3} = 1 + \frac{N(x)}{1-2x^6-107x^{12}-729x^{18}}$$

where

$$\begin{aligned} N(x) = {} & 2x^2 + x^3 + 3x^4 + 5x^5 + 4x^6 + 15x^7 + 4x^8 + 32x^9 \\ & + 21x^{10} + 55x^{11} + 83x^{12} + 90x^{13} + 27x^{14} + 81x^{15} \\ & + 243x^{16} + 729x^{18} \end{aligned}$$

Examples of multivariable rational functions with nonnegative coefficients.

$$A(x,y,z) = \frac{1}{1-2(x+y+z)+3(xy+xz+yz)}$$

$$B(x,y,z) = \frac{1}{1-x-y-z+4xyz}$$

$$C(x,y,z) = \frac{1}{1-x-y-xz-yz+4xyz}$$

These all have have nonnegative coefficients, but there is no known combinatorial interpretation for any of them.

Szegő, Kaluza 1933; Ismail and Tamhankar 1979

Note that

$$A(x,x,z) = \frac{1}{(1-3x)(1-x-2z)}$$

$$B(x,x,z) = \frac{1}{(1-2x)(1-z-2xz)}$$

$$C(x,x,z) = \frac{1}{(1-2x)(1-2xz)}$$

The nonnegativity of A is implied by that of B and the nonnegative of B is implied by that of C:

$$A(x,y,z) = \frac{1}{(1-x)(1-y)(1-z)} B\left(\frac{x}{1-x}, \frac{y}{1-y}, \frac{z}{1-z}\right)$$

and

$$B(x,y,z) = \frac{1}{1-z} C\left(x, y, \frac{z}{1-z}\right)$$

So it suffices to show that the coefficients of C are nonnegative. However, there is another way to show that the coefficients of B are nonnegative.

The coefficients of

$$D(x,y,z) = \frac{\sqrt{1-4xy}}{1-x-y-z+4xyz}$$

are nonnegative. In fact, there is an explicit formula for the coefficients:

$$D(x,y,z) = \sum_{a,b,c} \beta(a,b,c) x^a y^b z^c,$$

where

$$\beta(a,b,c) = \begin{cases} \dfrac{(a-b+c)!\,(a+b-c-1)!}{(a-b-c-1)!\,a!\,b!\,c!}, & \text{if } a > b+c \\ \beta(b,a,c), & \text{if } b > a+c \\ \dfrac{(c+a-b)!\,(c+b-a)!}{(c-a-b)!\,a!\,b!\,c!}, & \text{if } a+b \le c \\ 0, & \text{otherwise} \end{cases}$$

This follows from $(1-x-y-z+4xyz)D(x,y,z) = \sqrt{1-4xy}$ or from hypergeometric identities.

The numbers $\beta(a,b,c)$ are “super ballot numbers”. For $c=0$ they reduce to the ballot numbers:

$$\beta(a,b,0)=\frac{a-b}{a+b}\binom{a+b}{a}, \text{ for } a>b.$$

Also of interest is the special case of “super Catalan numbers”:

$$T(m,n)=\beta(m+n,n,m-1)=\frac{1}{2}\frac{(2m)!\,(2n)!}{m!\,n!\,(m+n)!}.$$

In particular,

$$T(1,n)=C_n=\frac{(2n)!}{n!\,(n+1)!}$$

$$T(2,n)=6\frac{(2n)!}{n!\,(n+2)!}=4C_n-C_{n+1}$$

Combinatorial interpretations of $T(2,n)$ have been found by Guoce Xin, Gilles Schaeffer, and Nicholas Pippenger and Kristin Schleich (but none are known for $T(3,n)$.)

Why are the coefficients of

$$C(x,y,z) = \frac{1}{1-x-y-xz-yz+4xyz}$$

nonnegative? They are essentially squares. More generally, let

$$\begin{aligned} &E(x,y,z;\lambda) \\ &\quad = \frac{1}{1-(1-\lambda)x-\lambda y-\lambda xz-(1-\lambda)yz+xyz} \\ &\quad = \sum_{i,j,k} \alpha(i,j,k;\lambda)x^i y^j z^k. \end{aligned}$$

Ismail and Tamhankar showed, using MacMahon's master theorem, that if $i+j<k$ then $\alpha(i,j,k;\lambda)=0$ and if $i+j \geq k$ then

$$\alpha(i,j,k;\lambda) = \lambda^{2i+j-k}(1-\lambda)^{k-i}\frac{(i+j-k)!\,k!}{i!\,j!} \times \left[\sum_l \binom{i}{l}\binom{j}{k-i+l}(1-\lambda^{-1})^l\right]^2$$

which is clearly nonnegative for $0<\lambda<1$. (Note that $C(x,y,z) = E(2x,2y,z;1/2)$.)

Ismail and Tamhankar's result can be generalized:

Let $A = (a_{ij})$ be an $m \times n$ matrix. Let $\mathbf{r} = (r_1, \ldots, r_n)$ and $\mathbf{s} = (s_1, \ldots, s_m)$ be sequences of nonnegative integers, and let $r = r_1 + \cdots + r_n$ and $s = s_1 + \cdots + s_m$. We use the notation $[z_1^{i_1} \cdots z_k^{i_k}]\, h(z_1, \ldots, z_k)$ to denote the coefficient of $z_1^{i_1} \cdots z_k^{i_k}$ in $h(z_1, \ldots, z_k)$. We define $F_A(\mathbf{r}, \mathbf{s})$ and $G_A(\mathbf{r}, \mathbf{s})$ by

$$F_A(\mathbf{r}, \mathbf{s}) = [y_1^{s_1} y_2^{s_2} \cdots y_m^{s_m}] \left(1 + \sum_{i=1}^{m} a_{i1} y_i\right)^{r_1} \cdots \left(1 + \sum_{i=1}^{m} a_{in} y_i\right)^{r_n}$$

and for $r \geq s$,

$$G_A(\mathbf{r}, \mathbf{s}) = [x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}] \left(\sum_{j=1}^{n} x_j\right)^{r-s} \left(\sum_{j=1}^{n} a_{1j} x_j\right)^{s_1} \cdots \left(\sum_{j=1}^{n} a_{mj} x_j\right)^{s_m}.$$

If $r < s$ then $G_A(\mathbf{r}, \mathbf{s}) = F_A(\mathbf{r}, \mathbf{s}) = 0$.

It's easy to show that

$$\binom{r}{r_1, r_2, \ldots, r_n} F_A(\mathbf{r}, \mathbf{s}) = \binom{r}{r - s, s_1, s_2, \ldots, s_m} G_A(\mathbf{r}, \mathbf{s}).$$

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $m \times n$ matrices. Let M be the $(n+m) \times (n+m)$ matrix

$$\begin{pmatrix} J & B^t \\ A & \mathbf{0} \end{pmatrix},$$

where J is an $n \times n$ matrix of ones and $\mathbf{0}$ is an $m \times m$ matrix of zeros, and let Z be the $(n+m) \times (n+m)$ diagonal matrix with diagonal entries $x_1, \ldots, x_n, y_1, \ldots, y_m$. Then

$$\sum_{\mathbf{r},\mathbf{s}} F_A(\mathbf{r}, \mathbf{s}) G_B(\mathbf{r}, \mathbf{s}) x_1^{r_1} \cdots x_n^{r_n} y_1^{s_1} \cdots y_m^{s_m}$$

$$= 1/\det(I - ZM).$$

So if $A = B$ then each coefficient of $1/\det(I - ZM)$ is a positive integer times the square of a polynomial in the a_{ij}, and if the a_{ij} are positive real numbers then $1/\det(I - ZM)$ has nonnegative coefficients.

Proof: Use MacMahon's Master Theorem.

Example:

Take $A = B = \begin{bmatrix} a & b \end{bmatrix}$, and write x for x_1, y for x_2, and z for y_1. Then the matrix is

$$I - \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} \begin{bmatrix} 1 & 1 & a \\ 1 & 1 & b \\ a & b & 0 \end{bmatrix} = \begin{bmatrix} 1-x & -x & -ax \\ -y & 1-y & -by \\ -az & -bz & 1 \end{bmatrix}$$

with determinant

$$1 - x - y - a^2xz - b^2yz + (a-b)^2xyz.$$

Thus the coefficients of

$$\frac{1}{1 - x - y - a^2xz - b^2yz + (a-b)^2xyz}$$

are positive integers times squares of polynomials in a and b.

This is really the same as Ismail and Tamhankar's example: replace x with $(1-\lambda)x$ and y with λy and set $a = \sqrt{\lambda/(1-\lambda)}$ and $b = -\sqrt{(1-\lambda)/\lambda}$.

We find also that (writing i for r_1, j for r_2, and k for s_1), we have

$$\frac{1}{1 - x - y - a^2xz - b^2yz + (a-b)^2xyz} = \sum_{i,j,k} F(i,j,k)G(i,j,k)x^i y^j z^k,$$

where

$$F(i,j,k) = [z^k]\,(1+az)^i(1+bz)^j = \sum_l \binom{i}{l}\binom{j}{k-l} a^l b^{k-l}$$

and

$$G(i,j,k) = [x^i y^j]\,(x+y)^{i+j-k}(ax+by)^k = \sum_l \binom{k}{l}\binom{i+j-k}{i-l} a^l b^{k-l}$$

To see that $F(i,j,k)/G(i,j,k)$ is a rational number, note that

$$\begin{aligned}\binom{i}{l}\binom{j}{k-l} &= i!\,j!\,\frac{1}{l!\,(i-l)!\;(k-l)!\,(j-k+l)!} \\ &= i!\,j!\,\frac{1}{l!\,(k-l)!\;(i-l)!\,(j-k+l)!} \\ &= \frac{i!\,j!}{k!\,(i+j-k)!}\binom{k}{l}\binom{i+j-k}{i-l}\end{aligned}$$

Note that setting $a=1$ and $b=-1$ gives

$$C(x,y,z) = \frac{1}{1-x-y-xz-yz+4xyz}.$$

Unfortunately, in the general case there does not seem to be an analogous nice specialization so we don't seem to get generalizations of the rational functions $A(x,y,z)$ and $B(x,y,z)$.

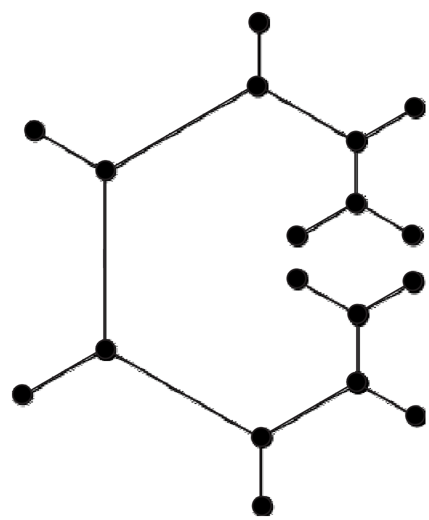
Pippenger and Schleich's combinatorial interpretation of $6\frac{(2n)!}{n!\,(n+2)!}$

An *ternary tree* is a tree in which every vertex has degree 3 or 1. (If there are n vertices of degree 3 then there are $n+2$ vertices of degree 1.) An *oriented ternary tree* is a plane drawing of an unlabeled ternary tree in which the edges incident with each internal vertex meet at angles of 120° and all edges are drawn at angles that are integral multiples of 60° from the vertical. Two such drawings are considered equivalent if they differ only by translations and lengthening or shortening of edges, but not by rotation.

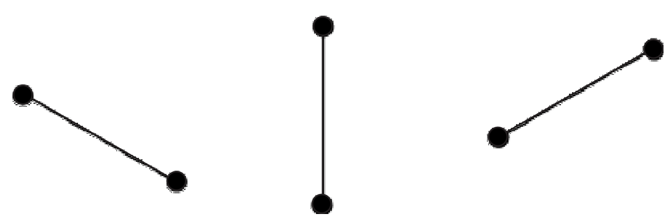
Theorem: The number of oriented ternary trees with n internal vertices and $n+2$ leaves is

$$6\frac{(2n)!}{n!\,(n+2)!}$$

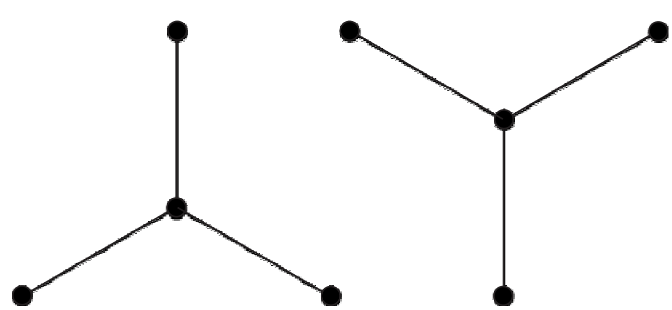
Example of an oriented ternary tree:



For $n = 0$ (three trees):



For $n = 1$ (two trees):



Sketch of proof: First we count labeled ternary trees with n vertices of of degree 3 from a set A of size n and $n+2$ vertices of degree 1 from a set B of size $n+2$. The Prüfer codes for such trees are sequences of length $2n$ in which every element of A appears twice (and the elements of B don't appear at all). There are $(2n)!/2^n$ such sequences.

Next, we embed these trees in the plane. To do this, for each vertex of degree 3 we choose a cyclic order of the three incident edges. There are 2 cyclic orders for each vertex of degree 3, so there are $(2n)!$ embeddings of these trees in the plane. Next we choose an orientation. Once one edge is oriented, the entire orientation is determined, and the number of ways to orient one edge is 6. So there are $6(2n)!$ oriented labeled trees.

To find the number of unlabeled oriented cubic trees we divide by the number of permutations of A and B which is $n!\,(n+2)!$ and we obtain $6(2n)!/n!\,(n+2)!$.