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Permutation Statistics and Partitions

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INTRODUCTION

This work is concerned with generating functions of multivariate distributions of certain permutation statistics.

To be precise, given a permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, we call $d(\sigma)$, $m(\sigma)$ and $i(\sigma)$ respectively the *number of descents*, the *major index* and the *number of inversions* of σ. That is

$$d(\sigma) = \sum_{i<n} \chi(\sigma_i > \sigma_{i+1}),^1$$

$$m(\sigma) = \sum_{i<n} i\chi(\sigma_i > \sigma_{i+1}),$$

$$i(\sigma) = \sum_{i<j} \chi(\sigma_i > \sigma_j).$$

For a given permutation σ there are five "statistics" that have been extensively studied in the literature. These are

$$i(\sigma),\ d(\sigma),\ d(\sigma^{-1}),\ m(\sigma),\ m(\sigma^{-1}). \tag{I.1}$$

The sequences of polynomials

$$E_n(t) = \sum_\sigma t^{d(\sigma)} \tag{I.2}$$

$$A_n{}^i(t, p) = \sum_\sigma p^{i(\sigma)} t^{d(\sigma)} \tag{I.3}$$

$$A_n{}^m(t, q) = \sum_\sigma q^{m(\sigma)} t^{d(\sigma)} \tag{I.4}$$

$$B_n(t_1, t_2) = \sum_\sigma t_1^{d(\sigma)} t_2^{d(\sigma^{-1})} \tag{I.5}$$

[1] Here and in the following we make the convention that for any statement $\mathcal{A}$ we let

$$\chi(\mathcal{A}) = 1 \quad \text{if } \mathcal{A} \text{ is true}$$
$$= 0 \quad \text{if } \mathcal{A} \text{ is false.}$$

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$$C_n(q_1, q_2) = \sum_\sigma q_1^{m(\sigma)} q_2^{m(\sigma^{-1})} \tag{I.6}$$

$$A_n^{i,m}(t, p, q) = \sum_\sigma p^{i(\sigma)} q^{m(\sigma)} t^{d(\sigma)}$$

have all known generating functions.

Of course in (I.2) we have the classical Eulerian polynomials [11], in (I.3) and (I.4) we have their "*inv*" and "*maj*" q-analogues (see [14] and [6]). The generating function of the joint distributions of $d(\sigma)$, $d(\sigma^{-1})$ and $m(\sigma)$, $m(\sigma^{-1})$ can be found in [2] and [10] respectively (see also [13]). Finally the trivariate distributions of $i(\sigma)$, $d(\sigma)$ and $m(\sigma^{-1})$ was first given in [6].

Our main result here is a generating function for the four-variate distributions of $d(\sigma)$, $d(\sigma^{-1})$, $m(\sigma)$ and $m(\sigma^{-1})$. More precisely we show here that if

$$H_n(t_1, t_2; q_1, q_2) = \sum_\sigma t_1^{d(\sigma)} t_2^{d(\sigma^{-1})} q_1^{m(\sigma)} q_2^{m(\sigma^{-1})} \tag{I.7}$$

then

$$\frac{1}{(1-t_1)(1-t_2)} \sum_{n\geq 0} u^n \frac{H_n(t_1, t_2; q_1, q_2)}{(1-t_1 q_1) \cdots (1-t_1 q_1{}^n)(1-t_2 q_2) \cdots (1-t_2 q_2{}^n)}$$
$$= \sum_{k_1\geq 0} \sum_{k_2\geq 0} t_1^{k_1} t_2^{k_2} \prod_{i\leq k_1} \prod_{j\leq k_2} \frac{1}{1-u q_1{}^i q_2{}^j} \tag{I.8}$$

It is not difficult to see that our methods can be used to derive as well the generating functions of the polynomials.

$$H_n(t_1, \ldots, t_k; q_1, \ldots, q_k) = \sum_{\sigma_1 \cdots \sigma_k = e} t_1^{d(\sigma_1)} \cdots t_k^{d(\sigma_k)} q_1^{m(\sigma_1)} \cdots q_k^{m(\sigma_k)}$$

where the sum is carried out over all k-tuples of permutations whose product is the identity.

A generating function for the joint distributions of the five statistics in (I.1) may also be obtained.

Our method of proof is based on the observation that permutation statistics have a very natural setting within the theory of partitions. This fact is implicit in the work of Mac Mahon [9] and has been given a general setting by Knuth [8] and Stanley [12].

To be more specific the method is to start with a family $\mathscr{F}$ of $\mathscr{P}$-partitions (that is the family of decreasing functions on a partially ordered set $\mathscr{P}$) and give two equally natural encodings for each $f \in \mathscr{F}$. We shall refer to them as the "*direct*" and the "*Mahonian*" encodings. The direct encoding leads to a generating function and the Mahonian leads to an expression involving permutation statistics. Equating the two results yields the desired identity.

We shall illustrate the power of this approach in three cases which are quite difficult to attack by any other methods and we shall see that in each case the result is obtained with the greatest of ease.

We should mention that the problem of finding the generating functions for the fourvariate distribution in (I.7) was stated in [3], furthermore, several symmetries of H_n have been obtained indirectly in [4].

Another feature of our derivation of formula (I.8) is that it yields a very explicit interpretation of the "*non-negativity*" of the integers $\lambda(m_1, \ldots, m_n)$ studied by Basil Gordon in [7]. Indeed, from this viewpoint $\lambda(m_1, \ldots, m_k)$ gives none other than the number of k-tuples of permutations $\sigma_1, \sigma_2, \ldots, \sigma_k$ whose product is the identity and the ith of which has major index equal to m_i. More precisely

$$\lambda(m_1, \ldots, m_k) = \sum_{\sigma_1 \cdots \sigma_k = e} \chi(d(\sigma_1) = m_1) \cdots \chi(d(\sigma_k) = m_k)$$

It is clear that these methods can be used in other situations. For instance, they may be used to produce q-analogues of set theoretical formulas. Indeed, comparing the derivation of the trivariate distribution in [5] with the one given here we see that *partitions* can play the same role played by *non-commuting variables*.

1. The Trivariate Distribution of $i(\sigma)$, $m(\sigma)$, $d(\sigma)$

We start by giving a new derivation of the generating function of the polynomials

$$A_n^{i,m}(t, p, q) = \sum_{\sigma} p^{i(\sigma)} q^{m(\sigma)} t^{d(\sigma)}. \tag{1.1}$$

To this end we shall work with the family $\mathscr{F}_n$ of non negative integer valued functions f on the set $\Omega_n = \{1, 2, \ldots, n\}$. For each $f \in \mathscr{F}_n$ we set

$$|f| = \sum_{i=1}^{n} f(i),$$
$$i(f) = \sum_{i<j} \chi(f(i) < f(j)). \tag{1.2}$$

We shall obtain the desired generating function by giving two different expressions for the formal power series

$$\sum_{f \in \mathscr{F}_n} p^{i(f)} q^{|f|} t^{\max f}. \tag{1.3}$$

To this end we shall first find an expression for the sum

$$\sum_{f \in \mathscr{F}_n} p^{i(f)} q^{|f|} \chi(\max f \leqslant k). \tag{1.4}$$

This is easily obtained. Indeed, if we set for each $f \in \mathscr{F}_n$

$$\omega(f) = x_{f(1)} x_{f(2)} \cdots x_{f(n)},$$

then Mac Mahon's [9] "*p-analogue*" of the multinomial identity gives

$$\sum_{f \in \mathscr{F}_n} p^{i(f)} \omega(f)\, \chi(\max f \leqslant k) = \sum_{\mu_0 + \cdots + \mu_k = n} \frac{[n]_p!}{[\mu_0]_p! \cdots [\mu_k]_p!} x_0^{\mu_0} \cdots x_k^{\mu_k} \tag{1.5}$$

where

$$[n]_p! = [n]_p [n-1]_p \cdots [1]_p$$

and

$$[n]_p = 1 + p + \cdots + p^{n-1}.$$

Note now that the sum in (1.4) can be obtained from the expression in (1.5) upon replacing x_i by q^i. The right hand side of (1.5) after this substitution becomes

$$[n]_p! \sum_{\mu_0 + \cdots + \mu_k = n} \frac{(q^0)^{\mu_0}}{[\mu_0]_p!} \cdots \frac{(q^k)^{\mu_k}}{[\mu_k]_p!}$$

and this is the same as $[n]_p!$ times the coefficient of u^n in the series

$$e[u]_p\, e[qu]_p \cdots e[q^k u]_p$$

where $e[u]_p$ is the p-analogue of the exponential function. That is

$$e[u]_p = \sum_{n \geqslant 0} \frac{u^n}{[n]_p!}.$$

We thus have

$$\frac{1}{[n]_p!} \sum_{f \in \mathscr{F}_n} p^{i(f)} q^{|f|} \chi(\max f \leqslant k) = e[u]_p \cdots e[q^k u]_p \,|_{u^n}{}^2$$

multiplying by t^k and summing we finally get

$$\frac{1}{[n]_p!} \frac{1}{1-t} \sum_{f \in \mathscr{F}_n} p^{i(f)} q^{|f|} t^{\max f} = \sum_{k \geqslant 0} t^k e[u]_p \cdots e[q^k u] \,|_{u^n} \tag{1.6}$$

We shall change now our point of view and obtain another expression for this series in terms of the polynomials in (1.1).

Let $f \in \mathscr{F}_n$ and let $\lambda_1 > \lambda_2 > \cdots > \lambda_{k+1} \geqslant 0$ be the different values taken by f, let $A_\nu = \{i : f(i) = \lambda_\nu\}$ and set

$$\sigma(f) = \uparrow A_1 \uparrow A_2 \cdots \uparrow A_{k+1} \tag{1.7}$$

where this expression is to mean that $\sigma(f)$ is the arrangement of the numbers

[2] "$|_{u^n}$" stands for "coefficient of u^n."

1, 2,..., n obtained by putting first the elements of A_1 in increasing order, then the elements of A_2 in increasing order, etc. ... . We shall call

$$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) = \sigma(f) \tag{1.8}$$

the permutation "*associated*" to f.

Let us also set

$$p(f) = (p_1, p_2, \ldots, p_n) \tag{1.9}$$

where

$$p_i = f(\sigma_i) - f(\sigma_{i+1}), \qquad p_n = f(\sigma_n). \tag{1.10}$$

Note that because of our definition (1.7) we shall necessarily have $f(\sigma_i) > f(\sigma_{i+1})$ at the *descents* of σ. That is

$$p_i \geqslant \chi(\sigma_i > \sigma_{i+1}) \qquad (\text{for } i \leqslant n-1) \tag{1.11}$$

Conversely, given $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ and $p = (p_1, p_2, \ldots, p_n)$ related by (1.11) the function defined by

$$f(\sigma_i) = p_i + p_{i+1} + \cdots + p_n \tag{1.12}$$

will have σ as its associated permutation.

The map $f \to (\sigma(f), p(f))$ amounts to an "encoding" of the elements of $\mathscr{F}_n$. Indeed, given σ and p the equations in (1.12) completely determine f.

Note now that we have

$$\begin{aligned} &\text{(a)} \quad |f| = p_1 + 2p_2 + \cdots + np_n \\ &\text{(b)} \quad \max f = p_1 + p_2 + \cdots + p_n \\ &\text{(c)} \quad i(f) = i(\sigma(f)). \end{aligned} \tag{1.13}$$

The first two of these relations are immediate consequences of (1.12). As for the last we note that in view of our definition (1.7) of $\sigma(f)$ a pair $i<j$ appears in reverse order in $\sigma(f)$ if and only if

$$f(i) < f(j)$$

Comparing with (1.2) we see that (1.13c) must hold true.

From these observations we must therefore conclude that

$$\sum_{\sigma(f)=\sigma} p^{i(f)} q^{|f|} t^{\max f} = p^{i(\sigma)} \sum_{p_1 \geqslant \epsilon_1} \cdots \sum_{p_n \geqslant \epsilon_n} t^{p_1 + \cdots + p_n} q^{p_1 + 2p_2 + \cdots + np_n} \tag{1.14}$$

where for convenience we have set

$$\epsilon_i = \chi(\sigma_i > \sigma_{i+1}), \qquad \epsilon_n = 0$$

Summing the right hand side of 1.14 gives

$$\frac{p^{i(\sigma)} t^{\epsilon_1 + \cdots + \epsilon_n} q^{\epsilon_1 + 2\epsilon_2 + \cdots + n\epsilon_n}}{(1-tq)(1-tq^2)\cdots(1-tq^n)} = \frac{p^{i(\sigma)} t^{d(\sigma)} q^{m(\sigma)}}{(1-tq)\cdots(1-tq^n)}$$

Using this expression in (1.14) and summing over σ we get

$$\sum_{f \in \mathscr{F}_n} p^{i(f)} q^{|f|} t^{\max f} = \frac{\sum_\sigma p^{i(\sigma)} q^{m(\sigma)} t^{d(\sigma)}}{(1-tq)(1-tq^2)\cdots(1-tq^n)}$$

Comparing with (1.6) we finally derive that

$$\begin{aligned} &\sum_{n \geqslant 0} \frac{u^n}{[n]_p!} \frac{\sum_{\sigma \in \mathscr{S}_n} p^{i(\sigma)} q^{m(\sigma)} t^{d(\sigma)}}{(1-t)(1-tq)\cdots(1-tq^n)} \\ &\quad = \sum_{k \geqslant 0} t^k e[u]_p \, e[qu]_p \cdots e[q^k u]_p \, . \end{aligned} \tag{1.15}$$

2. The Four-Variate Distribution of $d(\sigma)$, $d(\sigma^{-1})$, $m(\sigma)$, $m(\sigma^{-1})$

In this section we shall give a proof of the identity (1.8). The setting here will be entirely similar to that of the previous section only we shall replace the non-negative integer valued functions f by pairs of such functions

$$f = \begin{pmatrix} f_1(1) \cdots f_1(n) \\ f_2(1) \cdots f_2(n) \end{pmatrix} \tag{2.1}$$

subjected to the condition that for each $i < n$

$$\begin{aligned} &\text{(a)} \quad \text{either } f_1(i) > f_1(i+1) \\ &\text{(b)} \quad \text{or } f_1(i) = f_1(i+1) \text{ and } f_2(i) \geqslant f_2(i+1) \end{aligned} \tag{2.2}$$

These are the "*bipartite*" partitions studied by B. Gordon in [7].

Let us denote the family of all f of the form (2.1) which satisfy (2.2) by $\mathscr{B}_n$ and let us set

$$\mathscr{B}_n(k_1, k_2) = \{f \in \mathscr{B}_n : \max f_1 \leqslant k_1, \max f_2 \leqslant k_2\}.$$

We shall derive our identity by giving two different expressions for the formal power series

$$\sum_{f \in \mathscr{B}_n} q_1^{|f_1|} q_2^{|f_2|} t_1^{\max f_1} t_2^{\max f_2}, \tag{2.3}$$

where as before we have set

$$|f_\nu| = \sum_{i=1}^n f_\nu(i) \qquad (\nu = 1, 2)$$

Our first task is to find an expression for the sum

$$\sum_{f\in\mathscr{B}_n(k_1,k_2)} q_1^{|f_1|}q_2^{|f_2|}. \tag{2.4}$$

To this end we shall interpret each $f\in\mathscr{B}_n(k_1, k_2)$ as a "multisubset" of the "rectangle" of pairs $\binom{i}{j}$:

$$\left\{\binom{i}{j} : 0\leqslant i\leqslant k_1, 0\leqslant j\leqslant k_2\right\}.$$

Indeed, we can identify a given $f\in\mathscr{B}_n(k_1, k_2)$ with the monomial

$$w(f)=\prod_{i=0}^{k_1}\prod_{j=0}^{k_2}\binom{i}{j}^{\mu_{ij}(f)}$$

where $\mu_{ij}(f)$ denotes the "multiplicity" of $\binom{i}{j}$ in f, more precisely

$$\mu_{ij}(f)=\#\{\nu: f_1(\nu)=i, f_2(\nu)=j\}.$$

Conversely, given such a monomial we can obtain an $f\in\mathscr{B}_n(k_1, k_2)$ by ordering "lexicographically" the pairs $\binom{i}{j}$ occurring in it.

This gives

$$\sum_{f\in\mathscr{B}_n(k_1,k_2)} w(f)=\prod_{i=0}^{k_1}\prod_{j=0}^{k_2}\frac{1}{1-u\binom{i}{j}}\Bigg|_{u^n}. \tag{2.6}$$

Now note that since each pair $\binom{i}{j}$ contributes a $q_1{}^iq_2{}^j$ to the monomial $q_1^{|f_1|}q_2^{|f_2|}$, the sum in (2.4) can be obtained from the expression in (2.6) upon replacing $\binom{i}{j}$ by $q_1{}^iq_2{}^j$.

We thus have

$$\sum_{f\in\mathscr{B}_n(k_1,k_2)} q_1^{|f_1|}q_2^{|f_2|}=\prod_{i=0}^{k_1}\prod_{j=0}^{k_2}\frac{1}{1-uq_1{}^iq_2{}^j}\Bigg|_{u^n}.$$

Multiplying by $t_1^{k_1}t_2^{k_2}$ and summing we finally obtain

$$\sum_{f\in\mathscr{B}_n} q_1^{|f_1|}q_2^{|f_2|}\frac{t_1^{\max f_1}}{1-t_1}\frac{t_2^{\max f_2}}{1-t_2}$$

$$=\sum_{k_1\geqslant 0}\sum_{k_2\geqslant 0} t_1^{k_1}t_2^{k_2}\prod_{i=0}^{k_1}\prod_{j=0}^{k_2}\frac{1}{1-uq_1{}^iq_2{}^j}\Bigg|_{u^n} \tag{2.7}$$

This is one side of our identity.

To get the other side we resort to a "Mahonian" encoding of bipartite partitions given in essence by B. Gordon in [7]. Indeed we can work on the other side with the additional statistic

$$i(f)=\sum_{i<j}\chi(f_2(i)<f_2(j)). \tag{2.9}$$

More precisely we shall relate the formal series

$$\sum_{f\in\mathscr{B}_n} p^{i(f)}q_1^{|f_1|}q_2^{|f_2|}t_1^{\max f_1}t_2^{\max f_2} \tag{2.10}$$

to the five-variate distribution

$$\sum_{\sigma\in\mathscr{S}_n} p^{i(\sigma)}q_1^{m(\sigma)}q_2^{m(\sigma^{-1})}t_1^{d(\sigma)}t_2^{d(\sigma^{-1})} \tag{2.11}$$

To state the basic result here we need an additional definition. Given a permutation $\sigma=(\sigma_1, \sigma_2, \ldots, \sigma_n)$ we shall say that a partition $\lambda=(\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_n\geqslant 0)$ is σ-compatible if

$$\lambda_i-\lambda_{i+1}\geqslant\chi(\sigma_i>\sigma_{i+1}) \qquad (i\leqslant n-1), \tag{2.12}$$

that is λ is σ-compatible if it is "strict" at the descent set of σ.

Clearly, a partition λ is σ-compatible if and only if it is of the form

$$\lambda_i=p_i+p_{i+1}+\cdots+p_n$$

with

$$p_i\geqslant\epsilon_i=\begin{cases}1 & \text{if } \sigma_i>\sigma_{i+1},\quad i<n\\ 0 & \text{otherwise}\end{cases}$$

Thus if $\mathscr{P}(\sigma)$ denotes the set of all σ-compatible partitions we must have

$$\sum_{\lambda\in\mathscr{P}(\sigma)} t^{\max\lambda}q^{|\lambda|}=\sum_{p_1\geqslant\epsilon_1}\cdots\sum_{p_n\geqslant\epsilon_n} t^{p_1+\cdots+p_n}q^{p_1+2p_2+\cdots+np_n}$$

$$=\frac{t^{d(\sigma)}q^{m(\sigma)}}{(1-tq)(1-tq^2)\cdots(1-tq^n)} \tag{2.13}$$

This given we have the following basic fact.

THEOREM 2.1. *There is a bijection between the bipartite partitions and the triplets*

$$(\sigma, \lambda, \mu)$$

where σ is a permutation, λ is a partition compatible with σ and μ is a partition compatible with σ^{-1}. This bijection is simply given by the map

$$f=\Phi(\sigma,\lambda,\mu)=\begin{pmatrix}\lambda_1 & \lambda_2 & \cdots & \lambda_n\\ \mu_{\sigma_1} & \mu_{\sigma_2} & \cdots & \mu_{\sigma_n}\end{pmatrix}. \tag{2.14}$$

Furthermore when this holds we have

$$i(f) = i(\sigma). \tag{2.15}$$

Proof. We observe first that if $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n)$, $\mu = (\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n)$ and λ is σ-compatible then the pairs

$$\binom{\lambda_1}{\mu_{\sigma_1}}\binom{\lambda_2}{\mu_{\sigma_2}} \cdots \binom{\lambda_n}{\mu_{\sigma_n}}$$

are in lexicographic order. Indeed either $\lambda_i > \lambda_{i+1}$ or if $\lambda_i = \lambda_{i+1}$ the σ-compatibility of λ gives $\sigma_i < \sigma_{i+1}$ and so we must have $\mu_{\sigma_i} \geqslant \mu_{\sigma_{i+1}}$.

This means that Φ is indeed a map *into* $\mathscr{B}_n$.

Next we show that every bipartite partition

$$f = \begin{pmatrix} f_1(1) \cdots f_1(n) \\ f_2(1) \cdots f_2(n) \end{pmatrix}$$

has only one representation of the form 2.14. Let then

$$f_1(i) = \lambda_i,$$
$$f_2(i) = \mu_{\sigma_i}.$$

This gives

$$f_2(\sigma^{-1}(i)) = \mu_i. \tag{2.16}$$

Since some of the values of f_2 can be equal there are of course several permutations σ which satisfy (2.16). However, the σ^{-1} compatibility requirement on μ makes σ unique.

Indeed, let us interpret (2.16) as a "labelling" of the values of f_2. More precisely we interpret (2.16) as *saying* that

"$f_2(\sigma^{-1}(i))$ is the ith value of f_2".

Now, if μ is σ^{-1}-compatible the equality $\mu_i = \mu_{i+1}$ implies $\sigma^{-1}(i) < \sigma^{-1}(i+1)$. But this means that $\mu_i = \mu_{i+1}$ implies that the ith value of f_2 is to *left* of the $(i+1)$st.

Thus we get a simple rule for constructing the permutation σ. *We label the values of f_2 from left to right starting from the largest ones, the next largest ones, ..., and finally the smallest ones.*

For instance the bipartite partition

$$f = \begin{pmatrix} 5 & 5 & 3 & 3 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 & 2 & 4 & 3 \end{pmatrix}$$

is labelled

$$f = \begin{pmatrix} 5 & 5 & 3 & 3 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 & 2 & 4 & 3 \\ ⑤ & ⑦ & ① & ③ & ⑥ & ② & ④ \end{pmatrix}.$$

This done, the label of $f_2(i)$ will necessarily be σ_i. Thus in this example we obtain

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 7 & 1 & 3 & 6 & 2 & 4 \end{pmatrix}.$$

This gives that Φ is *one-to-one*. We show next that Φ is *onto*.

To this end we must show that, given a bipartite partition f, then if

(a) we set $\lambda_i = f_1(i)$

(b) we let $\mu = (\mu_1 \geqslant \cdots \geqslant \mu_n)$ be the non-increasing rearrangement of the values of f_2

(c) we construct σ by the above labelling procedure.

then λ will be σ-compatible and μ will be σ^{-1} compatible.

Now, note that the way we carry out our labelling procedure the inequality $\sigma_i > \sigma_{i+1}$ can only occur when $f_2(i)$ is *smaller* than $f_2(i+1)$, but then the "lexicographic" conditions 2.2 force $f_1(i) > f_1(i+1)$. In other words, we have $\sigma_i > \sigma_{i+1}$ only when $\lambda_i > \lambda_{i+1}$.

Finally, we observe that our labelling procedure puts the ith value of f_2 to the right of the $(i+1)$st (that is $\sigma^{-1}(i) > \sigma^{-1}(i+1)$) only when $f_2(\sigma^{-1}(i)) > f_2(\sigma^{-1}(i+1))$. That means we must have $\mu_i > \mu_{i+1}$ when $\sigma^{-1}(i) > \sigma^{-1}(i+1)$.

So μ is σ^{-1}-compatible. Thus Φ is onto.

To complete the proof we must show that

$$i(f) = i(\sigma). \tag{2.17}$$

Note that our labelling procedure gives for $i < j$

$$f_2(i) = f_2(j) \Rightarrow \sigma_i < \sigma_j$$
$$f_2(i) > f_2(j) \Rightarrow \sigma_i < \sigma_j$$
$$f_2(i) < f_2(j) \Rightarrow \sigma_i > \sigma_j.$$

Thus we have $\sigma_i > \sigma_j$ if and only if $f_2(i) < f_2(j)$ and this is 2.17.

The proof of the theorem is now complete.

Using the representation (2.14) we can immediately obtain the desired relation between the series in (2.10) and the polynomials in (2.11).

THEOREM 2.2

$$\sum_{f\in\mathscr{B}_n} p^{i(f)} t_1^{\max f_1} t_2^{\max f_2} q_1^{|f_1|} q_2^{|f_2|}$$
$$= \frac{\sum_\sigma p^{i(\sigma)} t_1^{d(\sigma)} t_2^{d(\sigma^{-1})} q_1^{m(\sigma)} q_2^{m(\sigma^{-1})}}{(1-t_1q_1)\cdots(1-t_1q_1{}^n)(1-t_2q_2)\cdots(1-t_2q_2{}^n)}. \tag{2.18}$$

Proof. Theorem 2.1 permits us to rewrite the left hand side of (2.18) in the form

$$\sum_\sigma \sum_{\lambda\in\mathscr{P}(\sigma)} \sum_{\mu\in\mathscr{P}(\sigma^{-1})} p^{i(\sigma)} t_1^{\max\lambda} t_2^{\max\mu} q_1^{|\lambda|} q_2^{|\mu|}.$$

Thus (2.18) is an immediate consequence of formula (2.13).

From this result we derive

THEOREM 2.3

$$\frac{\sum_\sigma t_1^{d(\sigma)} t_2^{d(\sigma^{-1})} q_1^{m(\sigma)} q_2^{m(\sigma^{-1})}}{(1-t_1)\cdots(1-t_1q_1{}^n)(1-t_2)\cdots(1-t_2q_2{}^n)}$$
$$= \sum_{k_1\geqslant 0}\sum_{k_2\geqslant 0} t_1^{k_1} t_2^{k_2} \prod_{i\leqslant k_1}\prod_{j\leqslant k_2} \frac{1}{1-uq_1{}^iq_2{}^j}\bigg|_{u^n} \tag{2.19}$$

Proof. Simply set $p = 1$ in (2.18) and compare with (2.7).

It may be worthwhile making some additional comments here.

Remark 2.1. (On the five-variate distribution). Clearly we can use (2.18) to "obtain" a generating function for the joint distribution of the five statistics in (I.1). We observe that if

$$w(f) = \prod_{i\leqslant k_1}\prod_{j\leqslant k_2} \binom{i}{j}^{\mu_{ij}(f)} \tag{2.20}$$

then $i(f_2)$ counts the number of pairs $\binom{i_1}{j_1}$, $\binom{i_2}{j_2}$ in (2.20) which satisfy the inequalities

$$i_1 > i_2, \qquad j_1 < j_2.$$

Thus if T is the operator which sends the monomial

$$w = \prod_{i\leqslant k_1}\prod_{j\leqslant k_2}\binom{i}{j}^{\mu_{ij}}$$

into

$$Tw = p^{\nu(w)} q_1^{\sum_{ij} i\mu_{ij}} q_2^{\sum_{ij} j\mu_{ij}}$$

where $\nu(w)$ denotes the number of such pairs then we have that the left hand side of (2.18) divided by $(1-t_1)(1-t_2)$ is also equal to

$$T\sum_{k_1\geqslant 0}\sum_{k_2\geqslant 0} t_1^{k_1}t_2^{k_2}\prod_{i\leqslant k_1}\prod_{j\leqslant k_2}\frac{1}{1-u\binom{i}{j}}\bigg|_{u^n}.$$

This yields a "form" of generating function for the five-variate distributions. However, it is quite possible that some other approach may yield a more explicit expression.

Remark 2.2. (On k-partite partitions). In [7] k-partite partitions are defined as n-tuples

$$f = \begin{pmatrix} f_1(1)\\ f_2(1)\\ \vdots\\ f_k(1)\end{pmatrix}\cdots\begin{pmatrix} f_1(n)\\ f_2(n)\\ \vdots\\ f_k(n)\end{pmatrix}$$

of k-vectors satisfying the conditions

(i) If $f_j(\nu) = f_j(\nu+1)\ \forall j \leqslant i$ and
$f_{i+1}(\nu) \neq f_{i+1}(\nu+1)$ then $f_{i+1}(\nu) > f_{i+1}(\nu+1)$.

Given such a k-partite partition we label the values of f_i for each $i \leqslant k$ starting from the *last* row ($i = k$) and proceeding on up in the following manner.

We label the values of f_k as in the proof of Theorem 2.1.

Then inductively, given that the values of f_{i+1} are labelled

$$\sigma^{(i+1)} = (\sigma_1^{(i+1)}, \sigma_2^{(i+1)},\ldots, \sigma_n^{(i+1)})$$

we label the values of f_i, starting with the largest ones, then the next largest ones, etc.

But this time breaking ties in a manner that is compatible with the permutation $\sigma^{(i+1)}$. For instance the tripartite partition

$$f = \begin{pmatrix} 5 & 5 & 5 & 3 & 3 & 3\\ 4 & 3 & 3 & 3 & 2 & 2\\ 1 & 2 & 1 & 3 & 1 & 1\end{pmatrix}$$

is labelled

$$f = \begin{pmatrix} 5 & 5 & 5 & 3 & 3 & 3\\ ① & ② & ③ & ④ & ⑤ & ⑥\\ 4 & 3 & 3 & 3 & 2 & 2\\ ① & ③ & ④ & ② & ⑤ & ⑥\\ 1 & 2 & 1 & 3 & 1 & 1\\ ③ & ② & ④ & ① & ⑤ & ⑥\end{pmatrix}$$

It is a property of the lexicographic ordering that this procedure always labels the entries in the first row in the natural order.

Now, let $\lambda^{(i)}$ denote the non-increasing rearrangement of the values of f_i and $\omega^{(i)}$ be the permutations defined by setting

$$\omega^{(i)} = \begin{pmatrix} \sigma_1^{(i)} & \cdots & \sigma_n^{(i)} \\ \sigma_1^{(i+1)} & \cdots & \sigma_n^{(i+1)} \end{pmatrix} \qquad \text{if} \quad i < k$$

and

$$\omega^{(k)} = \begin{pmatrix} \sigma_1^{(k)} & & \cdots & \sigma_n^{(k)} \\ 1 & 2 & \cdots & n \end{pmatrix}.$$

It can be shown that this gives a bijection of the family $\mathscr{B}^{(k)}$ of k-partite partitions and the $(2k)$-tuples

$$(\omega^{(1)},\dots, \omega^{(k)}; \lambda^{(1)},\dots, \lambda^{(k)})$$

where

(a) $\omega^{(1)},\dots, \omega^{(k)}$ are permutations whose product is the identity,

(b) $\lambda^{(1)},\dots, \lambda^{(k)}$ are partitions,

(c) $\lambda^{(i)}$ is $\omega^{(i)}$-compatible.

Using this fact one proves the identity

$$\sum_{h_1 \geqslant 0} \cdots \sum_{h_k \geqslant 0} t_1^{h_1} \cdots t_k^{h_k} \prod_{i_1 < h_1} \cdots \prod_{i_k < h_k} \frac{1}{1 - uq^{i_1} \cdots q^{i_k}} \Big|_{u^n}$$
$$= \frac{\sum_{\omega_1 \cdots \omega_k = e} q_1^{m(\omega_1)} \cdots q_k^{m(\omega_k)} t_1^{d(\omega_1)} \cdots t_k^{d(\omega_k)}}{\prod_{i=1}^k (1 - t_1) \cdots (1 - t_1 q_i{}^n)}$$

from which our statement concerning Basil Gordon's integers $\lambda_1(m_1, \dots, m_k)$ may be easily derived.

Remark 3.3. Historically speaking, Mac Mahon might have been the first to have used permutation statistics to encode partitions. Basil Gordon in [7] uses similar ideas to encode bipartite partitions. Some time later Carlitz, *et al.* [2] and Roselle in [10] essentially rediscovered Basil Gordon's encoding. However, roughly speaking, B. Gordon's work and [10] are only concerned with *major index* statistics and in [2] they are only concerned with *descents*. The amusing fact is that this encoding carries information also about the number of inversions.

Remark 3.4. It is interesting to point out that from our identities we can obtain a generating function proof that $d(\sigma)$, $m(\sigma)$ and $m(\sigma^{-1})$ have the same joint distribution as $d(\sigma)$, $i(\sigma)$, $m(\sigma)$. This remarkable result was essentially stated in [3] where it follows from purely combinatorial considerations.

To prove this here we let $t_2 = 1$, $t_1 = t$, $q_2 = p$, and $q_1 = q$ in (2.19) obtaining

$$\frac{\sum_{\sigma \in \mathscr{S}_n} t^{d(\sigma)} q^{m(\sigma)} p^{m(\sigma^{-1})}}{(1-t)(1-tq) \cdots (1-tq^n)[n]_p!} = (1-p)^n \sum_{k \geqslant 0} t^k \prod_{i<k} \prod_{j \geqslant 0} \frac{1}{1 - uq^i p^j} \Big|_{u^n}$$
$$= \sum_{k \geqslant 0} t^k \prod_{i<k} \prod_{j \geqslant 0} \frac{1}{1 - (1-p)\, p^j q^i u} \Big|_{u^n}$$

Observing that we do also have $e_p[u] = \prod_{j \geqslant 0} 1/1 - (1-p)\, p^j u$ and comparing with (1.15) the assertion follows immediately.

3. Major Index Statistics of Shuffle Permutations

Our third example should illustrate more clearly the scope of the method we alluded to in the introduction and put our previous examples into proper perspective.

The general setting is as follows. We are given a partially ordered set $\mathscr{P} = (\Omega, \leqslant)$ and a labelling $\omega\colon \Omega \to \{1, 2,\dots, n\}$ of its elements. For convenience we let the elements of Ω be $x_1, x_2, \dots, x_n$. The labelling ω is thus obtained by fixing some permutation

$$\omega = (\omega_1, \omega_2, \dots, \omega_n)$$

of $\{1, 2,\dots, n\}$ and setting

$$\omega(x_i) = \omega_i .$$

We *do not* assume that ω has any relation whatsoever with the partial ordering of $\mathscr{P}$.

Our object of study is then the family of non-negative, decreasing integer valued functions on Ω which are *strictly* decreasing when ω is *strictly* decreasing.

More precisely we set

$$\mathscr{F}_\omega(\mathscr{P}) = \{f\colon \Omega \to \mathbb{Z}_0{}^+\colon x \leqslant y \Rightarrow f(x) \geqslant f(y) \text{ with “}>\text{” when } \omega(x) > \omega(y)\}.$$

We call the elements of $A_\omega(\mathscr{P})$ the ω-compatible partitions.

This given let $f \in \mathscr{F}_\omega(\mathscr{P})$ and let $\lambda_1 > \lambda_2 \cdots > \lambda_k \geqslant 0$ be the different values taken by f. Let $A_\nu = \{x\colon f(x) = \lambda_\nu\}$ and set

$$\sigma(f) = \uparrow_\omega A_1 \uparrow_\omega A_2 \cdots \uparrow_\omega A_k , \tag{3.1}$$

where this expression is to mean that $\sigma(f)$ is the arrangement of the elements of Ω obtained by putting first the elements of A_1 in order of *increasing* ω, then the elements of A_2 in the same manner, etc. We call

$$\sigma(f) = x_{\sigma_1} x_{\sigma_2} \cdots x_{\sigma_n} \tag{3.2}$$

the "permutation of Ω associated to f". Let us also set

$$p(f) = (p_1, p_2, \ldots, p_n) \tag{3.3}$$

where

$$p_i = f(x_{\sigma_i}) - f(x_{\sigma_{i+1}}), \qquad p_n = f(x_{\sigma_n}). \tag{3.4}$$

One thing must be observed at once. Namely, for any $f \in \mathscr{F}_\omega(\mathscr{P})$ the resulting arrangement $\sigma(f)$ is always *compatible with the partial ordering of* $\mathscr{P}$.

For historical reasons we shall call such arrangements the "*Standard $\mathscr{P}$-tableaux*" and denote the set of all such arrangements by $\mathscr{S}(\mathscr{P})$.

Secondly we note that in view of the definition of $\sigma(f)$ the sequence $\{f(x_{\sigma_i})\}$ must decrease *strictly* at the descents of the permutation

$$\omega(x_\sigma) = (\omega(x_{\sigma_1}), \omega(x_{\sigma_2}), \ldots, \omega(x_{\sigma_n})).$$

In other words the components of $p(f)$ satisfy the inequalities

$$\begin{aligned} p_i \geqslant \epsilon_i &= 1 \quad \text{if} \quad \omega(x_{\sigma_i}) > \omega(x_{\sigma_{i+1}}) \qquad (i < n) \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{3.5}$$

Conversely, given any $x_\sigma \in \mathscr{S}(\mathscr{P})$ and a $p \in \mathbb{Z}^n$, whose components satisfy (3.5), the function f defined by setting

$$f(x_{\sigma_i}) = p_i + p_{i+1} + \cdots + p_n \tag{3.6}$$

will necessarily be in $\mathscr{F}_\omega(\mathscr{P})$. Furthermore, the map in (3.6) is a bijection between $\mathscr{F}_\omega(\mathscr{P})$ and the pairs (x_σ, p) related by (3.5). This is in essence the content of Stanley's theorem 6.2 in [12].

One of the consequences of this result is that if we set

$$|f| = \sum_{i=1}^{n} f(x_i)$$

$$i(f) = \sum_{\omega(x)<\omega(y)} \chi(f(x) < f(y))$$

we necessarily have as before

$$\sum_{f \in \mathscr{F}_\omega(\mathscr{P})} p^{i(f)} t^{\max f} q^{|f|} = \frac{\sum_{x \in \mathscr{S}_\sigma(\mathscr{P})} p^{i(\omega(x_\sigma))} t^{d(\omega(x_\sigma))} q^{m(\omega(x_\sigma))}}{(1-tq)(1-tq^2)\cdots(1-tq^n)} \tag{3.7}$$

We should recognize at once then that our first example corresponds to none other than the simplext case of this set up. Namely the case in which $\mathscr{P} = (\Omega_n, \Phi)$ where "Φ" denotes the "*empty*" relation and ω is the natural labelling of $\Omega_n = \{1, 2, \ldots, n\}$.

Our third example will correspond to the case in which $\mathscr{P}$ is a "*product of chains*". To see what (3.7) yields in this case we need some further terminology.

For a given *ordered* subset $\pi = \{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$ we set

$$m(\pi) = \sum_{\nu=1}^{k} \nu\chi(i_\nu > i_{\nu+1})$$

and call it the "*major index*" of π.

Furthermore, given a collection of $\{\pi_1, \pi_2, \ldots, \pi_k\}$ of *complementary* ordered subsets of $\Omega_n = \{1, 2, \ldots, n\}$ (i.e., they are disjoint as subsets and their union is Ω_n) we shall say that a permutation σ of $\{1, 2, \ldots, n\}$ is obtained by "shuffling" $\pi_1, \pi_2, \ldots, \pi_k$ if the elements of each π_i appear in σ in the same order as they did in π_i itself. That is, σ is obtained by shuffling in the ordinary sense (without cutting) the "decks" corresponding to $\pi_1, \pi_2, \ldots, \pi_k$.

This given we have the following remarkable extension of a result of Mac Mahon (see [1] theorem 3.7, page 42).

THEOREM 3.1. *Let $\pi_1, \pi_2, \ldots, \pi_k$ be ordered complementary subsets of $\{1, 2, \ldots, n\}$. Let $\mathscr{S}(\pi_1, \pi_2, \ldots, \pi_k)$ be the collection of permutations of $\{1, 2, \ldots, n\}$ obtained by shuffling $\pi_1, \pi_2, \ldots, \pi_k$, then*

$$\sum_{\sigma \in \mathscr{S}(\pi_1, \ldots, \pi_k)} q^{m(\sigma)} = \begin{bmatrix} n \\ \mu_1 \mu_2 \cdots \mu_k \end{bmatrix}_q q^{m(\pi_1) + \cdots + m(\pi_k)} \tag{3.8}$$

where $\mu_i = \operatorname{card} \pi_i$ and $\left[\begin{smallmatrix} n \\ \mu_1\mu_2\cdots\mu_k \end{smallmatrix}\right]_q$ denotes the q-analogue of the multinomial coefficient.

Proof. We take $\mathscr{P}$ to be the product of k chains $\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_k$ of respective lengths $\mu_1, \mu_2, \ldots, \mu_k$. We call the elements of the first chain $x_1, x_2, \ldots, x_{\mu_1}$ those of the second chain $x_{\mu_1+1}, \ldots, x_{\mu_1+\mu_2}$, etc. We then take $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ to be the juxtaposition of the ordered subsets $\pi_1, \pi_2, \ldots, \pi_k$ and set

$$\omega(x_i) = \omega_i.$$

In the figure below we illustrate the case in which $k = 3$, $n = 9$ and

$$\pi_1 = \{7, 1, 2, 6\}, \qquad \pi_2 = \{4, 8, 5\}, \qquad \pi_3 = \{3, 9\}.$$

$$\mathscr{P}: \begin{array}{l} \overset{7}{x_1} \to \overset{1}{x_2} \to \overset{2}{x_3} \to \overset{6}{x_4} \\ \overset{4}{x_5} \to \overset{8}{x_6} \to \overset{5}{x_7} \\ \overset{3}{x_8} \to \overset{9}{x_9} \end{array}$$

The label above x_i is the value of $\omega(x_i)$. This given we see that if we apply formula (3.7) (with $t = p = 1$) to the chain $\mathscr{P}_i$ we obtain

$$\sum_{f \in \mathscr{F}_{\pi_i}(\mathscr{P})} q^{|f|} = \frac{q^{m(\pi_i)}}{(1-q)(1-q^2)\cdots(1-q^{\mu_i})} \tag{3.9}$$

(there being only one standard $\mathscr{P}_i$-tableau).

On the other hand, applying the same formula to the poset $\mathscr{P}$ as a whole (and making the obvious identification between permutations in $\mathscr{S}(\pi_1, \pi_2, \ldots, \pi_n)$ and standard $\mathscr{P}$-tableaux) we obtain

$$\sum_{f \in \mathscr{F}_{\omega}(\mathscr{P})} q^{|f|} = \frac{\sum_{\sigma \in \mathscr{S}(\pi_1, \ldots, \pi_n)} q^{m(\sigma)}}{(1-q)(1-q^2)\cdots(1-q^n)} \tag{3.10}$$

Now it is not difficult to see that we must have

$$\sum_{f \in \mathscr{F}_{\omega}(\mathscr{P})} q^{|f|} = \prod_{i=1}^{k} \sum_{f_i \in \mathscr{F}_{\pi_i}(\mathscr{P}_i)} q^{|f_i|}.$$

Thus if we substitute (3.9) and (3.10) in this relationship we derive

$$\sum_{\sigma \in \mathscr{S}(\pi_1, \ldots, \pi_k)} q^{m(\sigma)} = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{\prod_{i=1}^{k}(1-q)\cdots(1-q^{\mu_i})} q^{m(\pi_1)+\cdots+m(\pi_k)}$$

and this is another way of writing formula (3.8).

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References

1. G. Andrews, The theory of partitions, *in* "*Encyclopedia of Mathematics and its Applications*," Vol. 2, Addison-Wesley, Reading, Mass., 1976.
2. L. Carlitz, D. P. Roselle, and R. A. Scovile, Permutations and sequences with repetitions by number of increases, *J. Combinatorial Theory* **1** (1966), 350–374.
3. D. Foata, Distributions Euleriennes et Mahoniennes sur le groupe des permutations, *in* "Higher Combinatorics." Proc. NATO Adv. Study Inst. (M. Aigner, Ed.), pp. 27–49, Reidel, Dordrecht, 1977.
4. D. Foata and M. P. Schutzenberger, Major index and inversion number of permutations, *Math. Nachr.*, in press.
5. A. M. Garsia, On the "maj" and "inv" q-analogues of Eulerian polynomials, *J. Linear Multilinear Algebra*, in press.

6. I. M. Gessel, "Generating functions and enumeration of sequences," M. I. T. doctoral thesis, 1977.
7. B. Gordon, Two theorems on multipartite partitions, *J. London Math. Soc.* **38** (1963), 459–464.
8. D. Knuth, A note on solid partitions, *Math. Comp.* **24** (1970) 955–962.
9. Mac Mahon, "Combinatory Analysis I-II," Cambridge Univ. Press, London/New York, 1916; Chelsea, New York, 1960.
10. D. P. Roselle, Coefficients associated with the expansion of certain products, *Proc. Amer. Math. Soc.* **46** (1974), 144–150.
11. J. Riordan, "An Introduction to Combinatorial Analysis," Wiley, New York, 1958.
12. R. Stanley, Ordered structures and partitions, *Mem. Amer. Math. Soc.* **119** (1972).
13. R. Stanley, The conjugate trace and trace of a plane partition, *J. Combinatorial Theory, Ser. A* **14** (1973), 53–65.
14. R. Stanley, Binomial posets Moebius inversion and permutation enumeration, *J. Combinatorial Theory Ser. A* **20** (1976), 336–356.

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