Permutation Statistics and Partitions

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INTRODUCTION

This work is concerned with generating functions of multivariate distributions of certain permutation statistics.

To be precise, given a permutation \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \), we call \( d(\sigma), m(\sigma) \) and \( \text{i}(\sigma) \) respectively the number of descents, the major index and the number of inversions of \( \sigma \). That is

\[
\begin{align*}
 d(\sigma) &= \sum_{i < j} \chi(\sigma_i > \sigma_j), \quad (1) \\
m(\sigma) &= \sum_{i < j} \text{i}(\sigma_i > \sigma_j), \\
\text{i}(\sigma) &= \sum_{i < j} \chi(\sigma_i > \sigma_j).
\end{align*}
\]

For a given permutation \( \sigma \) there are five "statistics" that have been extensively studied in the literature. These are

\[
\text{i}(\sigma), d(\sigma), d(\sigma^{-1}), m(\sigma), m(\sigma^{-1}).
\]

The sequences of polynomials

\[
\begin{align*}
 E_n(t) &= \sum_{\sigma} t^{d(\sigma)}, \\
 A_n^s(t, p) &= \sum_{\sigma} p^{d(\sigma)} t^{\text{i}(\sigma)}, \\
 A_n^m(t, q) &= \sum_{\sigma} q^{m(\sigma)} t^{d(\sigma)}, \\
 B_n(t_1, t_2) &= \sum_{\sigma} t_1^{d(\sigma)} t_2^{\text{i}(\sigma)}
\end{align*}
\]

have all known generating functions.

Of course in (1.2) we have the classical Eulerian polynomials [11], in (1.3) and (1.4) we have their "inv" and "maj" \( q \)-analogues (see [14] and [6]). The generating function of the joint distributions of \( d(\sigma), d(\sigma^{-1}) \) and \( m(\sigma), m(\sigma^{-1}) \) can be found in [2] and [10] respectively (see also [13]). Finally the trivariate distributions of \( i(\sigma), d(\sigma) \) and \( m(\sigma^{-1}) \) was first given in [6].

Our main result here is a generating function for the four-variate distributions of \( d(\sigma), d(\sigma^{-1}), m(\sigma) \) and \( m(\sigma^{-1}) \). More precisely we show here that if

\[
C_n(q_1, q_2) = \sum_{\sigma} q_1^{d(\sigma)} q_2^{m(\sigma^{-1})}
\]

then

\[
(1 - t_1)(1 - t_2) \sum_{k \geq 0} w^k \frac{H_n(t_1, t_2 \cdot q_1, q_2)}{(1 - t_1q_1^k)(1 - t_2q_2^k)} = \sum_{k \geq 0} \sum_{l \geq 0} t_1^l t_2^l \prod_{i<k} \prod_{i<k} \frac{1}{1 - w_1 q_1^i q_2^i}
\]

It is not difficult to see that our methods can be used to derive as well the generating functions of the polynomials

\[
H_n(t_1, ..., t_k; q_1, ..., q_k) = \sum_{\sigma} t_1^{d(\sigma)} ... t_k^{d(\sigma)} q_1^{m(\sigma)} ... q_k^{m(\sigma)}
\]

where the sum is carried out over all \( k \)-tuples of permutations whose product is the identity.

A generating function for the joint distributions of the five statistics in (1.1) may also be obtained.

Our method of proof is based on the observation that permutation statistics have a very natural setting within the theory of partitions. This fact is implicit in the work of MacMahon [9] and has been given a general setting by Knuth [8] and Stanley [12].

To be more specific the method is to start with a family \( \mathcal{F} \) of \( \rho \)-partitions (that is the family of decreasing functions on a partially ordered set) and give two equally natural encodings for each \( f \in \mathcal{F} \). We shall refer to them as the "direct" and the "Mahonian" encodings. The direct encoding leads to a generating function and the Mahonian leads to an expression involving permutation statistics. Equating the two results yields the desired identity.

We shall illustrate the power of this approach in three cases which are quite
difficult to attack by any other methods and we shall see that in each case the result is obtained with the greatest of ease.

We should mention that the problem of finding the generating functions for the fourvariable distribution in (1.7) was stated in [3], furthermore, several symmetries of $H_n$ have been obtained indirectly in [4].

Another feature of our derivation of formula (1.8) is that it yields a very explicit interpretation of the "non-negativity" of the integers $\lambda(m_1, \ldots, m_p)$ studied by Basil Gordon in [7]. Indeed, from this viewpoint $\lambda(m_1, \ldots, m_p)$ gives none other than the number of $p$-tuples of permutations $a_1, a_2, \ldots, a_p$ whose product is the identity and the $i$th of which has major index equal to $m_i$. More precisely

$$
\lambda(m_1, \ldots, m_p) = \sum_{\sigma_1 \cdots \sigma_p = 1} \chi(d(\sigma_1) = m_1) \cdots \chi(d(\sigma_p) = m_p)
$$

It is clear that these methods can be used in other situations. For instance, they may be used to produce $q$-analogues of set theoretical formulas. Indeed, comparing the derivation of the trivariate distribution in [5] with the one given here we see that partitions can play the same role played by non-commuting variables.

1. The Trivariate Distribution of $i(0)$, $m(0)$, $d(0)$

We start by giving a new derivation of the generating function of the polynomials

$$
A_n^{i,m}(t, p, q) = \sum_{\sigma} p(\sigma) q^{m(\sigma)} t^{d(\sigma)}.
$$

(1.1)

To this end we shall work with the family $\mathcal{F}_n$ of non negative integer valued functions $f$ on the set $\Omega_n = \{1, 2, \ldots, n\}$. For each $f \in \mathcal{F}_n$ we set

$$
|f| = \sum_{i=1}^{n} f(i),
$$

$$
i(f) = \sum_{i<j} x(f(i) < f(j)).
$$

(1.2)

We shall obtain the desired generating function by giving two different expressions for the formal power series

$$
\sum_{f \in \mathcal{F}_n} p^{i(f)} q^{d(f)} t^{\max f}.
$$

(1.3)

To this end we shall first find an expression for the sum

$$
\sum_{f \in \mathcal{F}_n} p^{i(f)} q^{d(f)} t^{\max f} (\max f \leq k).
$$

(1.4)

This is easily obtained. Indeed, if we set for each $f \in \mathcal{F}_n$

$$
\omega(f) = x(f(1)) x(f(2)) \cdots x(f(n)),
$$

then MacMahon's [9] "$p$-analogue" of the multinomial identity gives

$$
\sum_{f \in \mathcal{F}_n} p^{i(f)} q^{d(f)} t^{\max f} (\max f \leq k) = \sum_{\nu_1, \ldots, \nu_n = 1} \frac{[\nu_1]_p \cdots [\nu_k]_p t^{\nu_1} \cdots t^{\nu_k}}{[\nu_1]_p \cdots [\nu_k]_p}
$$

(1.5)

where

$$
[n]_p! = [n]_p [n-1]_p \cdots [1]_p,
$$

and

$$
[n]_p = 1 + p + \cdots + p^{n-1}.
$$

Note now that the sum in (1.4) can be obtained from the expression in (1.5) upon replacing $x_i$ by $q^i$. The right hand side of (1.5) after this substitution becomes

$$
\sum_{\nu_1, \ldots, \nu_n = 1} \frac{(q^\nu)^{\nu_1} \cdots (q^\nu)^{\nu_k}}{[\nu_1]_p \cdots [\nu_k]_p t^{\nu_1} \cdots t^{\nu_k}},
$$

and this is the same as $[\nu]_p!$ times the coefficient of $w^{\nu}$ in the series

$$
\epsilon[w]_p \epsilon[qu]_p \cdots \epsilon[q^k w]_p,
$$

where $\epsilon[w]_p$ is the $p$-analogue of the exponential function. That is

$$
\epsilon[w]_p = \sum_{n=0}^{\infty} w^n [n]_p^{-1}.
$$

We thus have

$$
\frac{1}{[n]_p!} \sum_{f \in \mathcal{F}_n} p^{i(f)} q^{d(f)} t^{\max f} (\max f \leq k) = \epsilon[w]_p \epsilon[qu]_p \cdots \epsilon[q^k w]_p.
$$

(1.6)

We shall change now our point of view and obtain another expression for this series in terms of the polynomials in (1.1).

Let $f \in \mathcal{F}_n$ and let $\lambda_1 > \lambda_2 > \cdots > \lambda_{k+1} > 0$ be the different values taken by $f$, let $A_i = \{i : f(i) = \lambda_j\}$ and set

$$
\sigma(f) = \uparrow A_1 \uparrow A_2 \cdots \uparrow A_{k+1}
$$

(1.7)

where this expression is to mean that $\sigma(f)$ is the arrangement of the numbers

1 "\lambda_\nu" stands for "coefficient of $w^\nu$".
1, 2, ..., n obtained by putting first the elements of \( A_1 \) in increasing order, then the elements of \( A_2 \) in increasing order, etc. ... We shall call
\[
\sigma = (\sigma_1, \sigma_2, ..., \sigma_n) = \sigma(f)
\]
the permutation "associated" to \( f \).

Let us also set
\[
p(f) = (p_1, p_2, ..., p_n)
\]
where
\[
p_i = f(\sigma_i) - f(\sigma_{i+1}), \quad p_n = f(\sigma_n).
\]
Note that because of our definition (1.7) we shall necessarily have \( f(\sigma_i) > f(\sigma_{i+1}) \) at the descents of \( \sigma \). That is
\[
p_i \geq \chi(\sigma_i > \sigma_{i+1}) \quad \text{(for } i \leq n - 1) \quad (1.11)
\]
Conversely, given \( \sigma = (\sigma_1, \sigma_2, ..., \sigma_n) \) and \( p = (p_1, p_2, ..., p_n) \) related by (1.11) the function defined by
\[
f(\sigma_i) = p_i + p_{i+1} + \cdots + p_n
\]
will have \( \sigma \) as its associated permutation.

The map \( f \rightarrow (\sigma(f), p(f)) \) amounts to an "encoding" of the elements of \( \mathcal{F}_n \).

Indeed, given \( \sigma \) and \( p \) the equations in (1.12) completely determine \( f \).

Note now that we have
\[
\begin{align*}
\text{(a)} & \quad |f| = p_1 + 2p_2 + \cdots + np_n \\
\text{(b)} & \quad \max f = p_1 + p_2 + \cdots + p_n \\
\text{(c)} & \quad i(f) = i(\sigma(f)).
\end{align*}
\]
The first two of these relations are immediate consequences of (1.12). As for the last we note that in view of our definition (1.7) of \( \sigma(f) \) a pair \( i < j \) appears in reverse order in \( \sigma(f) \) if and only if
\[
f(i) < f(j)
\]
Comparing with (1.2) we see that (1.13c) must hold true.

From these observations we must therefore conclude that
\[
\sum_{\omega(\sigma) = \omega} p_{(i_1)\cdots p_{(i_n)}} \sum_{p_{(i_1)} \leq \cdots \leq p_{(i_n)}} t^{q_{1(1)} + \cdots + q_{n(1)}} \cdot n^{q_{1(2)} + \cdots + q_{n(2)}} (1.14)
\]
where for convenience we have set
\[
\epsilon_i = \chi(\sigma_i > \sigma_{i+1}), \quad \epsilon_n = 0
\]
Summing the right hand side of 1.14 gives
\[
\sum_{\omega(\sigma) = \omega} p_{(i_1)\cdots p_{(i_n)}} q_{1(1)} \cdots q_{n(1)} \cdot n^{q_{1(2)} + \cdots + q_{n(2)}} = \prod_{t \leq q} (1 - t q^n \cdots (1 - tq^n)
\]
Using this expression in (1.14) and summing over \( \sigma \) we get
\[
\sum_{\omega(\sigma) = \omega} \sum_{\sigma(\sigma) = \omega} p_{(i_1)\cdots p_{(i_n)}} q_{1(1)} \cdots q_{n(1)} = \prod_{t \leq q} (1 - t q^n \cdots (1 - tq^n)
\]
Comparing with (1.6) we finally derive that
\[
\sum_{\omega(\sigma) = \omega} \sum_{\sigma(\sigma) = \omega} p_{(i_1)\cdots p_{(i_n)}} q_{1(1)} \cdots q_{n(1)} = \prod_{t \leq q} (1 - t q^n \cdots (1 - tq^n)
\]
(1.15)

2. THE FOUR-VARIATE DISTRIBUTION OF \( d(v), d(v^{-1}), m(v), m(v^{-1}) \)

In this section we shall give a proof of the identity (1.8). The setting here will be entirely similar to that of the previous section only we shall replace the nonnegative integer valued functions \( f \) by pairs of such functions
\[
\tilde{f} = (f_1(1), \ldots, f_1(n)) \quad (2.1)
\]
subjected to the condition that for each \( i < n \)
\[
\begin{align*}
\text{(a)} & \quad \text{either } f_1(i) = f_1(i + 1) \\
\text{(b)} & \quad \text{or } f_1(i) = f_1(i + 1) \text{ and } f_2(i) = f_2(i + 1)
\end{align*}
\]
These are the "bipartite" partitions studied by B. Gordon in \[7\].

Let us denote the family of all \( f \) of the form (2.1) which satisfy (2.2) by \( \mathcal{B}_n \) and let us set
\[
\mathcal{B}_n(k_1, k_2) = \{ f \in \mathcal{B}_n : \max f_1 \leq k_1, \max f_2 \leq k_2 \}
\]
We shall derive our identity by giving two different expressions for the formal power series
\[
\sum_{\tilde{f} \in \mathcal{B}_n} q_{1(1)} q_{1(2)} \cdots q_{n(1)} \cdots q_{n(2)} = \prod_{t \leq q} (1 - t q^n \cdots (1 - tq^n)
\]
where as before we have set
\[
|f_i| = \sum_{i=1}^{n} f(i) \quad (\nu = 1, 2)
\]
Our first task is to find an expression for the sum
\[ \sum_{f \in \mathcal{B}(k_1, k_2)} g_{i_1}^{j_1} g_{i_2}^{j_2} (2.4) \]
To this end we shall interpret each \( f \in \mathcal{B}(k_1, k_2) \) as a "multisubset" of the "rectangle" of pairs \((i, j)\):
\[ \{(i, j) : 0 \leq i \leq k_1, 0 \leq j \leq k_2\}. \]
Indeed, we can identify a given \( f \in \mathcal{B}(k_1, k_2) \) with the monomial
\[ w(f) = \prod_{i=0}^{k_1} \prod_{j=0}^{k_2} (i_{i,j})^\mu_{i,j}, \]
where \( \mu_{i,j} \) denotes the "multiplicity" of \((i, j)\) in \( f \), more precisely
\[ \mu_{i,j} = \# \{ v : f_i(v) = i, f_j(v) = j \}. \]
Conversely, given such a monomial we can obtain an \( f \in \mathcal{B}(k_1, k_2) \) by ordering "lexicographically" the pairs \((i, j)\) occurring in it.
This gives
\[ \sum_{f \in \mathcal{B}(k_1, k_2)} w(f) = \prod_{i=0}^{k_1} \prod_{j=0}^{k_2} \left( \frac{1}{1 - u(i_{i,j})} \right) \cdot \mu_{i,j}. \]
Now note that since each pair \((i, j)\) contributes a \( q_i^{j_1} q_j^{j_2} \) to the monomial \( g_{i_1}^{j_1} g_{i_2}^{j_2} \), the sum in (2.4) can be obtained from the expression in (2.6) upon replacing \((i, j)\) by \( q_i^{j_1} q_j^{j_2} \).
We thus have
\[ \sum_{f \in \mathcal{B}(k_1, k_2)} q_i^{j_1} q_j^{j_2} = \prod_{i=0}^{k_1} \prod_{j=0}^{k_2} \left( \frac{1}{1 - u q_i q_j} \right) \cdot \mu_{i,j}. \]
Multiplying by \( t_i^j t_j^i \) and summing we finally obtain
\[ \sum_{f \in \mathcal{B}(k_1, k_2)} q_1^{j_1} q_2^{j_2} \cdot \frac{t_1^{\max_i} t_2^{\max_j}}{1 - t_1} \frac{t_2^{\max_j}}{1 - t_2} = \sum_{i,j=0}^{k_1, k_2} t_i^j t_j^i \prod_{i=0}^{k_1} \prod_{j=0}^{k_2} \left( \frac{1}{1 - u q_i q_j} \right) \cdot \mu_{i,j}. \]
This is one side of our identity.
To get the other side we resort to a "Mahonian" encoding of bipartite partitions given in essence by B. Gordon in [7]. Indeed we can work on the other side with the additional statistic
\[ i(f) = \sum_{i<j} \chi(f_i(i) < f_j(j)). \]
More precisely we shall relate the formal series
\[ \sum_{f \in \mathcal{B}(k_1, k_2)} q_i^{j_1} q_j^{j_2} \cdot t_1^{\max_i} t_2^{\max_j} \]
to the five-variate distribution
\[ \sum_{f \in \mathcal{B}(k_1, k_2)} q_i^{j_1} q_j^{j_2} \cdot m_{i,j} t_1^{\max_i} t_2^{\max_j} \]
To state the basic result here we need an additional definition. Given a permutation \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \) we shall say that a partition \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0) \) is \( \sigma \)-compatible if
\[ \lambda_1 - \lambda_{\sigma_i} \geq \chi(\sigma_i > \sigma_{\sigma_i+1}) \quad (i \leq n - 1), \]
that is \( \lambda \) is \( \sigma \)-compatible if it is "strict" at the descent set of \( \sigma \).
Clearly, a partition \( \lambda \) is \( \sigma \)-compatible if and only if it is of the form
\[ \lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n \]
with
\[ \lambda_1 \geq \epsilon_1 = \begin{cases} 1 & \text{if } \sigma_1 > \sigma_{\sigma_1}, i < n \\ 0 & \text{otherwise} \end{cases} \]
Thus if \( \mathcal{P}(\sigma) \) denotes the set of all \( \sigma \)-compatible partitions we must have
\[ \sum_{\lambda \in \mathcal{P}(\sigma)} t_1^{\max_i} q_i^{j_1} = \sum_{\lambda \in \mathcal{P}(\sigma)} \sum_{\nu \in \mathcal{P}(\nu) \cap \mathcal{P}(\mu)} t_1^{\nu_1} \cdots t_n^{\nu_n} = \frac{t_1^{\sigma_1} \cdots t_n^{\sigma_n}}{(1 - t_1)(1 - t_2)^2 \cdots (1 - t_n^2)} \]
This given we have the following basic fact.

**Theorem 2.1.** There is a bijection between the bipartite partitions and the triplets
\[ (\sigma, \lambda, \mu) \]
where \( \sigma \) is a permutation, \( \lambda \) is a partition compatible with \( \sigma \) and \( \mu \) is a partition compatible with \( \sigma^{-1} \). This bijection is simply given by the map
\[ f = \Phi(\sigma, \lambda, \mu) = \left( \mu_{\sigma_1}, \mu_{\sigma_2}, \ldots, \mu_{\sigma_n} \right). \]
Furthermore when this holds we have
\[ i(f) = i(\sigma). \] (2.15)

**Proof.** We observe first that if \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n), \mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n) \) and \( \lambda \) is \( \sigma \)-compatible then the pairs
\[
\begin{pmatrix}
\lambda_1 \\
\mu_{\sigma_1} \\
\lambda_2 \\
\mu_{\sigma_2} \\
\vdots \\
\lambda_n \\
\mu_{\sigma_n}
\end{pmatrix}
\]
are in lexicographic order. Indeed either \( \lambda_i > \lambda_{i+1} \) or if \( \lambda_i = \lambda_{i+1} \) the \( \sigma \)-compatibility of \( \lambda \) gives \( \sigma_i < \sigma_{i+1} \) and so we must have \( \mu_{\sigma_i} \geq \mu_{\sigma_{i+1}} \).

This means that \( \Phi \) is indeed a map into \( S_n \).

Next we show that every bipartite partition
\[
f = \begin{pmatrix} f_1(1) & \cdots & f_1(n) \\ f_2(1) & \cdots & f_2(n) \end{pmatrix}
\]
has only one representation of the form 2.14. Let then
\[
f_1(i) = \lambda_i, \\
f_2(i) = \mu_i.
\]
This gives
\[ f_2(\sigma^{-1}(i)) = \mu_i. \] (2.16)

Since some of the values of \( f_2 \) can be equal there are of course several permutations \( \sigma \) which satisfy (2.16). However, the \( \sigma^{-1} \) compatibility requirement on \( \mu \) makes \( \sigma \) unique.

Indeed, let us interpret (2.16) as a "labelling" of the values of \( f_2 \). More precisely we interpret (2.16) as saying that

"\( f_2(\sigma^{-1}(i)) \) is the \( i \)th value of \( f_2 \)."

Now, if \( \mu \) is \( \sigma^{-1} \)-compatible the equality \( \mu_i = \mu_{i+1} \) implies \( \sigma^{-1}(i) < \sigma^{-1}(i+1) \). But this means that \( \mu_i = \mu_{i+1} \) implies that the \( i \)th value of \( f_2 \) is to left of the \( (i+1) \)st.

Thus we get a simple rule for constructing the permutation \( \sigma \). We label the values of \( f_2 \) from left to right starting from the largest ones, the next largest ones, ..., and finally the smallest ones.

For instance the bipartite partition
\[
f = \begin{pmatrix} 5 & 5 & 3 & 3 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 & 2 & 4 & 3 \end{pmatrix}
\]
is labelled
\[
f = \begin{pmatrix} 5 & 5 & 3 & 3 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 & 2 & 4 & 3 \end{pmatrix}
\]

This gives that \( \Phi \) is one-to-one. We show next that \( \Phi \) is onto.

Thus this example we obtain
\[
\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 7 & 1 & 3 & 6 & 2 & 4 \end{pmatrix}
\]

This gives that \( \Phi \) is onto.

To this end we must show that, given a bipartite partition \( f \), then if
\[
\begin{align*}
(a) & \quad \text{we set } \lambda_i = f_1(i) \\
(b) & \quad \text{we let } \mu = (\mu_1 \geq \cdots \geq \mu_n) \text{ be the non-increasing rearrangement of the values of } f_2 \\
(c) & \quad \text{we construct } \sigma \text{ by the above labelling procedure.}
\end{align*}
\]

then \( \lambda \) will be \( \sigma \)-compatible and \( \mu \) will be \( \sigma^{-1} \)-compatible.

Now, note that the way we carry out our labelling procedure the inequality \( \sigma_i > \sigma_{i+1} \) can only occur when \( f_2(i) \) is smaller than \( f_2(i+1) \), but then the "lexicographic" conditions 2.2 force \( f_2(i) > f_2(i+1) \). In other words, we have \( \sigma_i > \sigma_{i+1} \) only when \( \lambda_i > \lambda_{i+1} \).

Finally, we observe that our labelling procedure puts the \( i \)th value of \( f_2 \) to the right of the \( (i+1) \)st (that is \( \sigma^{-1}(i) > \sigma^{-1}(i+1) \)) only when \( f_2(\sigma^{-1}(i)) > f_2(\sigma^{-1}(i+1)) \). That means we must have \( \mu_i > \mu_{i+1} \) when \( \sigma^{-1}(i) > \sigma^{-1}(i+1) \).

So \( \mu \) is \( \sigma^{-1} \)-compatible. Thus \( \Phi \) is onto.

To complete the proof we must show that
\[ i(f) = i(\sigma). \] (2.17)

Note that our labelling procedure gives for \( i < j \)
\[
\begin{align*}
\sigma_i & < \sigma_j \\
\sigma_1 & > \sigma_i \\
\sigma_1 & > \sigma_j
\end{align*}
\]

Thus we have \( \sigma_i > \sigma_j \) if and only if \( f_1(i) < f_1(j) \) and this is 2.17.

The proof of the theorem is now complete.

Using the representation (2.14) we can immediately obtain the desired relation between the series in (2.10) and the polynomials in (2.11).
Theorem 2.2
\[
\sum_{f \in \mathcal{F}_s} p^{(\ell)} q_1^{m(\ell)} q_2^{m(\ell)} \prod_{i \in i(f)} q_1^{i(f)} q_2^{i(f)} = \sum_{f \in \mathcal{F}_s} p^{(\ell)} q_1^{m(\ell)} q_2^{m(\ell)} (1 - t_1q_1)(1 - t_2q_2)(1 - t_3q_3) \cdot (1 - t_4q_4) \cdot (1 - t_5q_5) \cdot (1 - t_6q_6). \tag{2.18}
\]

Proof. Theorem 2.1 permits us to rewrite the left hand side of (2.18) in the form
\[
\sum_{s} \sum_{\tau \in \mathcal{P}(s)} \sum_{a \in \mathcal{P}(s^{-1})} p^{(\ell)} q_1^{m(\ell)} q_2^{m(\ell)} \prod_{i \in i(f)} q_1^{i(f)} q_2^{i(f)}.
\]
Thus (2.18) is an immediate consequence of formula (2.13).

From this result we derive

Theorem 2.3
\[
\sum_{f \in \mathcal{F}_s} p^{(\ell)} q_1^{m(\ell)} q_2^{m(\ell)} (1 - t_1q_1)(1 - t_2q_2)(1 - t_3q_3) \cdot (1 - t_4q_4) \cdot (1 - t_5q_5) \cdot (1 - t_6q_6) \prod_{i \in i(f)} \prod_{i \in i(f)} \frac{1}{1 - u_i^s q_i^s} \tag{2.19}
\]

Proof. Simply set \( p = 1 \) in (2.18) and compare with (2.7).

It may be worthwhile making some additional comments here.

Remark 2.1. (On the five-variate distribution.) Clearly we can use (2.18) to "obtain" a generating function for the joint distribution of the five statistics in (1.1). We observe that if
\[
\omega(f) = \prod_{i \in i(f)} \prod_{i \in i(f)} \binom{t_f}{i(f)} \tag{2.20}
\]
then \( i(f_2) \) counts the number of pairs \( (i, i') \) in (2.20) which satisfy the inequalities
\[
i_i > i_{i'}, \quad j_i < j_{i'}.
\]

Thus if \( T \) is the operator which sends the monomial
\[w = \prod_{i \in i(f)} \prod_{i \in i(f)} \binom{t_f}{i(f)}\]
into
\[T w = p^{(\omega)} q_1^{m(\omega)} q_2^{m(\omega)} \]
where \( \omega(f) \) denotes the number of such pairs then we have that the left hand side of (2.18) divided by \( (1 - t_1q_1)(1 - t_2q_2) \) is also equal to
\[
T \sum_{f \in \mathcal{F}_s} \sum_{a \in \mathcal{P}(s)} \prod_{i \in i(f)} \prod_{i \in i(f)} \frac{1}{1 - u_i^s q_i^s} \tag{2.21}
\]

This yields a "form" of generating function for the five-variate distributions.

However, it is quite possible that some other approach may yield a more explicit expression.

Remark 2.2. (On \( k \)-partite partitions.) In [7] \( k \)-partite partitions are defined as \( n \)-tuples
\[
f = \left( \begin{array}{c}
f_1(1) \\
f_2(1) \\
\vdots \\
f_k(1)
\end{array} \right) \quad \left( \begin{array}{c}
f_1(n) \\
f_2(n) \\
\vdots \\
f_k(n)
\end{array} \right)
\]
of \( k \)-vectors satisfying the conditions
\[
\text{if } f_i(v) = f_i(v + 1) \text{ for all } i \leq j \text{ and } f_{i+1}(v) \neq f_{i+1}(v + 1) \text{ then } f_{i+1}(v) > f_{i+1}(v + 1).
\]

Given such a \( k \)-partite partition we label the values of \( f_i \) for each \( i \leq k \) starting from the last row \( (i = k) \) and proceeding up in the following manner.

We label the values of \( f_k \) as in the proof of Theorem 2.1.

Then inductively, given that the values of \( f_{i+1} \) are labelled
\[\sigma_i^{(i+1)} = \sigma_1^{(i+1)}, \sigma_2^{(i+1)}, \ldots, \sigma_n^{(i+1)}\]
we label the values of \( f_i \), starting with the largest ones, then the next largest ones, etc.

But this time breaking ties in a manner that is compatible with the permutation \( \sigma_i^{(i+1)} \). For instance the tripartite partition
\[
\begin{pmatrix}
5 & 5 & 5 & 3 & 3 \\
4 & 3 & 3 & 2 & 2 \\
1 & 2 & 1 & 3 & 1
\end{pmatrix}
\]
is labelled
\[
\begin{pmatrix}
5 & 5 & 5 & 3 & 3 \\
1 & 2 & 1 & 3 & 1 \\
3 & 2 & 4 & 1 & 5
\end{pmatrix}
\]
It is a property of the lexicographic ordering that this procedure always labels the entries in the first row in the natural order.

Now, let \( \lambda^{(i)} \) denote the non-increasing rearrangement of the values of \( f_i \) and \( \omega^{(i)} \) be the permutations defined by setting

\[
\omega^{(i)} = \left( \sigma_i^{(i)} \cdots \sigma_n^{(i)} \right) \text{ if } i < k
\]

and

\[
\omega^{(i)} = \left( \begin{array}{c}
\sigma_1^{(i)} \\
1 \\
\cdots \\
n
\end{array} \right)
\]

It can be shown that this gives a bijection of the family \( \mathcal{B}^{(k)} \) of \( k \)-partite partitions and the \( (2k) \)-tuples

\[
(\omega^{(1)}, \ldots, \omega^{(k)}) \cup (\lambda^{(1)}, \ldots, \lambda^{(k)})
\]

where

(a) \( \omega^{(1)}, \ldots, \omega^{(k)} \) are permutations whose product is the identity,
(b) \( \lambda^{(1)}, \ldots, \lambda^{(k)} \) are partitions,
(c) \( \lambda^{(i)} \) is \( \omega^{(i)} \)-compatible.

Using this fact one proves the identity

\[
\sum_{\lambda^{(1)}, \ldots, \lambda^{(k)}} \tau_1^{\lambda_1} \cdots \tau_k^{\lambda_k} \prod_{i=1}^{k} \frac{1}{1 - \tau_i^{\lambda_i}} \prod_{i=2}^{k} \frac{1}{1 - \tau_i \tau_1^{\lambda_i}} = \sum_{\lambda^{(1)}, \ldots, \lambda^{(k)}} \frac{g_1(\lambda_1) \cdots g_k(\lambda_k) \tau_1^{\lambda_1} \cdots \tau_k^{\lambda_k}}{\prod_{i=1}^{k} (1 - \tau_i) (1 - \tau_i^{\lambda_i})}
\]

from which our statement concerning Basil Gordon's integers \( \lambda(m_1, \ldots, m_k) \) may be easily derived.

**Remark 3.3.** Historically speaking, MacMahon might have been the first to have used permutation statistics to encode partitions. Basil Gordon in [7] uses similar ideas to encode bipartite partitions. Some time later Carlitz, et al. [2] and Roselle in [10] essentially rediscovered Basil Gordon’s encoding. However, roughly speaking, B. Gordon’s work and [10] are only concerned with major index statistics and in [2] they are only concerned with descents. The amusing fact is that this encoding carries information also about the number of inversions.

**Remark 3.4.** It is interesting to point out that from our identities we can obtain a generating function proof that \( d(\sigma), m(\sigma) \) and \( m(\sigma^{(1)}) \) have the same joint distribution as \( d(\sigma), i(\sigma), m(\sigma) \). This remarkable result was essentially stated in [3] where it follows from purely combinatorial considerations.

To prove this here we let \( t_1 = 1, t_2 = 1, q_2 = q, \) and \( q_1 = q \) in (2.19) obtaining

\[
\frac{\sum_{\lambda^{(1)}, \ldots, \lambda^{(k)}} \tau_1^{\lambda_1} \cdots \tau_k^{\lambda_k} \prod_{i=1}^{k} \frac{1}{1 - \tau_i^{\lambda_i}}}{(1 - t_1) (1 - t_1^{q_1}) (1 - t_2) (1 - t_2^{q_2}) \prod_{i=1}^{k} \frac{1}{1 - \tau_i^{\lambda_i}}} = (1 - p)^e \sum_{\lambda^{(1)}, \ldots, \lambda^{(k)}} \frac{1}{\prod_{i=1}^{k} \frac{1}{1 - \tau_i^{\lambda_i}}}
\]

Observe that we do also have \( e_{\lambda}(u) = \prod_{i=1}^{k} 1/(1 - \tau_i^{\lambda_i}) \) and comparing with (1.15) the assertion follows immediately.

### 3. Major Index Statistics of Shuffle Permutations

Our third example should illustrate more clearly the scope of the method we alluded to in the introduction and put our previous examples into proper perspective.

The general setting is as follows. We are given a partially ordered set \( \mathcal{P} = (\Omega, \leq) \) and a labelling \( \omega: \Omega \to \{1, 2, \ldots, n\} \) of its elements. For convenience we let the elements of \( \Omega \) be \( x_1, x_2, \ldots, x_n \). The labelling \( \omega \) is thus obtained by fixing some permutation

\[\omega = (\omega_1, \omega_2, \ldots, \omega_n)\]

of \( \{1, 2, \ldots, n\} \) and setting

\[\omega(x_i) = \omega_i.\]

We do not assume that \( \omega \) has any relation whatsoever with the partial ordering of \( \mathcal{P} \).

Our object of study is then the family of non-negative, decreasing integer valued functions on \( \Omega \) which are strictly decreasing when \( \omega \) is strictly decreasing.

More precisely we set

\[\mathcal{F}_\mathcal{P}(\mathcal{P}) = \{ f: \Omega \to \mathbb{Z}^+ : x < y \implies f(x) \geq f(y) \text{ with } "\geq" \text{ when } \omega(x) > \omega(y) \}.\]

We call the elements of \( A_\mathcal{P}(\mathcal{P}) \) the \( \omega \)-compatible partitions.

This given let \( f \in \mathcal{F}_\mathcal{P}(\mathcal{P}) \) and let \( \lambda_1 > \lambda_2 > \cdots > \lambda_k \geq 0 \) be the different values taken by \( f \). Let \( A_\mathcal{P}(f) = \{ x : f(x) = \lambda_1 \} \) and set

\[\sigma(f) = 1_\mathcal{P} A_1 1_\mathcal{P} A_2 \cdots 1_\mathcal{P} A_k, \quad \text{(3.1)}\]

where this expression is to mean that \( \sigma(f) \) is the arrangement of the elements of \( \Omega \) obtained by putting first the elements of \( A_1 \) in order of increasing \( \omega \), then the elements of \( A_2 \) in the same manner, etc. We call

\[\sigma(f) = x_1 x_2 \cdots x_n, \quad \text{(3.2)}\]
the "permutation of $\Omega$ associated to $f". Let us also set

$$p(f) = (p_1, p_2, \ldots, p_n)$$

(3.3)

where

$$p_i = f(x_i) - f(x_{i+1}), \quad p_n = f(x_n).$$

(3.4)

One thing must be observed at once. Namely, for any $f \in \mathcal{P}(\Omega)$ the resulting arrangement $\sigma(f)$ is always compatible with the partial ordering of $\Omega$.

For historical reasons we shall call such arrangements the "Standard $\mathcal{P}$-tableaux" and denote the set of all such arrangements by $\mathcal{P}(\Omega)$.

Secondly we note that in view of the definition of $\sigma(f)$ the sequence $\{f(x_i)\}$ must decrease strictly at the descents of the permutation

$$\omega(x_i) = (\omega(x_{i+1}), \omega(x_{i+2}), \ldots, \omega(x_n)).$$

In other words the components of $p(f)$ satisfy the inequalities

$$p_i \geq p_i + 1 \quad \text{if} \quad \omega(x_i) > \omega(x_{i+1}) \quad (i < n)$$

$$= 0 \quad \text{otherwise}.$$ 

(3.5)

Conversely, given any $x_i \in \mathcal{P}(\Omega)$ and a $p \in \mathbb{Z}^n$, whose components satisfy (3.5), the function $f$ defined by setting

$$f(x_i) = p_i + p_{i+1} + \cdots + p_n$$

(3.6)

will necessarily be in $\mathcal{P}(\Omega)$. Furthermore, the map in (3.6) is a bijection between $\mathcal{P}(\Omega)$ and the pairs $(x_i, p)$ related by (3.5). This is in essence the content of Stanley's theorem 6.2 in [12].

One of the consequences of this result is that if we set

$$|f| = \sum_{i=1}^n f(x_i)$$

$$i(f) = \sum_{\omega(x \leq y \in f)} x(f(x) < f(y))$$

we necessarily have as before

$$\sum_{f \in \mathcal{P}(\Omega)} p(f) \max |f|^{q^{|f|}} = \sum_{f \in \mathcal{P}(\Omega)} p^{\omega(x_{i+1})} q^{\omega(x_{i+2})} \cdots (1 - q^n)$$

(3.7)

We should recognize at once then that our first example corresponds to none other than the simplest case of this set up. Namely the case in which $\mathcal{P} = (\Omega, \Phi)$ where "$\Phi" denotes the "empty" relation and $\omega$ is the natural labelling of $\Omega = (1, 2, \ldots, n)$.

Our third example will correspond to the case in which $\mathcal{P}$ is a "product of chains". To see what (3.7) yields in this case we need some further terminology.

For a given ordered subset $\pi = (i_1, i_2, \ldots, i_k)$ of $\{1, 2, \ldots, n\}$ we set

$$m(\pi) = \sum_{i=1}^k p(x_i > i_{i+1})$$

and call it the "major index" of $\pi$.

Furthermore, given a collection of $\{\pi_1, \pi_2, \ldots, \pi_k\}$ of complementary ordered subsets of $\Omega_n = (1, 2, \ldots, n)$ (i.e., they are disjoint as subsets and their union is $\Omega_n$) we shall say that a permutation $\sigma$ of $\{1, 2, \ldots, n\}$ is obtained by "shuffling" $\pi_1, \pi_2, \ldots, \pi_k$ if the elements of each $\pi_i$ appear in $\sigma$ in the same order as they did in $\pi_i$ itself. That is, $\sigma$ is obtained by shuffling in the ordinary sense (without cutting) the "decks" corresponding to $\pi_1, \pi_2, \ldots, \pi_k$.

This given we have the following remarkable extension of a result of MacMahon (see [1] theorem 3.7, page 42).

**Theorem 3.1.** Let $\pi_1, \pi_2, \ldots, \pi_k$ be ordered complementary subsets of $\{1, 2, \ldots, n\}$. Let $\mathcal{P}(\pi_1, \pi_2, \ldots, \pi_k)$ be the collection of permutations of $\{1, 2, \ldots, n\}$ obtained by shuffling $\pi_1, \pi_2, \ldots, \pi_k$, then

$$\sum_{\sigma \in \mathcal{P}(\pi_1, \pi_2, \ldots, \pi_k)} q^{\omega(\pi)} = \left[ \begin{array}{c} n \\ \mu_1 \mu_2 \cdots \mu_k \end{array} \right] q^{m(\pi_1) + \cdots + m(\pi_k)}$$

(3.8)

where $\mu_i = \text{card } \pi_i$ and $[\nu_1 \nu_2 \cdots \nu_k]$ denotes the $q$-analogue of the multinomial coefficient.

**Proof.** We take $\mathcal{P}$ to be the product of $k$ chains $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ of respective lengths $\mu_1, \mu_2, \ldots, \mu_k$. We call the elements of the first chain $x_1, x_2, \ldots, x_{\mu_1}$ those of the second chain $x_{\mu_1+1}, x_{\mu_1+2}, \ldots, x_{\mu_1+\mu_2}$ etc. We then take $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ to be the juxtaposition of the ordered subsets $\pi_1, \pi_2, \ldots, \pi_k$ and set

$$\omega(x_i) = \omega_i.$$ 

In the figure below we illustrate the case in which $k = 3, n = 9$ and

$$\pi_1 = (7, 1, 2, 6), \quad \pi_2 = (4, 8, 5), \quad \pi_3 = (3, 9).$$

![Diagram of shuffling chains]

We should recognize at once then that our first example corresponds to none other than the simplest case of this set up. Namely the case in which $\mathcal{P} = (\Omega_n, \Phi)$ where "$\Phi" denotes the "empty" relation and $\omega$ is the natural labelling of $\Omega_n = (1, 2, \ldots, n)$.
The label above $x_i$ is the value of $u(x_i)$. This given we see that if we apply formula (3.7) (with $r = p = 1$) to the chain $\mathcal{P}$, we obtain
\[
\sum_{i \in \mathcal{P}(x_i)} q^{m(x_i)} = \frac{1 - q(1 - q^2) \cdots (1 - q^n)}{1 - q(1 - q^2) \cdots (1 - q^n)}
\] (3.9)

(there being only one standard $\mathcal{P}$-tableau).

On the other hand, applying the same formula to the poset $\mathcal{P}$ as a whole (and making the obvious identification between permutations in $\mathcal{P}(x_1, x_2, \ldots, x_n)$ and standard $\mathcal{P}$-tableaux) we obtain
\[
\sum_{i \in \mathcal{P}(x_1, x_2, \ldots, x_n)} q^{m(x_i)} = \frac{\sum_{i \in \mathcal{P}(x_1, x_2, \ldots, x_n)} q^{m(x_i)}}{1 - q(1 - q^2) \cdots (1 - q^n)}
\] (3.10)

Now it is not difficult to see that we must have
\[
\sum_{i \in \mathcal{P}(x_1, x_2, \ldots, x_n)} q^{m(x_i)} = \prod_{i=1}^{n} \sum_{i \in \mathcal{P}(x_i)} q^{m(x_i)}
\]

Thus if we substitute (3.9) and (3.10) in this relationship we derive
\[
\sum_{\sigma \in \mathcal{P}(x_1, x_2, \ldots, x_n)} q^{m(x)} = \frac{1 - q(1 - q^2) \cdots (1 - q^n) q^{\sigma(x_1) \cdots \sigma(x_n)}}{\prod_{i=1}^{n} (1 - q) \cdots (1 - q^n)}
\]

and this is another way of writing formula (3.8).

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