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A PROBABILISTIC METHOD FOR LATTICE PATH ENUMERATION

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Abstract: A probabilistic method for obtaining functional equations for lattice path enumeration problems is studied.

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1. Introduction

We study a probabilistic method for obtaining functional equations for generating functions for lattice path enumeration problems.

The method is best explained by a simple example. Fix positive integers r and μ, and let d_n be the number of paths in the plane, with unit steps in the positive horizontal and vertical directions, from the origin to $(n, r+\mu n)$ which never touch the line $y=r+\mu x$ except at the endpoint. We now consider a particle which starts at the origin and successively moves with probability p one unit to the right and with probability $q=1-p$ one unit up. The particle stops if it touches the line $y=r+\mu x$.

First we find the probability that the particle eventually stops. The probability that the particle stops at $(n, r+\mu n)$ is $p^n q^{r+\mu n} d_n$. Thus if we set $d(t)=\sum_{n=0}^{\infty} d_n t^n$, the probability that the particle eventually stops is $q^r d(pq^\mu)$.

However it is not hard to show that if p is sufficient small (actually, $p \le 1/(\mu+1)$) then the particle will eventually reach the line $y=r+\mu x$ with probability 1. Thus for p small, we have

$$q^r d(pq^\mu)=1, \tag{1}$$

where $q=1-p$.

This is a functional equation for d. It can be solved by Lagrange inversion (Goulden and Jackson (1983), p. 17) to yield the explicit formula

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$$d_n=\frac{r}{(\mu+1)n+r}\binom{(\mu+1)n+r}{n},\tag{2}$$

a well-known result. (See, for example, Mohanty (1979), p. 9.)

The simplest case, in which $\mu=1$, is worth examining more closely. Here (1) becomes $(1-p)^r d(p-p^2)=1$. If we set $t=p-p^2$, then we have $p=\frac{1}{2}(1\pm\sqrt{1-4t})$. Since $p\le\frac{1}{2}$, we must have in fact $p=\frac{1}{2}(1-\sqrt{1-4t})$ and a little algebra gives

$$d(t)=\left[\frac{1-\sqrt{1-4t}}{2t}\right]^r.$$

Note that $d(t)$ has a branch point at $t=\frac{1}{4}$, which corresponds to $p=\frac{1}{2}$.

It is natural to ask what the probability is that the particle will eventually stop if $p>\frac{1}{2}$. Our argument shows that this probability is $q^r d(pq)$. We know that $q^r d(pq)=1$ for $p\le\frac{1}{2}$, so by the symmetry between p and q, we must have $p^r d(pq)=1$ for $p\ge\frac{1}{2}$, so $d(pq)=p^{-r}$. Thus for $p\ge\frac{1}{2}$, $q^r d(pq)=(q/p)^r$. A similar argument has been given by Cohen and Katz (1980). For $\mu=2$ one can find the probability that the particle stops for $p>\frac{1}{3}$ by solving the quadratic equation.

Before proceeding to further examples, we make several observations. Dwass (1967) also used a probabilistic argument to derive functional equations for generating functions to count paths, but in a somewhat different way. For further applications of Dwass's method, see Aneja and Sen (1972), Handa and Mohanty (1979), Mohanty and Handa (1970), and Mohanty (1979), pp. 108-115.

We note that we can avoid the use of probability in deriving (1), at the cost of some extra work. With $e_{m,n}$ the number of paths from (0,0) to (m,n) that never touch the line $y=r+\mu x$ and d_n as before, a combinatorial argument yields

$$\binom{m+n}{m}=e_{m,n}+\sum_{i=0}^{K}d_i\binom{m+n-r-(\mu+1)i}{m-i},\tag{3}$$

where $K=\min(m,\lfloor(n-r)/\mu\rfloor)$. With p an indeterminate, multiply (3) by p^m and sum on m, obtaining

$$\frac{1}{(1-p)^{n+1}}=\sum_{m>(n-r)/\mu}e_{m,n}p^m+\sum_{i\le(n-r)/\mu}d_ip^i/(1-p)^{n-r-\mu i+1}.\tag{4}$$

Now multiply both sides of (4) by $(1-p)^{n+1}$ and take the limit as $n\to\infty$. We obtain

$$1=(1-p)^r\sum_{i=0}^{\infty}d_i[p(1-p)^{\mu}]^i,$$

which is (1).

The same argument works with an arbitrary upper boundary: let $u_0,u_1,u_2,\ldots$ be arbitrary integers satisfying $0\le u_0\le u_1\le\cdots$, and let f_i be the number of paths from (0,0) to (i,u_i) which never touch any (j,u_j) for $j<i$. Then as formal power series in p,

$$\sum_{i=0}^{\infty}f_ip^i(1-p)^{u_i}=1.\tag{5}$$

Moreover, equating coefficients in (5) gives a recurrence from which the f_i can be determined. If p is a number between 0 and 1, then the left side of (5) may be interpreted as the probability that a particle will eventually reach one of the points (i,u_i). If the u_i grow too rapidly (for example, if $u_i=1+i^2$), then for $p>0$ this probability is always less than 1. Thus in these cases we have the unusual phenomenon that (5) is true as a formal power series identity, but is false whenever p is a real number, $0<p<1$, even though the sum exists.

2. Another simple example

We give one more application of the method to a well-known problem before considering some harder problems.

We consider the boundary lines $y=x+a$ and $y=x-b$, where $a,b>0$. Let f_m be the number of paths from (0,0) to $(m,a+m)$ which never touch any boundary points except at the end, and let g_n be the number of analogous paths ending at $(b+n,n)$. Let $f(t)=\sum_{m=0}^{\infty}f_mt^m$ and $g(t)=\sum_{n=0}^{\infty}g_nt^n$. Then the probabilistic method yields the functional equation

$$q^af(pq)+p^bg(pq)=1,\tag{6}$$

where $q=1-p$, for all $0\le p\le 1$. Since (6) holds for all p, we may interchange p and q to obtain

$$p^af(pq)+q^bg(pq)=1.\tag{7}$$

Solving (6) and (7), we find

$$f(pq)=(q^b-p^b)/(q^{a+b}-p^{a+b}),$$

$$g(pq)=(q^a-p^a)/(q^{a+b}-p^{a+b}).$$

Now let $t=p(1-p)$. We can solve for p and q to get $p,q=\frac{1}{2}(1\pm\sqrt{1-4t})$.

The well-known expressions for f_m and g_n as alternating sums of binomial coefficients can be obtained by expanding in powers of $\frac{1}{2}(1-\sqrt{1-4t})$ and using the formula

$$\left(\frac{1-\sqrt{1-4t}}{2}\right)^k=\sum_{n=k}^{\infty}\frac{k}{2n-k}\binom{2n-k}{n-k}t^n.$$

Our approach also gives the probabilities of reaching either of the lines first.

3. Half-integer slopes

Suppose we want to count paths which never touch or cross the line $y=r+\mu x$ where r and μ are arbitrary nonnegative real numbers. To apply our method we need to find the boundary points of the allowed region, that is, the lattice points (i,u_i) such that (i,u_i) is outside the region but (i,u_i-1) is inside. We find that $u_i=\lceil r+\mu i\rceil$, where $\lceil z\rceil$ is the least integer not less than z. If μ is irrational, little more can be said. However, if μ is rational, with denominator d, then the points $(i,\lceil r+\mu i\rceil)$ lie on d lines, according to the residue class of i (mod d). Thus (5) can be written as a functional equation involving d unknown functions. We consider here the simplest nontrivial case, $d=2$. A similar analysis holds in the general case.

Let f_n be the number of paths from $(0,0)$ to $(n,\lceil r+\mu n\rceil)$ which never touch any point $(i,\lceil r+\mu i\rceil)$ until the end. We assume that $\mu=\lambda+\frac{1}{2}$ where λ is a nonnegative integer. It is easy to see that without loss of generality we may take r to be either an integer or a half-integer. For simplicity we take $r=s-\frac{1}{2}$ where s is a positive integer; the case in which r is an integer is similar. Let $P_i=(i,\lceil r+\mu i\rceil)=(i,s+\lambda i+\lceil\frac{1}{2}(i-1)\rceil)$. So

$$P_{2i}=(2i,s+(2\lambda+1)i)\quad\text{and}\quad P_{2i+1}=(2i+1,s+\lambda+(2\lambda+1)i).$$

Let $g_i=f_{2i}$ and $h_i=f_{2i+1}$, and let $g(t)=\sum_{i=0}^{\infty}g_it^i$ and $h(t)=\sum_{i=0}^{\infty}h_it^i$. Then (5) becomes

$$\sum_{i=0}^{\infty}g_ip^{2i}q^{s+(2\lambda+1)i}+\sum_{i=0}^{\infty}h_ip^{2i+1}q^{s+\lambda+(2\lambda+1)i}=1,$$

where $q=1-p$, and thus we have

$$q^sg(p^2q^{2\lambda+1})+pq^{s+\lambda}h(p^2q^{2\lambda+1})=1. \tag{8}$$

It is easy to see that (8) is valid when p is a sufficiently small probability. Now set $t=pq^{\lambda+1/2}$. We may rewrite (8) as

$$q^sg(t^2)+pq^{s+\lambda}h(t^2)=1. \tag{9}$$

For p sufficiently small, we may invert $t=p(1-p)^{\lambda+1/2}$ to express p as a power series in t, $p=p(t)$. Let us write $\bar p$ for $p(-t)$ and similarly for $\bar q$. Then changing t to $-t$ in (9) yields

$$\bar q^sg(t^2)+\bar p\bar q^{s+\lambda}h(t^2)=1. \tag{10}$$

Solving (9) and (10) for $g(t^2)$ and $h(t^2)$, we find

$$g(t^2)=\frac{p^{-1}q^{-s-\lambda}-\bar p^{-1}\bar q^{-s-\lambda}}{p^{-1}q^{-\lambda}-\bar p^{-1}\bar q^{-\lambda}}=\frac{q^{1/2-s}+\bar q^{1/2-s}}{q^{1/2}+\bar q^{1/2}} \tag{11}$$

and

$$h(t^2)=\frac{q^{-s}-\bar q^{-s}}{pq^{\lambda}-\bar p\bar q^{\lambda}}=\frac{q^{-s}-\bar q^{-s}}{t(q^{-1/2}+\bar q^{-1/2})}. \tag{12}$$

By Lagrange inversion we find that

$$q^{-\alpha}=\sum_{n=0}^{\infty}\frac{\alpha}{(\lambda+\frac{3}{2})n+\alpha}\binom{(\lambda+\frac{3}{2})n+\alpha}{n}t^n.$$

Therefore

$$g(t)=\frac{\displaystyle\sum_{n=0}^{\infty}\frac{s-\frac{1}{2}}{(2\lambda+3)n+s-\frac{1}{2}}\binom{(2\lambda+3)n+s-\frac{1}{2}}{2n}t^n}{\displaystyle-\sum_{n=0}^{\infty}\frac{1}{2(2\lambda+3)n-1}\binom{(2\lambda+3)n-\frac{1}{2}}{2n}t^n} \tag{13}$$

and

$$h(t)=\frac{\displaystyle\sum_{n=0}^{\infty}\frac{s}{(\lambda+\frac{3}{2})(2n+1)+s}\binom{(\lambda+\frac{3}{2})(2n+1)+s}{2n+1}t^n}{\displaystyle\sum_{n=0}^{\infty}\frac{1}{2(2\lambda+3)n+1}\binom{(2\lambda+3)n+\frac{1}{2}}{2n}t^n} \tag{14}$$

Since

$$g(t^2)=\frac{q^{1/2-s}+q^{-1/2-s}}{q^{1/2}+q^{-1/2}}=\sum_{i=0}^{2s-2}(-1)^iq^{-(i+1)/2}\bar q^{-(2s-1-i)/2}$$

and

$$h(t^2)=\frac{q^{-s}-\bar q^{-s}}{t(q^{-1/2}+\bar q^{-1/2})}=t^{-1}\sum_{i=0}^{2s-1}(-1)^iq^{-(2s-1-i)/2}\bar q^{-i/2},$$

we can express h_n and g_n explicitly as (at worst) double sums. In particular, for $s=1$ we have

$$h(t^2)=\frac{(q^{-1/2}-\bar q^{-1/2})}{t}=\sum_{n=0}^{\infty}\frac{1}{(2\lambda+3)n+\lambda+2}\binom{(2\lambda+3)n+\lambda+2}{2n+1}t^{2n}. \tag{15}$$

This formula for h_n can also be derived combinatorially (see [10, p. 13]).

4. Paths in a wedge

We now consider the problem of counting paths in the region bounded by the two lines $y=2x$ and $x=2y$. For simplicitly, we consider the case in which the paths start at (k,k), so both boundaries will have the same generating function. Let a_n be the number of paths from (k,k) to $(k+n, 2k+2n)$ which do not touch either of the lines $y=2x$ or $x=2y$ except at the endpoint. Let $a(t)=\sum_{n=0}^{\infty} a_n t^n$. Then as before we obtain the functional equation

$$q^k a(pq^2)+p^k a(p^2q)=1, \tag{16}$$

where $p+q=1$ and either $p\le\frac{1}{3}$ or $p\ge\frac{2}{3}$. It seems unlikely that $a(t)$ can be expressed in terms of familiar functions or that there is a simple formula for a_n. However, we can extract some information from (20).

First, by expanding in powers of p and equating coefficients, we can obtain a recurrence for a_n. Next, we can use (16) to find some information about the analytic behavior of $a(t)$ which can be used to obtain asymptotic information about a_n. It is not difficult to show that $a(t)$ has radius of convergence 4/27, and that on the circle of convergence the only singularity is a branch point at $t=4/27$. The method of Darboux (Greene and Knuth, 1981) can thus be applied to obtain an asymptotic formula for a_n.

Finally, as suggested by J. Wimp, (16) can be solved as follows. Multiplying (16) by p^kq^k, we obtain

$$(pq^2)^k a(pq^2)+(p^2q)_k a(p^2q)=(pq)^k. \tag{17}$$

Now set $b(z)=z^k a(z)$ and set $pq^2=t$, $p^2q=u$. Then $pq=pq(q+p)=pq^2+p^2q=t+u$, so (17) may be rewritten as

$$b(t)+b(u)=(t+u)^k. \tag{18}$$

Now suppose p and t are small. Then p can be expanded as a power series in t, and hence so can u, say $u=\alpha(t)=t^2+\text{higher powers}$. Thus

$$\begin{aligned} b(t) &= [t+\alpha(t)]^k - b(\alpha(t)) \\ &= [t+\alpha(t)]^k - [\alpha(t)+\alpha(\alpha(t))]^k + b(\alpha(\alpha(t))) \\ &= \sum_{i=0}^{m} (-1)^i[\alpha_i(t)+\alpha_{i+1}(t)]^k + (-1)^{m+1} b(\alpha_{m+1}(t)), \end{aligned}$$

where $\alpha_0(t)=t$ and $\alpha_{i+1}(t)=\alpha(\alpha_i(t))$. Since $\alpha(t)$ starts with t^2, $\lim_{n\to\infty}\alpha_n(t)=0$ as a formal power series, so

$$b(t)=t^k a(t)=\sum_{i=0}^{\infty}(-1)^i[\alpha_i(t)+\alpha_{i+1}(t)]^k. \tag{19}$$

Note that for $k=1$ this gives $b(t)=t$, as it should.

By Lagrange inversion we find that

$$[\alpha(t)]^j = t^{2j}\sum_{n=0}^{\infty}\frac{j}{n+j}\binom{3n+3j}{n}t^n. \tag{20}$$

Using (20) we can find an explicit expression for $[\alpha_k(t)]^j$ as a k-fold sum, and thus we can find an explicit, but not very useful, expression for $b(t)$ as a multiple sum.

5. Kreweras's three-candidate ballot problem

In this section we consider a difficult lattice path problem with a nice explicit solution. It was first solved by Kreweras (1965), and another solution was given by Niederhausen (1983). (For a closely related problem, see Kreweras and Niederhausen (1981).) The solution given here is very different from either of these. It is complicated and we omit some of the details.

We consider paths in three dimensions with the steps (1,0,0), (0,1,0), and (0,0,1). Let $a_{m,n}$ be the number of paths from (0,0,1) to $(m, m+n+1, m+n+1)$ which never touch the planes $x=z$ or $y=z$ except at the endpoint. (This differs from the usual three-candidate ballot problem in that here paths may touch the plane $x=y$.) Let

$$b_{m,n}=2^{2m}\frac{(2n+2)!}{n!(n+1)!}\,\frac{(3m+2n)!}{m!(2m+2n+2)!}.$$

We shall show that $a_{m,n}=b_{m,n}$. Let

$$a(t,u)=\sum_{m,n=0}^{\infty}a_{m,n}t^m u^n \quad\text{and}\quad b(t,u)=\sum_{m,n=0}^{\infty}b_{m,n}t^m u^n.$$

We apply the probabilistic method to a particle starting at (0,0,1) and moving in the x-, y-, or z-directions with probabilities p, q, and r, until it hits either the plane $x=z$ or the plane $y=z$. Using x-y symmetry, we obtain the functional equation

$$pa(pqr,pr)+qa(pqr,qr)=1, \tag{21}$$

as long as r is sufficiently small. (It is sufficient to take $r\le\max(p,q)$.)

We shall verify that $b(t,u)$ satisfies (21). Since it is easily seen by equating coefficients that (21) determines $a(t,u)$ uniquely, it will follow that $b(t,u)=a(t,u)$.

It can be shown, for example, by multivariable Lagrange inversion (Goulden and Jackson (1983), pp. 21-22) that b satisfies

$$b(\alpha(1-\alpha)^2/4,\ \beta(1-\beta)(1-\alpha)^2)=(1-\alpha)^{-2}(1-\beta)^{-2}(1-\tfrac{3}{2}\alpha-\beta+\alpha\beta) \tag{22}$$

for α and β sufficiently small. We want to show that (22) implies that

$$pb(pqr,pr)+qb(pqr,qr)=1, \tag{23}$$

where $p+q+r=1$ and r is small. To do this we set

$$\alpha(1-\alpha)^2/4=pqr, \tag{24}$$

$$\beta(1-\beta)(1-\alpha)^2=pr, \tag{25}$$

$$\gamma(1-\gamma)(1-\alpha)^2=qr. \tag{26}$$

Then for r small, α, β, and γ are uniquely determined by p, q, and r. By (22) and (24)-(26), (23) may be rewritten as

$$p(1-\alpha)^{-2}(1-\beta)^{-2}(1-\tfrac{3}{2}\beta-\beta+\alpha\beta)+q(1-\alpha)^{-2}(1-\gamma)^{-2}(1-\tfrac{3}{2}\alpha-\gamma+\alpha\gamma)=1. \tag{27}$$

We must show that (27) is implied by the conditions on the variables. First we solve for p and q from (24), (25), and (26):

$$p=\alpha/4\gamma(1-\gamma), \tag{28}$$

$$q=\alpha/4\beta(1-\beta). \tag{29}$$

In (24) we substitute $1-p-q$ for r and use (28) and (29) to obtain an equation involving only α, β, and γ. It is a quadratic equation in α. The two solutions, which turn out to be rational, are

$$\alpha=\frac{4\beta(1-\beta)\gamma(1-\gamma)}{4\beta^2\gamma^2-4(\beta^2\gamma+\beta\gamma^2)+6\beta\gamma-\beta-\gamma+1} \tag{30}$$

and

$$\alpha=\frac{4\beta(1-\beta)\gamma(1-\gamma)}{4\beta^2\gamma^2-4(\beta^2\gamma+\beta\gamma^2)+2\beta\gamma+\beta+\gamma}. \tag{31}$$

To determine which is the right value of α, we solve for r and find that

$$r=\frac{\text{denom.}+\beta^2+\gamma^2-\beta-\gamma}{\text{denom.}},$$

where denom. is the denominator of (30) or (31). This is small (for small α and β) only with (31), so (31) must be the correct solution. Finally, with (28), (29), and (31), (27) reduces to a 'trivial' algebraic identity.

It is natural to ask if this approach will work for paths starting at points other than $(0,0,1)$. Even though we would expect a closely related generating function, it seems difficult to work backwards to find the exact form. Another question is whether there is a combinatorial or probabilistic interpretation to equation (22).

6. Further examples

One may ask whether the example described in Section 5 is an isolated peculiarity or whether it is a special case of a general theory waiting to be discovered. Here we give two examples which resemble that of Section 5 in that they involve functional equations with simple solutions which seem difficult to derive algebraically. However, these solutions, unlike that of the previous section, are not hard to derive combinatorially. But they do suggest that there is more to be learned about functional equations of these types.

For our first example we take the ordinary three-candidate ballot problem. We take our initial point to be $(0,1,2)$ and our boundary to consist of the two planes $x=y$ and $y=z$. The probabilistic method gives the functional equation

$$pf(pqr,r)+qg(pqr,qr)=1, \tag{32}$$

where f is the generating function for paths ending at points $(m+1,m+1,m+n+2)$ and g is the generating function for paths ending at points $(m,m+n+2,m+n+2)$. Here $p+q+r=1$ and r is small. By the well-known solution to the many-candidate ballot problem (see, for example, MacMahon (1960), Vol. 1, p. 132) we have

$$f(t,u)=\sum_{m,n=0}^{\infty}(n+1)(n+2)\frac{(3m+n)!}{m!(m+1)!(m+n+2)!}t^m u^n \tag{33}$$

and

$$g(t,u)=\sum_{m,n=0}^{\infty}(n+1)(n+2)\frac{(3m+n)!}{m!(m+n+1)!(m+n+2)!}t^m u^n. \tag{34}$$

It seems to be difficult to verify directly that (33) and (34) satisfy (32).

Four our last example, we consider paths starting at $(1,1)$ with steps $(0,\pm 1)$, $(\pm 1,0)$ and with boundary lines $x=0$ and $y=0$. We suppose that the particle takes the steps $(0,1),(1,0),(0,-1),(-1,0)$ with probabilities p,q,r,s, respectively, where $p+q+r+s=1$. For the particle to hit the boundary with probability 1, it suffices to have $s>q$ or $r>p$. Let $c_{l,m,n}$ be the number of paths from $(1,1)$ to the point $(0,m)$ which consist of l steps to the right, $l+1$ steps to the left, $m+n$ steps up, and n steps down, and which touch the boundary only at the endpoint. Let

$$c(t,u,v)=\sum_{l,m,n=0}^{\infty}c_{l,m,n}t^l u^m v^n.$$

Using x-y symmetry we find the functional equation

$$sc(qs,p,pr)+rc(pr,q,qs)=1, \tag{35}$$

where $p+q+r+s=1$ and p and q are small.

It is easy to find $c_{l,m,n}$ directly: A path counted by $c_{l,m,n}$ consists of horizontal steps and vertical steps. If the vertical steps are removed, what remains is a path consisting of l steps to the right and $l+1$ steps to the left that touches the $y=$axis only at its endpoint. The number of these paths is the Catalan number $\binom{2l}{l}/(l+1)$. If the horizontal steps are removed, what remains is a path counted by the ballot number

$$\frac{m+1}{m+n+1}\binom{m+2n}{n}.$$

The horizontal and vertical paths may be 'shuffled' together arbitrarily to obtain a path counted by $c_{l,m,n}$ as long as the last step is to the left. Thus

$$c_{l,m,n}=\binom{m+2n+2l}{2l}\cdot\frac{1}{l+1}\binom{2l}{l}\cdot\frac{m+1}{m+n+1}\binom{m+2n}{n}$$

$$= (m+1)\frac{(m+2n+2l)!}{l!(l+1)!n!(m+n+1)!}. \tag{36}$$

As before, it is not easy to verify directly that $c(t,u,v)$ as defined by (36) satisfies (35).

The two examples we have just given are not in their most general form. In both we could have taken the starting point to be anywhere. The first example generalizes to paths in any number of dimensions, and the second generalizes to any problem which factors into lower dimensional problems which can be solved explicitly.

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