GENERALIZED ROOK POLYNOMIALS AND ORTHOGONAL POLYNOMIALS

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Abstract. We consider several generalizations of rook polynomials. In particular we develop analogs of the theory of rook polynomials that are related to general Laguerre and Charlier polynomials in the same way that ordinary rook polynomials are related to simple Laguerre polynomials.

1. Introduction.

Suppose that $p_0(x), p_1(x), \ldots$ is a sequence of polynomials orthogonal with respect to a measure $d\mu$. Many authors have considered the problem of finding a combinatorial interpretation of the integral

$$\int \prod_{i=1}^{r} p_{n_i}(x)\, d\mu.$$

See, for example, Askey and Ismail [1], Askey, Ismail, and Koornwinder [2], Azor, Gillis, and Victor [3], de Sainte-Catherine and Viennot [4], Even and Gillis [5], Foata and Zeilberger [6, 7], Godsil [9], Ismail, Stanton, and Viennot [16], Jackson [17], Viennot [26], and Zeng [27].

The earliest of these papers, that of Even and Gillis [5] in 1976, showed that an integral of a product of simple Laguerre polynomials counts "generalized derangements": permutations of a set of objects of different "colors" with the property that the each object goes to an object of a different color. Jackson [17] observed that the result of Even and Gillis follows easily from the theory of rook polynomials, and in fact an equivalent result (not stated explicitly in terms of integrals of Laguerre

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polynomials) had been published by Kaplansky [19] in 1944. A similar interpretation of the integral of a product of Hermite polynomials in terms of matching polynomials was found by Godsil [9].

The motivating goal of this paper is to find analogous interpretations for other orthogonal polynomials, but we also discuss generalizations of rook polynomials that are not directly related to orthogonal polynomials. The theory of rook polynomials deals with the problem of counting permutations with restricted position, that is, permutations satisfying restrictions of the form $\pi(i) \neq j$ for certain pairs (i, j). In Section 2 we review the classical theory of rook polynomials. The following properties of rook polynomials are relevant to our generalizations:

(a) If $r(x)$ is a rook polynomial then $\Phi\bigl(r(x)\bigr)$ counts a set of permutations, where Φ is the linear functional on polynomials defined by $\Phi(x^n) = n!$. This set of permutations is specified in terms of conditions of the form $\pi(i) = j$.
(b) The combinatorial interpretation of $\Phi\bigl(r(x)\bigr)$ can be derived by inclusion-exclusion.
(c) The product of two rook polynomials is a rook polynomial.
(d) Every rook polynomial is a linear combination with nonnegative integer coefficients of particular rook polynomials $l_n(x)$ (which are, up to normalization, simple Laguerre polynomials).

In Section 3 we describe a setting for generalized rook polynomials, with property (b) and an analog of (a) but possibly with objects other than permutations, and a different linear functional Φ. (Properties (c) and (d) are not discussed explicitly, but analogous conditions hold for all of our examples.) We also describe how in some cases the inclusion-exclusion may be simplified by Möbius inversion. In Section 4 we discuss some simple examples of generalized rook polynomials for counting permutations, with $\Phi(x^n) = n!$, which satisfy all four properties. In Section 5 we discuss a further generalization of rook polynomials, still with $\Phi(x^n) = n!$, which satisfies properties (a), (c), and (d). This generalization is described in terms of certain functors, which we call "rook functors," whose decomposition into irreducibles gives a natural setting for property (d). In Section 6 we consider a slight variant of the theory of rook polynomials in which the objects are "linear permutations" with restrictions on adjacent elements. Section 7 gives a brief sketch of the theory of matching (or matchings) polynomials, in which the objects counted are partitions of a set with every block of size 2 and the conditions are defined by which pairs may be blocks. Matching polynomials are associated with Hermite polynomials. In Section 8 we discuss "cycle rook polynomials," which are essentially rook polynomials with an additional parameter that counts cycles of permutations. We show how the combinatorial interpretation of the integral of a product of general Laguerre polynomials found by Foata and Zeilberger [6] can be interpreted in terms of cycle rook polynomials. In Section 9 we consider generalized rook polynomials associated with partitions of a set, which we call "partition polynomials." These polynomials may be used to count partitions of a set in which certain elements are forbidden to be in singleton blocks and certain pairs are forbidden to be in the same block. The combinatorial interpretation of the integral of a product of Charlier polynomials given by Zeng [27] can be explained in terms of partition polynomials. In the special case in which there are no restrictions on singletons, the partition polynomial turns out to be the same as the chromatic polynomial of an associated graph.

2. Rook polynomials.

In this section we review the classical theory of rook polynomials. More details can be found in Kaplansky and Riordan [20] and Riordan [22, pp. 163–237]. For some recent work on rook polynomials, see Goldman, Joichi, and White [10–14] and Joni and Rota [18]. Let B be a subset of $[n] \times [n]$, where $[n] = \{1, 2, \ldots, n\}$. We call B a *board*. We wish to count permutations π of $[n]$ satisfying none of the conditions $\pi(i) = j$ for (i, j) in B.

It is not difficult to count these permutations using inclusion-exclusion. For any subset S of B, let U_S be the number of permutations π of $[n]$ satisfying $\pi(i) = j$ for all (i, j) in S and let V_S be the number of permutations π of $[n]$ satisfying $\pi(i) = j$ for all (i, j) in S but for no other (i, j) in B. Then

$$U_S = \sum_{S \subseteq T \subseteq B} V_T,$$

so by inclusion-exclusion,

$$V_\varnothing = \sum_{T \subseteq B} (-1)^{|T|} U_T. \tag{2.1}$$

It is easy to evaluate U_T: if T contains distinct pairs (i, j) and (i', j') with $i = i'$ or $j = j'$, then $U_T = 0$. Otherwise, an easy argument shows that $U_T = (n - |T|)!$.

Let us state our result in a different way. A subset S of B is *compatible* if no two elements of S agree in either coordinate. Let r_k be the number of compatible k-subsets of B. (So $r_0 = 1$ and $r_1 = |B|$). Then (2.1) implies that

$$V_\varnothing = \sum_{k=0}^{n} (-1)^k r_k (n-k)!\,. \tag{2.2}$$

Let us define the *rook polynomial* of B (sometimes called the *associated rook polynomial*) to be

$$r_B(x) = \sum_{k=0}^{n} (-1)^k r_k x^{n-k}. \tag{2.3}$$

Now let Φ be the linear functional on polynomials in x defined by $\Phi(x^n) = n!$. Note that Φ has the integral representation

$$\Phi\big(p(x)\big) = \int_0^\infty e^{-x} p(x)\, dx.$$

We can now write (2.2) in the form

$$V_\varnothing = \Phi\big(r_B(x)\big).$$

It might appear that the introduction of the linear functional Φ and the rook polynomial $r_B(x)$ is an unnecessary complication. The usefulness of the rook polynomial arises from the fact that the product of two rook polynomials is a rook polynomial: Suppose that B_1 and B_2 are boards in $[n_1]\times[n_1]$ and $[n_2]\times[n_2]$. We define the board $B_1 \oplus B_2$ in $[n_1+n_2]\times[n_1+n_2]$ to be

$$B_1 \cup \{\, (n_1+i, n_1+j) \mid (i,j) \in B_2 \,\}.$$

Then it follows from (2.3) that $r_{B_1\oplus B_2}(x) = r_{B_1}(x) r_{B_2}(x)$.

Of particular interest is the case in which B is all of $[n]\times[n]$. It is easy verified that the rook polynomial for this board is

$$l_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 k!\, x^{n-k}.$$

In terms of the simple Laguerre polynomial $L_n(x)$ as usually normalized,

$$l_n(x) = (-1)^n n!\, L_n(x).$$

Of course $\Phi\big(l_n(x)\big)$ is not very interesting since it is 0 for $n > 0$, but Φ applied to a product of the $l_n(x)$ is interesting:

$$\Phi\Big(\prod_{i=1}^{r} l_{n_i}(x)\Big) \tag{2.4}$$

is the number of permutations of $n_1 + \cdots + n_r$ objects, with n_i of color i, such that each object goes to an object of a different color. It is easy to see combinatorially that

$$\Phi\big(l_m(x) l_n(x)\big) = \begin{cases} m!^2, & \text{if } m = n; \\ 0, & \text{if } m \neq n. \end{cases} \tag{2.5}$$

Analytically (2.5) is interesting because it expresses the orthogonality of the Laguerre polynomials. Moreover, (2.5) implies that

$$k!^{-2}\Phi\big(l_m(x) l_n(x) l_k(x)\big)$$

is the coefficient of $l_k(x)$ in the expansion of $l_m(x)l_n(x)$ as a linear combination of the $l_i(x)$. (In Section 5 we will see another interpretation for this coefficient.) The combinatorial intepretation of (2.4) has been studied by Kaplansky [19], Even and Gillis [5], Askey and Ismail [1], and Jackson [17].

3. Generalized rook polynomials.

In this section we describe the setting for a generalization of rook polynomials. Our account is descriptive rather than axiomatic; the details left out here will be clear in the examples.

Let $T_0, T_1, \ldots$ be sets with cardinalities $M_0, M_1, \ldots$. For technical reasons we assume that the sequence $\{M_i\}$ does not satisfy any linear homogeneous recurrence with constant coefficients. For each n we have a set C_n of *conditions*, and we assume that $C_0 \subseteq C_1 \subseteq C_2 \cdots$. Associated to each condition c in C_n is a subset T_n^c of T_n. We say that the elements of T_n^c *satisfy* the condition c. For $A \subseteq C_n$ we define T_n^A to be $\bigcap_{a\in A} T_n^a$. We need the following property:

> *If $A \subseteq C_n$, then either $T_m^A = \varnothing$ for every $m \geq n$ (in which case we call A incompatible) or there exists a nonnegative integer $\rho(A)$ such that for every $m \geq n$ there is a bijection $T_m^A \to T_{m-\rho(A)}$.*

Note that the restriction on $\{M_i\}$ implies that for each set A of conditions A, $\rho(A)$ is uniquely determined.

Now let B be a set of conditions in C_n (and thus in C_m for all $m \geq n$). We want to count elements of T_m, for $m \geq n$, satisfying none of the conditions in B. We denote this set by T_m/B. By inclusion-exclusion,

$$T_m/B = \sum_{\substack{A\subseteq B\\ \text{compatible}}} (-1)^{|A|} M_{m-\rho(A)}. \tag{3.6}$$

We now define the generalized rook polynomial of B to be

$$r_B(x) = \sum_{\substack{A\subseteq B\\ \text{compatible}}} (-1)^{|A|} x^{n-\rho(A)}. \tag{3.7}$$

It follows that $|T_m/B| = \Phi\big(r_B(x)x^{m-n}\big)$, where $\Phi(x^n) = M_n = |T_n|$, and the restriction on $\{M_i\}$ implies that this equation uniquely determines $r_B(x)$.

To make the theory interesting, the product of two generalized rook polynomials should be a generalized rook polynomial. This will be clear in the examples. Also, in our main examples, the sets T_i will be weighted, and rather than working with cardinalities we will work with sums of weights. The bijection $T_m^A \to T_{m-\rho(A)}$ will be "almost weight-preserving" in a sense that will be explained.

In some cases the inclusion-exclusion sum in (3.6) and (3.7) can be simplified by Möbius inversion. We may define a closure operator on compatible sets of conditions as follows: If A is a compatible set of conditions in C_n, we define $\bar{A}$ to be the set of conditions implied by all the conditions in A. In other words, $c \in \bar{A}$ if and only if for all $m \geq n$, $T_m^A \subseteq T_m^c$. It is clear that $T_m^A = T_m^{\bar{A}}$ for $m \geq n$ and that $\rho(A) = \rho(\bar{A})$. The partial order of closed subsets of C_n ordered by inclusion has a unique minimal element, the empty set, and every interval is a lattice, but if there are incompatible sets of conditions then joins will not always exist. By Möbius inversion (3.6) may be replaced with

$$T_n/B = \sum_{A} \mu(\varnothing, A) M_{n-\rho(A)} \tag{3.8}$$

and the uniqueness of $r_B(x)$ implies that

$$r_B(x) = \sum_A \mu(\varnothing, A) x^{n-\rho(A)}, \tag{3.9}$$

where the sum is over all closed compatible subsets A of B and the Möbius function is in the poset of closed sets of conditions under inclusion. Thus the generalized rook polynomial is like a characteristic polynomial [24], but ρ is more general than a rank function (and in fact the poset need not be ranked). We may also derive (3.9) directly from (3.7) using the fact that

$$\mu(\varnothing, A) = \sum_S (-1)^{|S|},$$

where the sum is over all compatible sets of conditions S with closure A. This is a special case of a well-known theorem on Möbius functions. (See, for example, Rota [24, Proposition 1].) In the case of ordinary rook polynomials, every compatible set of conditions is closed, but we shall see an example in the next section where this is not the case.

4. Some simple examples.

Before looking at the main examples, let us consider two simple examples closely related to ordinary rook polynomials that illustrate some aspects of the theory of the previous section. In both of these, T_n is the set of permutations of $[n]$.

In the usual theory of rook polynomials we take as our conditions the "basic conditions" $\pi(i) = j$. However, we may also consider more general conditions which correspond to intersections of these basic conditions. For example, let $B = \{b_1, b_2\}$, where the conditions b_1 and b_2 in C_3 be defined by

$$\begin{aligned} &b_1 : \pi(1) = 1 \\ &b_2 : \pi(2) = 3 \text{ and } \pi(3) = 2 \end{aligned} \tag{4.10}$$

Here "$b_1 : \pi(1) = 1$" means that b_1 is defined by $T_m^{b_1} = \{\, \pi \mid \pi(1) = 1 \,\}$ for $m \geq 3$. Then for $m \geq 3$,

$$T_m/B = \{\, \pi \in T_m \mid \pi(1) \neq 1 \text{ and } (2\ 3) \text{ is not a cycle of } \pi \,\}.$$

We have

$$\begin{aligned} \rho(\varnothing) &= 0 \\ \rho(\{b_1\}) &= 1 \\ \rho(\{b_2\}) &= 2 \\ \rho(\{b_1, b_2\}) &= 3 \end{aligned}$$

Thus $r_B(x) = x^3 - x^2 - x + 1$. This is not a rook polynomial in the usual sense, since the coefficients do not alternate in sign. Nevertheless, $r_B(x)$ can be expressed as a nonnegative integer linear combination of the renormalized Laguerre polynomials

$l_n(x)$: $r_B(x) = l_3(x) + 8l_2(x) + 13l_1(x) + 4l_0(x)$. The combinatorial interpretation of the coefficients in this decomposition is explained in the next section.

We now give another example, which illustrates the Möbius function formula (3.9). Let $B = \{b_1, b_2, b_3\}$, where the conditions b_1, b_2, and b_3 in C_3 be defined by

$$b_1 : \pi(1) = 2 \text{ and } \pi(2) = 3$$
$$b_2 : \pi(2) = 3 \text{ and } \pi(3) = 1$$
$$b_3 : \pi(3) = 1 \text{ and } \pi(1) = 2$$

Here all sets of conditions are compatible. We find that $\rho(\{b_i\}) = 2$ for each i, $\rho(\{b_i, b_j\}) = 3$ for each $i \neq j$, and $\rho(\{b_1, b_2, b_3\}) = 3$. Then from the definition of the generalized rook polynomial (3.7) we obtain

$$r_B(x) = x^3 - 3x + 3 - 1 = x^3 - 3x + 2.$$

Alternatively, we may use formula (3.9): Since any two of the conditions imply the third, the sets $\{b_i, b_j\}$ are not closed—the closed sets of conditions are $\varnothing$, the singletons $\{b_i\}$, and $\{b_1, b_2, b_3\}$. We have $\mu(\varnothing, \{b_i\}) = -1$ and $\mu(\varnothing, \{b_1, b_2, b_3\}) = 2$, so $r_B(x) = x^3 - 3x + 2$ as before.

5. Rook functors.

In this section we discuss a further generalization of ordinary rook polynomials for counting permutations with certain restrictions. In the classical theory of permutations with restricted position we count permutations satisfying none of a set of conditions, each of the form $\pi(i) = j$. In the previous section we considered the problem of counting permutations satisfying none of a set of conditions, each of which is an intersection of the basic conditions $\pi(i) = j$. We now consider a more general problem: counting permutations satisfying any Boolean expression in the basic conditions $\pi(i) = j$. We shall see that any problem of this type has associated to it a rook polynomial which is a nonnegative linear combination of the renormalized Laguerre polynomials $l_n(x)$. Moreover, these rook polynomials have the same multiplicative property that ordinary rook polynomials do.

The most elegant way to formalize this concept seems to be through the language of categories and functors. Let A be a set of size n and let E be a Boolean expression in the n^2 terms "$\pi(i) = j$" for i, j in A. Then the *rook functor* R_E, from the category of finite sets with bijections to itself, is defined as follows: if S is any finite set disjoint from A, then $R_E(S)$ is the set of permutations of $A \cup S$ satisfying E. (In the special case of an ordinary rook polynomial, the Boolean expression E is a conjunction of negations of terms $\pi(i) = j$.) It can be shown that a finite sum of rook functors, in the sense of disjoint union, is a rook functor (though if this were not the case we would simply define a rook functor to be a sum of rook functors of the form R_E).

Of particular importance are the *Laguerre functors* L_n: $L_n(S)$ is the set of all permutations of $[n] \cup S$ such that $\pi(i) \neq j$ for all i, j in $[n]$. A fundamental fact is that every rook functor can be expressed uniquely as a sum of Laguerre functors.

To see this, let E be as in the previous paragraph and write E in disjunctive normal form; i.e., as a disjunction of conjunctions of the form

$$\bigwedge_{i,j\in A} t_{ij},$$

where each term t_{ij} is either $\pi(i) = j$ or $\pi(i) \neq j$. Some of these conjunctions will be incompatible (contradictory); those which are not will correspond to Laguerre functors with disjoint images.

As an example, take $n = 2$ and let E be the Boolean expression "$\pi(1) = 2$ or $\pi(2) = 1$." The disjunctive normal form of E is a disjunction of 12 conjunctions. Of these, 9 conjunctions are incompatible, for example,

$$\big(\pi(1) = 1\big) \wedge \big(\pi(1) \neq 2\big) \wedge \big(\pi(2) = 1\big) \wedge \big(\pi(2) = 2\big).$$

The remaining three conjunctions may be expressed more comprehensibly in the equivalent forms (with the understanding that, e.g., $\pi(1) = 2$ implies $\pi(1) \neq 1$)

$$\begin{gathered}\big(\pi(1) = 2\big) \wedge \big(\pi(2) = 1\big)\\ \big(\pi(1) = 2\big) \wedge \big(\pi(2) \notin \{1,2\}\big)\\ \big(\pi(1) \notin \{1,2\}\big) \wedge \big(\pi(2) = 1\big).\end{gathered}$$

The first conjunction corresponds to the Laguerre functor L_0 and the other two to the Laguerre functor L_1. Thus $R_E = L_0 + 2L_1$.

A more concise description of the decomposition of a rook functor into Laguerre functors is as follows: Let us define a *partial permutation of A* to be an injection from a subset of A to A. Then the multiplicity of the Laguerre functor L_{n-k} in R_E is the number of partial permutations of A satisfying E with domain of size k. In particular, the multiplicity of L_0 in R_E is the number of "complete" permutations of A satisfying E.

Now given a rook functor R we define its rook polynomial $r(x)$ to be the unique polynomial for which $\Phi\big(r(x)x^m\big) = |R([m])|$ for all m, where $\Phi(x^n) = n!$. The uniqueness of such a polynomial is clear, and its existence follows from the decomposition of R into Laguerre functors. It is also clear that the rook polynomial of a sum of rook functors is the sum of their rook polynomials.

Thus if

$$l_i(x)l_j(x) = \sum_{k=0}^{i+j} c_k l_{i+j-k}(x)$$

then c_k is the number of partial permutations π of size k of $X_1 \cup X_2$, where X_1 and X_2 are disjoint sets of sizes i and j, such that if $p \in X_r$ then $\pi(p) \notin X_r$. The reader can check combinatorially that this number is $(i+j-k)!^{-2}$ times the combinatorial interpretation for $\Phi\big(l_i(x)l_j(x)l_{i+j-k}(x)\big)$ given in Section 2, as it should be. It is not hard to derive the explicit formula

$$c_k = \sum_t \binom{i}{t}\binom{j}{t} t! \cdot \binom{j}{k-t}\binom{i}{k-t}(k-t)!\,.$$

Another easily interpreted formula is the recurrence for Laguerre polynomials,

$$x l_n(x) = l_{n+1}(x) + (2n+1) l_n(x) + n^2 l_{n-1}(x).$$

The reader may find it instructive to interpret the following formula (see, e.g. Rainville [21, p. 209]) in terms of rook functors, where m is a positive integer:

$$l_n(mx) = \sum_{k=0}^{n} \binom{n}{k}^2 k!\, m^{n-k} (m-1)^k l_{n-k}(x).$$

6. Linear permutations.

In this section we consider "linear permutations": linear arrangements of the integers 1 to n in some order. Here we take our basic conditions to be of the form "i is immediately followed by j." Just as in the case of ordinary rook polynomials, we have $M_n = \Phi(x^n) = n!$. If we take all conditions we get the polynomial $l_n^*(x)$ given by $l_0^*(x) = 1$ and for $n > 0$,

$$l_n^*(x) = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} \binom{n-1}{k} k!\, x^{n-k}.$$

In terms of the general Laguerre polynomials $L_n^{(\alpha)}$, we have

$$l_n^*(x) = n!\, (-1)^n L_n^{(-1)}(x).$$

Then

$$\Phi\left(\prod_{i=1}^{r} l_{n_i}^*(x)\right)$$

is the number of linear arrangements of $n_1 + \cdots + n_r$ objects, with n_i of color i, such that adjacent objects have different colors. It is easy to see combinatorially that

$$\Phi\big(l_m^*(x) l_n^*(x)\big) = \begin{cases} 2\, m!^2, & \text{if } m = n; \\ m!\, n!, & \text{if } |m-n| = 1; \\ 0, & \text{if } |m-n| > 1. \end{cases}$$

Thus the polynomials $l_n^*(x)$ are almost, but not quite, orthogonal.

An example of generalized rook polynomials in the sense of Section 5 for linear permutations can be found in Goulden and Jackson [15, p. 228, Exercise 3.5.2(b); Solution, pp. 451–453]. They gave the generating function for the generalized rook polynomials $p_n(x)$ for linear permutations with the property that for every i in $[n]$ there is some j in $[n]$ such that i and j are adjacent:

$$\sum_{n=0}^{\infty} p_n(x) \frac{u^n}{n!} = \exp\left(x \frac{u^2 + u^3}{1 + u^3} \right).$$

Thus

$$\Phi\Big(\prod_{i=1}^{r} p_{n_i}(x)\Big)$$

is the number of linear permutations of $n_1+\cdots+n_r$ objects, with n_i of color i, such that every object is adjacent to another object of the same color, and $\Phi\big(p_m(x)l_n^*(x)\big)$ is the number of linear permutations of m red and n blue objects with every red object adjacent to another red object and no two blue objects adjacent.

Rook polynomials for linear permutations were studied by Roselle [23]. Roselle also considered the case of cyclic permutations with conditions of the form $\pi(i)=j$. Results for cyclic permutations arise as a consequence of the theory discussed in Section 8.

7. Matching Polynomials.

Other than rook polynomials, the only case which has been extensively studied in the literature is that of matching (or matchings) polynomials. The relevant part of the theory of matching polynomials has been given by Godsil [9]. In this case T_n is the set of all complete matchings of $[n]$, that is, the set of all partitions of $[n]$ in which every block has size 2. There are, of course, no such partitions unless n is even. We have

$$|T_n| = \begin{cases} \dfrac{n!}{2^{n/2}(n/2)!}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

The linear functional Φ for which $\Phi(x^n)=|T_n|$ has the integral representation

$$\Phi\big(p(x)\big) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}p(x)\,dx.$$

The basic conditions are of the form "$\{i,j\}$ is a block." If all possible conditions are taken then the matching polynomial is the Hermite polynomial

$$H_n(x) = \sum_{k=0}^{n}(-1)^k\frac{n!}{2^k k!\,(n-2k)!}x^{n-2k}.$$

(A different normalization for Hermite polynomials is often used.)

Orthogonality arises from the fact that if there exists a partition of the set $X_1\cup X_2$, where X_1 and X_2 are disjoint, into 2-element blocks in which there is no block of the form $\{i,j\}$ with i and j both in X_1 or both in X_2, then every block must consist of one element of X_1 and one element of X_2, and thus $|X_1|=|X_2|$.

The combinatorial interpretation of the integral of a product of Hermite polynomials was also found by Azor, Gillis, and Victor [3].

As an application of the ideas of Sections 4 and 5, we can find a combinatorial interpretation to the coefficients c_i in the expansion

$$H_n(x) = \sum_{k=0}^{n} c_k l_{n-k}(x).$$

Clearly $H_n(x)$ is the generalized rook polynomial for the forbidden conditions "$\pi(i)=j$ and $\pi(j)=i$" for all $i<j$. Thus c_i is the number of partial permutations of $[n]$ with domain of size i containing no 2-cycles.

8. Cycle rook polynomials.

We now consider *cycle rook polynomials,* a generalization of rook polynomials in which the number of cycles of a permutation is taken into account. Just as the simple Laguerre polynomials arise as ordinary rook polynomials, the general Laguerre polynomials arise as cycle rook polynomials. These polynomials provide a setting for the combinatorial interpretation of the integral of a product of general Laguerre polynomials found by Foata and Zeilberger [6].

We count permutations in which a permutation with j cycles is assigned the weight α^j. As in the case of ordinary rook polynomials, the set T_n is the set of all permutations of $[n]$, but M_n is the sum of their weights, so

$$M_n = \alpha(\alpha+1)\cdots(\alpha+n-1) = (\alpha)_n.$$

Just as in the case of ordinary rook polynomials, the basic conditions are of the form $\pi(i) = j$, and we identify a set of conditions of this form with the corresponding board in $[n] \times [n]$. (We can also allow more general conditions, as in Section 4.)

If A is a compatible set of conditions, then we have a bijection

$$T_n^A \to T_{n-|A|},$$

but this bijection need not preserve the number of cycles. However the number of cycles is transformed in a very simple way: the number of cycles of a permutation in T_n^A is the number of cycles of the corresponding permutation in $T_{n-|A|}$ plus the number of cycles of A (i.e., the number of directed cycles in the digraph associated to A).

Thus given a board B, we define its cycle rook polynomial $r_B(x,\alpha)$ by

$$r_B(x,\alpha) = \sum_{\substack{A\subseteq B\\ \text{compatible}}} (-1)^{|A|}\alpha^{c(A)}x^{n-|A|},$$

where $c(A)$ is the number of cycles of A.

We define the linear functions Φ by $\Phi\big(x^n\big) = (\alpha)_n$. If α is a real number greater than or equal to 1 then Φ has the integral representation

$$\Phi\big(p(x)\big) = \frac{1}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}e^{-x}p(x)\,dx.$$

As in the case of ordinary rook polynomials, we have the result that $\Phi\big(r_B(x,\alpha)\big)$ is the weighted sum of all permutations of $[n]$ with none of the conditions in B, and the multiplicative property of ordinary rook polynomials carries over without change to cycle rook polynomials.

Let us look at a simple example. Suppose $n = 2$ and take as our conditions

$$b_1 : \ \pi(1) = 1$$
$$b_2 : \ \pi(1) = 2$$
$$b_3 : \ \pi(2) = 1$$

The compatible sets of conditions are $\varnothing$, $\{b_1\}$, $\{b_2\}$, $\{b_3\}$, and $\{b_2, b_3\}$. The sets $\{b_1\}$ and $\{b_2, b_3\}$ each contain one cycle, and the other compatible sets of conditions contain no cycles. Thus the cycle rook polynomial for B is $x^2 - (\alpha+2)x + \alpha$.

If we take all possible conditions for B on $[n]$ then we obtain the cycle rook polynomial

$$l_n(x,\alpha) = \sum_{k=0}^{n} (-1)^k x^{n-k} \binom{n}{k} (\alpha+n-k)_k.$$

In terms of the usual normalization for the general Laguerre polynomial, this polynomial is $n!\,(-1)^n L_n^{(\alpha-1)}(x)$. Then $\Phi\bigl(l_m(x,\alpha) l_n(x,\alpha)\bigr)$ counts by cycles permutations of the set $X_1 \cup X_2$ such that if $j \in X_i$ then $\pi(j) \notin X_i$ where X_1 and X_2 are disjoint sets with $|X_1| = m$ and $|X_2| = n$. It is easily seen that this is zero if $m \neq n$ and $m!\,(\alpha)_m$ if $m = n$. A similar interpretation arises for

$$\Phi\biggl(\prod_{i=1}^{r} l_{n_i}(x,\alpha)\biggr),$$

which was first given by Foata and Zeilberger [6].

As an interesting example of cycle rook polynomials, we consider the derangement numbers. The ordinary derangement numbers are obtained from the board

$$B = \{(1,1), (2,2), \ldots, (n,n)\},$$

which has the rook polynomial $r_B(x) = (x-1)^n$. The derangement number D_n is then

$$D_n = \Phi\bigl((x-1)^n\bigr) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

The same result is obtained for the board

$$B^* = \{(1,2), (2,3), \ldots, (n,1)\}.$$

However, when cycles are taken into account, these two boards give different results. For the board B, we have $r_B(x,\alpha) = (x-\alpha)^n$, which yields the cycle derangement numbers

$$D_n(\alpha) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \alpha^k (\alpha)_{n-k}. \tag{8.11}$$

These numbers count derangements (permutations with no fixed points) by the number of cycles. From (8.11) we obtain the exponential generating function

$$\sum_{n=0}^{\infty} D_n(\alpha) \frac{x^n}{n!} = \left(\frac{e^{-x}}{1-x}\right)^{\alpha}, \tag{8.12}$$

which can also be derived by the usual methods for working with exponential generating functions. The cycle rook polynomial for B^* is

$$r_{B^*}(x,\alpha) = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} x^{n-k} + (-1)^n \alpha = (x-1)^n + (-1)^n(\alpha - 1)$$

for $n \geq 1$, since the only cycle in B^* is obtained by taking all of B^*. Thus

$$D_n^*(\alpha) = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} (\alpha)_{n-k} + (-1)^n \alpha, \quad n \geq 1. \tag{8.13}$$

From (8.13) we obtain the exponential generating function

$$\sum_{n=1}^{\infty} D_n^*(\alpha) \frac{x^n}{n!} = e^{-x} \left(\alpha - 1 + \frac{1}{(1-x)^\alpha} \right) - \alpha. \tag{8.14}$$

Formula (8.14) is interesting because, unlike (8.12), it does not follow from the usual methods for working with exponential generating functions.

The coefficient of α in $D_n^*(\alpha)$ is the number of cyclic permutations π of $[n]$ with no occurrences of $\pi(i) \equiv i+1 \pmod{n}$. For further information about these numbers, see Stanley [25, Exercise 8, p. 88; Solution, p. 93] and the references cited there. From (8.13) we obtain the formula

$$\sum_{k=0}^{n-1} (-1)^k \binom{n}{k} (n-k-1)! + (-1)^n, \quad n \geq 1$$

for these numbers and from (8.14) we obtain the exponential generating function

$$e^{-x} \left(1 + \log \frac{1}{1-x} \right) - 1.$$

9. Partition polynomials. We now turn our attention from permutations to partitions. We take T_n to be the set of partitions of $[n]$, and we weight a partition with k blocks by β^k. Then

$$M_n = \Phi(x^n) = \sum_{k=0}^{n} S(n,k) \beta^k,$$

where $S(n,k)$ is the Stirling number of the second kind. Note that if we set $\beta = 1$ then M_n reduces to the Bell number B_n. We can also represent Φ as a sum (or discrete integral):

$$\Phi\bigl(p(x)\bigr) = e^{-\beta} \sum_{l=0}^{\infty} p(l) \frac{\beta^l}{l!}. \tag{9.15}$$

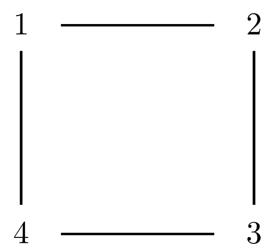
The basic conditions we work with are of two types:

(1) i is in a singleton block.
(2) i and j are in the same block.

(Condition (1) could be generalized to the condition that a given subset of $[n]$ is a block.) We call the corresponding generalized rook polynomials *partition polynomials*.

First let us consider the case in which all conditions are of the second type. We may represent a set of conditions as a graph G in which the condition that i and j are in the same block is represented by the edge $\{i, j\}$. We write $r_G(x)$ for the partition polynomial corresponding to the graph G. Note that all conditions are compatible. For any set A of edges, $n - \rho(A)$ is the number of components of the spanning subgraph of $[n]$ with edge set A.

Thus if $n = 4$ and the conditions correspond to the pairs $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, and $\{4, 1\}$, then the graph is the 4-cycle:



For this graph we have

$$r_G(x) = x^4 - 4x^3 + 6x^2 - 4x + x = x^4 - 4x^3 + 6x^2 - 3x,$$

and this polynomial is also the chromatic polynomial of G.

It turns out that for any graph G, $r_G(x)$ is equal to the chromatic polynomial of G. One way to prove this is by Möbius inversion: Suppose that A is a set of edges of G. If π is a partition of $[n]$ that satisfies all the conditions corresponding to the edges in A, then both endpoints of any edge in A must lie in the same block of A, and thus each connected component of the spanning subgraph of G with edge set A must be contained in a block of π. It follows that the closed sets of conditions correspond to partitions of $[n]$ in which the induced subgraph of G on each block is connected, and the partial ordering of inclusion of sets of conditions corresponds to the partial ordering of refinement of partitions.

Thus by (3.9), we have

$$r_G(x) = \sum_{\pi} \mu(\hat{0}, \pi) x^{|\pi|}, \tag{9.16}$$

where the sum and the Möbius function are taken over the lattice of all partitions π of $[n]$ in which every block is a connected subgraph of G, ordered by refinement. But the right side of (9.16) is well known to be the chromatic polynomial of G. (See, for example, Stanley [25, p. 162, Exercise 44; Solution, p. 187] or Rota [24].)

We can also give a more direct proof that the partition polynomial of a graph G is equal to its chromatic polynomial: Let u_i be the number of partitions of $[n]$ with i blocks in which vertices which are adjacent in G are in different blocks. Then

$$\sum_{i=0}^{n} u_i \beta^i = \Phi\bigl(r_G(x)\bigr). \tag{9.17}$$

A proper coloring of G in l colors may be obtained from such a partition by choosing i of the colors and assigning each one to one of the i blocks. Thus

$$P_G(l) = \sum_{i=1}^{n} \binom{l}{i} i!\, u_i. \tag{9.18}$$

Multiplying (9.18) by $\beta^l/l!$ and summing on l yields

$$\sum_{l=0}^{\infty} P_G(l) \frac{\beta^l}{l!} = e^{\beta} \sum_{i=0}^{n} u_i \beta^i. \tag{9.19}$$

Comparing (9.19) with (9.17) and (9.15) yields

$$\Phi\big(P_G(x)\big) = \Phi\big(r_G(x)\big)$$

and it is clear that when restricted to polynomials in x that do not involve β, Φ is invertible.

In particular, it follows that the partition polynomial for the complete graph K_n is $x(x-1)\cdots(x-n+1)$.

To take care of singleton blocks, we note that a singleton creates a new block with weight β, and any singleton is incompatible with all pair conditions. So in the general case, $r_B(x)$ can be expressed as an alternating sum of powers of β times chromatic polynomials of subgraphs of G.

If we take all conditions we get the polynomial

$$c_n(x,\beta) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \beta^i x(x-1)\cdots\big(x-(n-i)+1\big) = (-\beta)^n C_n(x;b),$$

where $C_n(x;b)$ is the Charlier polynomial in the usual normalization. We have orthogonality here since

$$\Phi\big(c_m(x,\beta)c_n(x,\beta)\big)$$

is the number of partitions of $X_1 \cup X_2$, where X_1 and X_2 are disjoint sets of cardinalities m and n, with no singletons and no two elements of X_i in the same block for $i = 1, 2$. Thus each block must contain one element of X_1 and one element of X_2, so

$$\Phi\big(c_m(x,\beta)c_n(x,\beta)\big) = \begin{cases} m! & \text{if } m = n; \\ 0 & \text{if } m \neq n. \end{cases}$$

The same reasoning leads to a combinatorial interpretation for

$$\Phi\Big(\prod_{i=1}^{r} c_{n_i}(x,\beta)\Big),$$

which was found by Stanton and Ismail (unpublished) and Zeng [27].

As in the case of rook polynomials, we can allow more general conditions which are intersections of the basic conditions. For example, let p be a prime. We wish to count partitions in which 1, 2, ... , p are not all singletons, and they are not all in the same block. The partition polynomial is $x^p - x - \beta^p$. Thus $\Phi\big((x^p - x - \beta^p)x^n\big)$ is the weighted sum of partitions of $\{1, 2, \dots, p+m\}$ satisfying these conditions. Now the cyclic group C_p of order p, with its action on $[p]$ transported in the natural way to these partitions, has no fixed points, and thus the orbits all have size p. We therefore have the congruence $\Phi\big((x^p - x - \beta^p)x^m\big) \equiv 0 \pmod p$, or equivalently,

$$S(m+p,k) - S(m+1,k) - S(m,k-p) \equiv 0 \pmod p.$$

Taking $\beta = 1$ yields the well-known congruence for Bell numbers

$$B_{m+p} - B_{m+1} - B_m \equiv 0 \pmod p.$$

Similarly, $\Phi\big((x^p - x - \beta^p)^j x^m\big)$ counts partitions of $[jp+m]$ in which for $0 \le i < j$, the numbers $ip+1$, $ip+2$, ... , $ip+p$ are not all singletons and not all in the same block. It can be shown that under the natural action of the product of cyclic groups C_p^j, each orbit has size divisible by $p^{\lceil j/2 \rceil}$, and thus

$$\Phi\big((x^p - x - \beta^p)^j x^m\big) \equiv 0 \pmod{p^{\lceil j/2 \rceil}}.$$

Moreover, by using a p-Sylow subgroup of the symmetric group S_{jp} one can show that the congruence is valid modulo a higher power of p for $p > 2$. This method is studied in great detail in Gessel [8], and further references to the literature can be found there. Many of the examples given in [8] can be viewed as instances of generalized rook polynomials.

10. Final remarks.

There are many posssible aspects of the theory of generalized rook polynomials that we have not discussed.

We have considered only the problem of counting objects satisfying none of a given set of conditions; however, the approach could easily be used to count the number of objects satisfying a specified number of conditions.

We have not studied the recurrences which generalized rook polynomials satisfy, which are of considerable interest in the case of ordinary rook polynomials and matching polynomials.

It seems likely that a generalized rook polynomial interpretation can be given for the Meixner polynomials when normalized so that the exponential generating function for their moments is

$$\left(\frac{1}{1-\alpha(e^z-1)}\right)^{\beta},$$

which counts sets of cycles of sets. There is a more elegant interpretation of Meixner polynomials (with a different normalization) in terms of descents of permutations.

(See Askey and Ismail [1], Viennot [26], and Zeng [27].) However the method of this paper does not seem to work well for this interpretation, nor for any combinatorial objects whose weight depends on a total ordering of the underlying set, as in Ismail, Stanton, and Viennot's treatment of q-Hermite polynomials [16].

Joni and Rota [18] studied a q-analog of permutations with restricted position. The problem they considered was that of counting the number of (unordered) bases of a finite-dimensional vector space over a finite field in which certain vectors are forbidden as basis elements. This problem does not lead to orthogonal polynomials, but other q-analogs related to vector spaces are possible. For example, another natural q-analog of permutations with restricted position is the problem of counting invertible linear transformations π from a vector space to itself satisfying none of a set of conditions of the form $\pi(i) = j$.

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