THE SMITH COLLEGE DIPLOMA PROBLEM

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The following problem, proposed by Russell Maurer, appeared in the American Mathematical Monthly in 1974 [2]:

> At Smith College, the graduation exercises traditionally proceed as follows: Although each diploma is made out to a particular girl, all the diplomas are initially given out at random. All of the girls who do not get their own diplomas then form a circle, and each passes the diploma she has to the girl on her right. Those who now have their own diplomas drop out, and the remaining girls again pass their diplomas to the right, and so on. This procedure is repeated until each girl has her own diploma. If there are n girls in the graduating class what is the probability that it takes precisely k passes before each girl has her own diploma?

The published solution, by Don West [5], uses recurrences to prove by induction that of the $n!$ ways of distributing the diplomas, the number which require k passes is given by the formula

$$\sum_{r=0}^{k}(-1)^r\binom{n+1}{r}(k+1-r)^n. \tag{1}$$

As noted in the editor's comment on West's solution, the numbers given by this formula are the well-known *Eulerian numbers*; in the most common notation, (1) is equal to $A_{n,k+1}$. The Eulerian numbers have several combinatorial interpretations, of which we shall discuss three, and in particular we shall relate the number of passes directly to one of these: $A_{n,k}$ is the number of permutations π of $\{1,\dots,n\}$ with k *strong excedances*, which are values of i for which $i<\pi(i)$.

Before we take up the Smith College diploma problem, we explain the connection between this interpretation of the Eulerian numbers and one that is somewhat

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better known: $A_{n,k+1}$ is also the number of permutations π of $[n] = \{1, 2, \ldots, n\}$ with k *rises* (or *ascents*), which are values of i for which $\pi(i) < \pi(i+1)$.

Let $\mathfrak{S}_n$ be the set of permutations of $\{1, \ldots, n\}$. Following the approach of Foata and Schützenberger [1, Chapter 1], we describe a bijection $\mathfrak{S}_n \to \mathfrak{S}_n$ that takes a permutation π with k rises to a permutation $\hat{\pi}$ with k strict excedances.

Let us first recall several notations for permutations. The simplest is the "two-line representation" $\binom{1\,2\,3\,4\,5\,6}{6\,2\,4\,3\,1\,5}$ which represents the permutation π with $\pi(1) = 6, \pi(2) = 2$, and so on. Removing the top line, gives the "one-line representation" 6 2 4 3 1 5. Finally we have the "cycle representation" (3 4)(2)(1 6 5). Although the cycle representation of π may be written in many different ways—thus in our example we could also write (2)(6 5 1)(4 3))—we get a canonical cycle representation by requiring that the least element of each cycle be written first, and that the cycles be arranged so that their least elements decrease from left to right, as in the example. Then if we remove the parentheses from the canonical cycle representation we obtain the one-line representation $\hat{\pi}(1)\ \hat{\pi}(2) \cdots \hat{\pi}(n)$ for a permutation $\hat{\pi}$. (In our example, $\hat{\pi}$ is 342165.) It is not hard to show that the map $\pi \mapsto \hat{\pi}$ gives a bijection $\mathfrak{S}_n \to \mathfrak{S}_n$. Moreover, $\hat{\pi}$ has the property that with $\hat{\pi}(i) = j$, we have $j < \pi(j)$ if and only if $\hat{\pi}(i) < \hat{\pi}(i+1)$: This is clear if j is not the last element in its cycle, since in this case, $\hat{\pi}(i+1) = \pi(j)$. But if j is the last element in its cycle then $\pi(j)$ is the first, and hence smallest, element of this cycle, so $j \geq \pi(j)$; and either $i+1 = n$ (so that i is not a rise) or $\hat{\pi}(i+1)$ is the first and smallest element of the next cycle, and is therefore less than $j = \hat{\pi}(i)$. Thus if π has k rises then $\hat{\pi}$ has k strict excedances.

The solution of the Smith College diploma problem is closely related to another property of Eulerian numbers. A *weak excedance* of a permutation π of $\{1, \ldots, n\}$ is an i such that $i \leq \pi(i)$. Thus every strong excedance is a weak excedance, but not conversely: if i is *fixed point* of π; i.e., $\pi(i) = i$, then i is a weak but not strong excedance of π. It turns out that the number of permutations of $\{1, \ldots, n\}$ with k weak excedances is equal to the number of permutations of $\{1, \ldots, n\}$ with $k-1$ strong excedances. Although this can be explained by a variation of the correspondence described above,[1] we give a more direct proof that also provides the key to our solution of the Smith College diploma problem. We describe a bijection $\Theta : \mathfrak{S}_n \to \mathfrak{S}_n$ that takes a permutation with k weak excedances to a permutation with $k-1$ strong excedances:

Given $\pi \in \mathfrak{S}_n$, we define $\Theta\pi \in \mathfrak{S}_n$ by

$$(\Theta\pi)(i) = \begin{cases} \pi(i)+1, & \text{if } \pi(i) \neq n \\ 1 & \text{if } \pi(i) = n. \end{cases}$$

Lemma. *If $\pi \in \mathfrak{S}_n$ has k weak excedances then $\Theta\pi$ has $k-1$ strong excedances.*

Proof. If i is a weak excedance of π and $\pi(i) \neq n$ then i is a strong excedance of $\Theta\pi$. If $\pi(i) = n$, then i is a weak excedance of π but not a strong excedance of

[1] We start each cycle with its *greatest* element and arrange the cycles so that the greatest elements are in *increasing* order.

$\Theta\pi$. Conversely, if i is a strong excedance of $\Theta\pi$, then i is a weak excedance of π. Thus the number of strong excedances of $\Theta\pi$ is one less than the number of weak excedances of π. $\square$

Corollary. *The number of permutations in $\mathfrak{S}_n$ with k weak excedances is the Eulerian number $A_{n,k}$.*

Other proofs of this relation between weak and strong excedances have been given by Riordan [3, p. 215], Foata and Schützenberger [1, p. 29, equation (1)], and Stanley [4, p. 23, Proposition 1.3.12].

For the Smith College diploma problem, we will need to apply Θ to permutations of sets of integers that are not of the form $\{1, \ldots, n\}$. We therefore generalize it in an obvious way: If π is a permutation of a set $S = \{s_1 < s_2 < \cdots < s_m\}$ then $\Theta\pi$ is a permutation of S defined as follows: suppose that $\pi(s_i) = s_j$. Then

$$(\Theta\pi)(s_i) = \begin{cases} s_{j+1}, & \text{if } j \neq m \\ s_1 & \text{if } j = m. \end{cases}$$

Thus, for example, $\Theta\left(\begin{smallmatrix} 2 & 4 & 5 & 7 \\ 4 & 2 & 7 & 5 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 & 4 & 5 & 7 \\ 5 & 4 & 2 & 7 \end{smallmatrix}\right)$, Excedances of permutations of S are defined as before, and Θ has the same property of decreasing by one the number of strong excedances in a permutation without fixed points.

Note that in a permutation without fixed points, weak excedances and strong excedances are the same. Thus if π has k strong excedances and no fixed points, then $\Theta\pi$ has $k-1$ strong excedances.

We now have all the tools we need for the Smith College diploma problem. Let us suppose that the students are numbered from 1 to n around the circle, and oriented so that student j passes her diploma to student $j+1$ modulo n. We assign to each diploma the number of its owner. Then a distribution of diplomas can be described by the permutation π for which diploma i is held by student $\pi(i)$.

Theorem. *Suppose that an initial distribution of diplomas is described by a permutation π. Then the number of passes needed to deliver all diplomas to their owners is equal to the number of strong excedances of π.*

Proof. Suppose that at some point in the graduation exercise, the distribution of diplomas corresponds to a permutation σ, with at least one nonfixed point, of a set S. After the students who have their own diplomas drop out, the remaining distribution of diplomas corresponds to the restriction of σ to the set of nonfixed points of σ, which we denote by σ'. Then after diplomas are passed, the corresponding permutation is $\Theta\sigma'$. It follows from the lemma and the fact that σ' has no fixed points that if σ has k strong excedances then $\Theta\sigma'$ has $k-1$ strong excedances.

Thus each pass reduces the number of excedances of the corresponding permutation by 1. Since a permutation consists entirely of fixed points if and only if it has no strong excedances, the number of passes required for an initial distribution of diplomas is equal to the number of strong excedances of its corresponding permutation. $\square$

Corollary. *The probability that n diplomas, distributed randomly, require k passes is $A_{n,k+1}/n!$.*

References

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