Signed Mahonians

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Abstract

A classical result of MacMahon gives a simple product formula for the generating function of major index over the symmetric group. A similar factorial-type product formula for the generating function of major index together with sign was given by Gessel and Simion. Several extensions are given in this paper, including a recurrence formula, a specialization at roots of unity and type B analogues.

1 Introduction

1.1 Outline

Enumeration over the symmetric group S_n and related combinatorial objects, taking into account also the *sign* of each permutation, was studied by Simion and Schmidt [36] and others (see, e.g., [35, 42, 39, 5, 28]).

The polynomial

$$\sum_{\pi \in S_n} \operatorname{sign}(\pi)\, q^{\operatorname{des}(\pi)}$$

was called the *signed Eulerian* by Désarmenien and Foata [15]. An elegant formula for signed Eulerians, conjectured by Loday [24], was proved by

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Désarmenien and Foata [15] and by Wachs [41]. Type B analogues were given by Reiner [33].

MacMahon showed, about a hundred years ago, that the generating function for major index over the symmetric group has a simple product formula. The *signed Mahonian* will be defined as

$$\sum_{\pi \in S_n} \mathrm{sign}(\pi)\, q^{\mathrm{maj}(\pi)}.$$

An elegant factorial-type product formula for the signed Mahonians was given by Gessel and Simion [41, Cor. 2] (Theorem 1.3 below). Various extensions of this theorem are given in this paper.

First, a recurrence for the joint distribution of the inversion number, major index, and last digit of a permutation is given (Theorem 2.1 below). It is shown that these parameters give rise to a multiplicative, factorial-type formula, if the parameter for inversion number is set equal to 1 or to -1 (Theorem 3.2 below).

An extension in a different direction gives a factorization of the bivariate generating function of major index and inversion number at roots of unity (Theorem 4.4 below). The proof applies a remarkable identity which follows from results of Gordon [21], Roselle [34], and Foata-Schützenberger [18]. The identity was independently proved by Gessel [19, Theorem 8.5].

These extensions imply two different new proofs of Theorem 1.3.

Then Theorem 1.3 is extended to the group of signed permutations B_n, where the generating function of the flag-major index with each of the one-dimensional characters is shown to have a similar factorial type formula (Theorems 5.1, 6.1 and 6.2 below).

These results yield explicit simple generating functions for the (flag) major index on subgroups of index 2 of S_n and B_n, such as the alternating groups and the Weyl groups of type D. See Section 7.

The rest of the paper is organized as follows. Necessary background and statements of main results are given in the rest of this section. In Section 2, a multivariate recurrence formula for length, major index and last digit is proved (Theorem 2.1). Then, in Section 3, this formula is applied to prove a new extension (Theorem 3.2) of the Gessel-Simion Theorem. A second proof of the Gessel-Simion Theorem, via specialization at roots of unity, is given in Section 4. The type B analogue (Theorem 5.1) is proved in Section 5. The distribution of the (flag) major index on index 2 subgroups is then deduced in Sections 6 and 7.

1.2 Background

The Coxeter generators $\{ s_i = (i, i+1) \mid 1 \le i \le n-1 \}$ of S_n give rise to various combinatorial statistics. For $\pi \in S_n$ let the *length*, $l(\pi)$, be the standard length of π with respect to these generators, which is the same as the number of inversions of π. This notion is defined similarly for other Coxeter groups. The generating function of length in a Coxeter group W is called the *Poincaré polynomial* of W [23, Ch. 3].

For a positive integer n define

$$[n]_q := \frac{1-q^n}{1-q}.$$

Then

Theorem 1.1 [23, §3.15]

$$\sum_{\pi \in S_n} q^{l(\pi)} = [1]_q [2]_q \cdots [n]_q.$$

Another statistic on S_n, which has a Coxeter group interpretation, is the descent number. Given a permutation π in the symmetric group S_n, the *descent set* of π is

$$\mathrm{Des}(\pi) := \{ i \mid l(\pi) > l(\pi s_i) \} = \{ i \mid \pi(i) > \pi(i+1) \}$$

and the corresponding *descent number* is $\mathrm{des}(\pi) := |\mathrm{Des}(\pi)|$. The *major index* of π is the following weighted enumeration of the descents

$$\mathrm{maj}(\pi) := \sum_{i \in \mathrm{Des}(\pi)} i.$$

A well-known classical result asserts that the length function and major index of a permutation are equidistributed over the symmetric group S_n.

Theorem 1.2 (MacMahon [25])

$$\sum_{\pi \in S_n} q^{l(\pi)} = \sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} = [1]_q [2]_q \cdots [n]_q.$$

A similar simple factorial-type product formula for the signed Mahonians was given by Gessel and Simion [41, Cor. 2].

The *sign* of an element w in a Coxeter group W is

$$\mathrm{sign}(w) := (-1)^{l(w)}.$$

Theorem 1.3 (The Gessel-Simion Theorem)

$$\sum_{\pi\in S_n} \mathrm{sign}(\pi) q^{\mathrm{maj}(\pi)} = [1]_q[2]_{-q}[3]_q[4]_{-q}\cdots[n]_{(-1)^{n-1}q}.$$

Recall that B_n denotes the group of all bijections σ of the set $[-n,n]\setminus\{0\}$ onto itself such that

$$\sigma(-a) = -\sigma(a)$$

for all $a \in [-n,n]\setminus\{0\}$, with composition as the group operation. This group is usually known as the group of "signed permutations" on $[n]$, or as the *hyperoctahedral group* of rank n, or as the classical Weyl group of type B and rank n.

It is well known (see, e.g., [8, Proposition 8.1.3]) that B_n is a Coxeter group with respect to the generating set $\{s_0, s_1, s_2, \ldots, s_{n-1}\}$, where

$$s_0 := [-1, 2, \ldots n]$$

and

$$s_i := [1, 2, \ldots, i-1, i+1, i, i+2, \ldots n]$$

for $i = 1, \ldots, n-1$. Let $l(\sigma)$ be the standard length of $\sigma \in B_n$ with respect to its Coxeter generators.

Theorem 1.4 [23, §3.15]

$$\sum_{\pi\in B_n} q^{l(\pi)} = [2]_q[4]_q\cdots[2n]_q.$$

Despite the fact that an increasing number of enumerative results of this nature have been generalized to the hyperoctahedral group B_n (see, e.g., [9, 16, 31, 32, 37]) and that several "major index" statistics have been introduced and studied for B_n [10, 11, 12, 17, 29, 30, 40] no generalization of MacMahon's result to B_n has been found until a new statistic, the *flag major index*, was introduced.

The *flag-major* of $\sigma \in B_n$ is defined as

$$\text{flag-major}(\sigma) \ := \ 2\cdot \mathrm{maj}(\sigma) + \mathrm{neg}(\sigma)$$

where

$$\mathrm{neg}(\sigma) \ := \ \#\{\, 1 \le i \le n \mid \sigma(i) < 0 \,\}$$

and $\mathrm{maj}(\sigma)$ is the major index of the sequence $(\sigma(1), \ldots, \sigma(n))$, with respect to the order

$$-1 < \cdots < -n < 1 < \cdots < n.$$

A type B analogue of Theorem 1.2 was given in [4].

Theorem 1.5 [4]

$$\sum_{\pi \in B_n} q^{l(\pi)} = \sum_{\pi \in B_n} q^{\text{flag-major}(\pi)} = [2]_q [4]_q \cdots [2n]_q.$$

For a unified definition of the classical major index and the flag-major index as a length of a distinguished canonical expression see [4]. The flag-major index has many combinatorial and algebraic properties which are shared with the classical major index on S_n [4, 1, 22, 2, 3, 7, 13]. In this paper we will give a type B analogue of the Gessel-Simion Theorem (Theorem 5.1 below), as well as other new extensions of this theorem.

1.3 Main Results

We find a recurrence (theorem 2.1 below) for the joint distribution of length, major index, and last digit, which leads to the following result. Let

$$\mathrm{last}(\pi) \ := \ \pi(n) - 1.$$

Then

Theorem 1.6 (see Theorem 3.2 below)
For $\varepsilon = \pm 1$,

$$\sum_{\pi \in S_n} \varepsilon^{l(\pi)} q^{\mathrm{maj}(\pi)} z^{\mathrm{last}(\pi)} = [1]_q \cdot [2]_{\varepsilon q} \cdot [3]_q \cdot [4]_{\varepsilon q} \cdots [n-1]_{\pm q} \cdot [n]_{\pm \varepsilon q/z} \cdot z^{n-1}.$$

This theorem shows that the distribution of (signed) major index over permutations with prescribed last digit is essentially independent of this digit (Corollary 3.4). Letting $\varepsilon = -1$ and $z = 1$ gives Theorem 1.3.

A second new proof of Theorem 1.3 uses a known identity (Theorem 4.3 below) involving the generating function for length and major index. This also leads to a factorization at roots of unity other than ± 1.

Let

$$A_n(t, q) \ := \ \sum_{\pi \in S_n} t^{l(\pi)} q^{\mathrm{maj}(\pi)}.$$

For a positive integer n define

$$(q)_n \ := \ (1-q)(1-q^2) \cdots (1-q^n).$$

Theorem 1.7 (see Theorem 4.4 below)
Let n and m be positive integers. Let ζ be a primitive mth root of unity and assume that $n = mk + i$ with $0 \le i < m$. Then

$$A_n(\zeta, q) = A_i(\zeta, q) \frac{(q)_n}{(q)_i (1 - q^m)^k}.$$

The case $m = 2$ gives Theorem 1.3.

A type B analogue of Theorem 1.3 is :

Theorem 1.8 (see Theorem 5.1 below)

$$\sum_{\pi \in B_n} \mathrm{sign}(\pi) \cdot q^{\text{flag-major}(\pi)} = [2]_{-q} [4]_q \cdots [2n]_{(-1)^n q}.$$

Explicit generating functions of the major index and flag major index on distinguished subgroups follow from Theorems 1.3 and 1.8. See Corollaries 7.1 and 7.2 below.

2 A Recurrence Formula

Let S_n be the symmetric group. For $\pi \in S_n$ define the following statistics:

$$\begin{aligned} \mathrm{inv}(\pi) &:= \text{inversion number of } \pi \\ &(= \text{length of } \pi \text{ w.r.t. the usual Coxeter generators of } S_n) \\ \mathrm{maj}(\pi) &:= \text{major index of } \pi = \sum \{\, 1 \le i \le n-1 \mid \pi(i) > \pi(i+1) \,\} \\ \mathrm{last}(\pi) &:= \pi(n) - 1, \text{ one less than the last digit in } \pi \end{aligned}$$

Define the multivariate generating function

$$f_n(x, y, z) := \sum_{\pi \in S_n} x^{\mathrm{inv}(\pi)} y^{\mathrm{maj}(\pi)} z^{\mathrm{last}(\pi)}. \tag{1}$$

Theorem 2.1 (Recurrence Formula)

$$f_1(x, y, z) = 1$$

and, for $n \ge 2$,

$$\begin{aligned} (x - z) f_n(x, y, z) &= (x^n y^{n-1} - z^n) \cdot f_{n-1}(x, y, 1) \\ &\quad + x^{n-1}(1 - y^{n-1}) z \cdot f_{n-1}(x, y, z/x). \end{aligned}$$

Proof. The case $n = 1$ is clear. Assume $n \geq 2$.
Given a permutation

$$\pi = (\pi(1), \ldots, \pi(n-1)) \in S_{n-1},$$

append k ($1 \leq k \leq n$) as the nth digit, while adding 1 to each existing digit between k and $n-1$, to get a permutation

$$\bar{\pi} = (\bar{\pi}(1), \ldots, \bar{\pi}(n-1), k) \in S_n$$

where, for $1 \leq i \leq n-1$,

$$\bar{\pi}(i) = \begin{cases} \pi(i), & \text{if } \pi(i) < k; \\ \pi(i) + 1, & \text{otherwise.} \end{cases}$$

The new statistics for $\bar{\pi}$ are:

$$\begin{aligned} \operatorname{inv}(\bar{\pi}) &= \operatorname{inv}(\pi) + (n-k) \\ \operatorname{maj}(\bar{\pi}) &= \begin{cases} \operatorname{maj}(\pi), & \text{if } k > \pi(n-1); \\ \operatorname{maj}(\pi) + (n-1), & \text{otherwise.} \end{cases} \\ \operatorname{last}(\bar{\pi}) &= k-1 \end{aligned}$$

We can therefore compute

$$\begin{aligned} f_n &= f_n(x, y, z) \\ &= \sum_{\pi \in S_{n-1}} \sum_{k=1}^{n} x^{\operatorname{inv}(\bar{\pi})} y^{\operatorname{maj}(\bar{\pi})} z^{\operatorname{last}(\bar{\pi})} \\ &= \sum_{\pi \in S_{n-1}} x^{\operatorname{inv}(\pi)+n-1} y^{\operatorname{maj}(\pi)} \\ &\quad \times \left[y^{n-1} \sum_{k=1}^{\operatorname{last}(\pi)+1} x^{1-k} z^{k-1} + \sum_{k=\operatorname{last}(\pi)+2}^{n} x^{1-k} z^{k-1} \right] \\ &= (1 - z/x)^{-1} \sum_{\pi \in S_{n-1}} x^{\operatorname{inv}(\pi)+n-1} y^{\operatorname{maj}(\pi)} \\ &\quad \times \left[y^{n-1} \left(1 - (z/x)^{\operatorname{last}(\pi)+1} \right) + \left((z/x)^{\operatorname{last}(\pi)+1} - (z/x)^n \right) \right] \\ &= (1 - z/x)^{-1} [(x^{n-1} y^{n-1} - x^{-1} z^n) f_{n-1}(x, y, 1) \\ &\quad + x^{n-2} (1 - y^{n-1}) z f_{n-1}(x, y, z/x) \,. \end{aligned}$$

Multiplying both sides by $x - z$ gives the claimed recurrence.

□

3 A Multiplicative Generating Function

In general, the generating function from the previous section is a complicated polynomial of its variables. However, assuming in addition that $x^2 = 1$ leads to surprisingly simple results.

Corollary 3.1 *The first few values of f_n, assuming $x = \varepsilon = \pm 1$, are:*

$$\begin{aligned} f_1(\varepsilon, q, z) &= 1 \\ f_2(\varepsilon, q, z) &= z + \varepsilon q \\ f_3(\varepsilon, q, z) &= (1+\varepsilon q)(z^2 + qz + q^2) \\ f_4(\varepsilon, q, z) &= (1+\varepsilon q)(1+q+q^2)(z^3 + \varepsilon q z^2 + q^2 z + \varepsilon q^3) \end{aligned}$$

The case $\varepsilon = z = 1$ is a well-known result of MacMahon.

Theorem 3.2 *For $\varepsilon = \pm 1$,*

$$\begin{aligned} \sum_{\pi \in S_n} \varepsilon^{\mathrm{inv}(\pi)} q^{\mathrm{maj}(\pi)} z^{\mathrm{last}(\pi)} &= \left(\prod_{i=1}^{n-1} [i]_{\varepsilon^{i-1} q} \right) \cdot [n]_{\varepsilon^{n-1} q/z} \cdot z^{n-1} \\ &= [1]_q [2]_{\varepsilon q} [3]_q [4]_{\varepsilon q} \cdots [n-1]_{\pm q} \cdot [n]_{\pm \varepsilon q/z} z^{n-1}. \end{aligned}$$

Proof. By induction on n. By Corollary 3.1, the claim is true for $n = 1$ (as well as for $n = 2, 3, 4$). Assume now that the claim holds for $n-1$, where $n \geq 2$. Thus

$$f_{n-1}(\varepsilon, q, z) = \left(\prod_{i=1}^{n-2} [i]_{\varepsilon^{i-1} q} \right) \cdot [n-1]_{\varepsilon^{n-2} q/z} \cdot z^{n-2}.$$

Substituting in the recurrence formula of Theorem 2.1 and eliminating the factor

$$\left(\prod_{i=1}^{n-2} [i]_{\varepsilon^{i-1} q} \right),$$

it remains to show that

$$\begin{aligned} & (\varepsilon - z)[n-1]_{\varepsilon^{n-2} q} [n]_{\varepsilon^{n-1} q/z} \cdot z^{n-1} \\ = \; & (\varepsilon^n q^{n-1} - z^n)[n-1]_{\varepsilon^{n-2} q} + \varepsilon^{n-1}(1 - q^{n-1}) z [n-1]_{\varepsilon^{n-1} q/z} \cdot (z/\varepsilon)^{n-2}. \end{aligned}$$

Using the definition of $[k]_q$, this is equivalent to

$$\begin{aligned} & \frac{(\varepsilon - z)(1 - (\varepsilon^{n-2} q)^{n-1})(z^n - (\varepsilon^{n-1} q)^n)}{(1 - \varepsilon^{n-2} q)(z - \varepsilon^{n-1} q)} \\ = \; & \frac{(\varepsilon^n q^{n-1} - z^n)(1 - (\varepsilon^{n-2} q)^{n-1})}{1 - \varepsilon^{n-2} q} + \frac{\varepsilon z (1 - q^{n-1})(z^{n-1} - (\varepsilon^{n-1} q)^{n-1})}{z - \varepsilon^{n-1} q}. \end{aligned}$$

Clearing denominators and using the fact that $(n-2)(n-1)$ is even, we can transform this equation into

$$\begin{aligned}(\varepsilon - z)(1-q^{n-1})(z^n - q^n) &= (\varepsilon^n q^{n-1} - z^n)(1-q^{n-1})(z-\varepsilon^{n-1}q) \\ &+ \varepsilon z(1-q^{n-1})(z^{n-1}-\varepsilon^{n-1}q^{n-1})(1-\varepsilon^{n-2}q).\end{aligned}$$

Dividing by $(1-q^{n-1})$ one gets

$$(\varepsilon - z)(z^n - q^n) = (\varepsilon^n q^{n-1} - z^n)(z-\varepsilon^{n-1}q) + \varepsilon z(z^{n-1}-\varepsilon^{n-1}q^{n-1})(1-\varepsilon^{n}q),$$

completing the proof.

□

Letting $z=1$, one gets

Corollary 3.3

$$\begin{aligned}\sum_{\pi\in S_n} q^{\mathrm{maj}(\pi)} &= [n]_q! := [1]_q[2]_q\cdots[n]_q \\ \sum_{\pi\in S_n} \mathrm{sign}(\pi)\, q^{\mathrm{maj}(\pi)} &= [n]_{\pm q}! := [1]_q[2]_{-q}[3]_q[4]_{-q}\cdots[n]_{(-1)^{n-1}q}\end{aligned}$$

The first formula is a classical result of MacMahon [25], and the second was first proved by Gessel and Simion [41, Cor. 2].

Corollary 3.4 *The distributions of* maj *and of* maj *with* sign *over all permutations with a prescribed last digit are essentially independent of this digit, namely: if*

$$S_n(k) := \{\, \pi\in S_n \mid \pi(n)=k \,\} \qquad (1\le k\le n)$$

then, for $\varepsilon=\pm1$,

$$\begin{aligned}\sum_{\pi\in S_n(k)} \varepsilon^{\mathrm{inv}(\pi)} q^{\mathrm{maj}(\pi)} &= f_{n-1}(\varepsilon,q,1)\cdot(\varepsilon^{n-1}q)^{n-k} \\ &= \left(\prod_{i=1}^{n-1}[i]_{\varepsilon^{i-1}q}\right)\cdot(\varepsilon^{n-1}q)^{n-k}.\end{aligned}$$

4 Specialization at Roots of Unity

A proof of Theorem 1.7 is given in this section.

Suppose that we have a sequence $f_0(q), f_1(q), \ldots$ of polynomials in q defined by the Eulerian generating function

$$F(u;q) = \sum_{n=0}^{\infty} f_n(q) \frac{u^n}{(q)_n}, \tag{2}$$

where $(q)_n := (1-q)(1-q^2)\cdots(1-q^n)$. We would like to study the values of $f_n(q)$ at a root of unity. We cannot simply evaluate (2) at a root of unity, since this would make denominators vanish. Instead we take a less direct approach.

Fix a positive integer m, and let $\phi_m(q)$ be the cyclotomic polynomial of order m in q (whose roots are all the primitive mth roots of unity). If $f(q)$ and $g(q)$ are polynomials in q with rational coefficients and ζ is a primitive mth root of unity, then $f(q) \equiv g(q) \pmod{\phi_m(q)}$ if and only if $f(\zeta) = g(\zeta)$, since $\phi_m(q)$ is irreducible over the rationals and $\phi_m(\zeta) = 0$.

Given two Eulerian generating functions $F(u;q) = \sum_{n=0}^{\infty} f_n(q)u^n/(q)_n$ and $G(u;q) = \sum_{n=0}^{\infty} g_n(q)u^n/(q)_n$, by $F(u;q) \equiv G(u;q)$ we mean that $f_n(q) \equiv g_n(q) \pmod{\phi_m(q)}$ for all n. Henceforth we take all congruences to be modulo $\phi_m(q)$.

The basic facts about these congruences are contained in the following lemma:

Lemma 4.1 *Let $u_i := u^i/(q)_i$.*

(i) If $0 \le i, j < m$ and $i + j \ge m$ then $u_i u_j \equiv 0$.

(ii) If $0 \le i < m$ then

$$u_{mk+i} \equiv \frac{u_m^k}{k!} u_i.$$

Proof. Let ζ be a primitive mth root of unity. For (i), we have

$$u_i u_j = \frac{u^{i+j}}{(q)_i (q)_j} = \frac{(q)_{i+j}}{(q)_i (q)_j} u_{i+j}.$$

The quotient in the right-hand-side is a polynomial in q (a q-binomial coefficient, see below). Since $(q)_{i+j}$ vanishes for $q = \zeta$ but $(q)_i (q)_j$ does not, (i) follows.

For (ii), we have

$$\frac{u_m^k}{k!}u_i = \frac{u^{mk+i}}{(q)_m^k k!\,(q)_i} = \frac{(q)_{mk+i}}{(q)_m^k k!\,(q)_i}u_{mk+i},$$

so it suffices to show that

$$\left.\frac{(q)_{mk+i}}{(q)_m^k k!\,(q)_i}\right|_{q=\zeta} = 1.$$

To prove this, we show that

$$\left.\frac{(q)_{mk+i}}{(q)_{mk}(q)_i}\right|_{q=\zeta} = 1$$

and that

$$\left.\frac{(q)_{mk}}{(q)_m^k}\right|_{q=\zeta} = k!.$$

For the first equality, we have

$$\frac{(q)_{mk+i}}{(q)_{mk}(q)_i} = \frac{1-q^{mk+1}}{1-q}\frac{1-q^{mk+2}}{1-q^2}\cdots\frac{1-q^{mk+i}}{1-q^i}.$$

Since $\zeta^{mk+j} = \zeta^j \neq 1$ for $j = 1, 2, \ldots, i$, the equality follows.

For the second equality, let us write

$$(q)_{mk} = \prod_{\substack{1\leq l\leq mk \\ m\nmid l}}(1-q^l)\cdot\prod_{j=1}^{k}(1-q^{mj}),$$

so

$$\frac{(q)_{mk}}{(q)_m^k} = \frac{\displaystyle\prod_{1\leq l\leq mk,\, m\nmid l}(1-q^l)}{(q)_{m-1}^k}\cdot\prod_{j=1}^{k}\frac{1-q^{mj}}{1-q^m}.$$

We may evaluate the first factor on the right at $q=\zeta$ by simply setting $q=\zeta$, since neither the numerator nor the denominator vanishes, and we see easily that this factor becomes 1. Writing the second factor as

$$\prod_{j=1}^{k}(1+q^m+\cdots+q^{m(j-1)}),$$

we see that setting $q=\zeta$ in it yields $k!$.

□

Now recall that the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the polynomial in q defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q)_n}{(q)_k (q)_{n-k}}$$

for $0 \le k \le n$, with $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ for $n < k$. As a consequence of Lemma 4.1 we obtain a frequently rediscovered result of Gloria Olive [26, (1.2.4)] about the evaluation of q-binomial coefficients at roots of unity:

Corollary 4.2 *Let m be a positive integer and let ζ be a primitive mth root of unity. Let a_1, a_2, b_1, and b_2 be nonnegative integers with $0 \le b_1, b_2 < m$. Then*

$$\begin{bmatrix} (ma_1 + b_1) + (ma_2 + b_2) \\ ma_1 + b_1 \end{bmatrix}_\zeta = \binom{a_1 + a_2}{a_1} \begin{bmatrix} b_1 + b_2 \\ b_1 \end{bmatrix}_\zeta .$$

Proof. With the notation of Lemma 4.1 we have

$$\begin{aligned} \begin{bmatrix} (ma_1 + b_1) + (ma_2 + b_2) \\ ma_1 + b_1 \end{bmatrix}_q u_{(ma_1+b_1)+(ma_2+b_2)} &= \frac{u^{ma_1+b_1}}{(q)_{ma_1+b_1}} \frac{u^{ma_2+b_2}}{(q)_{ma_2+b_2}} \\ &= u_{ma_1+b_1} u_{ma_2+b_2}. \end{aligned}$$

By Lemma 4.1(ii) this is congruent modulo $\phi_m(q)$ to

$$\frac{u_m^{a_1}}{a_1!} u_{b_1} \frac{u_m^{a_2}}{a_2!} u_{b_2}.$$

If $b_1 + b_2 \ge m$ then, by Lemma 4.1(i), this is congruent to 0. Otherwise we have, by Lemma 4.1(ii),

$$\begin{aligned} \frac{u_m^{a_1}}{a_1!} u_{b_1} \frac{u_m^{a_2}}{a_2!} u_{b_2} &= \binom{a_1 + a_2}{a_1} \frac{u_m^{a_1+a_2}}{(a_1 + a_2)!} \cdot \begin{bmatrix} b_1 + b_2 \\ b_1 \end{bmatrix}_q u_{b_1+b_2} \\ &\equiv \binom{a_1 + a_2}{a_1} \begin{bmatrix} b_1 + b_2 \\ b_1 \end{bmatrix}_q u_{m(a_1+a_2)+(b_1+b_2)}, \end{aligned}$$

and the result follows.

□

Our proof of Theorem 1.7 is based on the generating function for the bivariate distribution of length and major index:

Theorem 4.3 *Let the polynomials $A_n(q,r)$ be defined by*

$$A(u;q) := \prod_{i,j=0}^{\infty} \frac{1}{1-q^i r^j u} = \sum_{n=0}^{\infty} \frac{A_n(q,r)}{(q)_n (r)_n} u^n. \tag{3}$$

Then

$$A_n(q,r) \;=\; \sum_{\pi \in S_n} q^{l(\pi)} r^{\mathrm{maj}(\pi)}.$$

Historical Note: Theorem 4.3 was first proved by Gessel [19, Theorem 8.5]. (For a refinement that also includes the number of descents, see [20].) Basil Gordon [21] had earlier given a combinatorial interpretation to the coefficients of $A_n(q,r)$, but did not describe it very explicitly. (In fact, he considered the generalization $\prod_{i,j,\ldots,k=0}^{\infty}(1-q^i r^j \cdots s^k u)^{-1}$.) D. P. Roselle [34] explained Gordon's combinatorial interpretation more explicitly. His result is equivalent to

$$A_n(q,r) = \sum_{\pi \in S_n} q^{\mathrm{maj}(\pi^{-1})} r^{\mathrm{maj}(\pi)}.$$

Then D. Foata and M.-P. Schützenberger [18] gave a bijective proof that

$$\sum_{\pi \in S_n} q^{\mathrm{maj}(\pi^{-1})} r^{\mathrm{maj}(\pi)} = \sum_{\pi \in S_n} q^{l(\pi)} r^{\mathrm{maj}(\pi)},$$

which, together with the result of Gordon and Roselle, implies Theorem 4.3.

Theorem 4.4 *Let n and m be positive integers. Let ζ be a primitive mth root of unity, and assume that $n = mk + i$ with $0 \le i < m$. Then*

$$A_n(\zeta, r) = A_i(\zeta, r) \frac{(r)_n}{(r)_i (1-r^m)^k}.$$

Proof. To find a congruence modulo $\phi_m(q)$ for the polynomials $A_n(q,r)$, think of (3) as an Eulerian generating function in which the coefficient of $u^n/(q)_n$ is $A_n(q,r)/(r)_n$. By taking logarithms and exponentiating, we see that

$$\begin{aligned} A(u;q) &= \prod_{i,j=0}^{\infty} \frac{1}{1-q^i r^j u} = \exp\Big(-\sum_{i,j=0}^{\infty} \ln(1-q^i r^j u)\Big) \\ &= \exp\Big(\sum_{i,j=0}^{\infty}\sum_{t=1}^{\infty} \frac{(q^i r^j u)^t}{t}\Big) = \exp\Big(\sum_{t=1}^{\infty} \frac{u^t}{t(1-q^t)(1-r^t)}\Big). \end{aligned}$$

Now

$$\sum_{t=1}^{\infty} \frac{u^t}{t(1-q^t)(1-r^t)} = \sum_{t=1}^{\infty} \frac{(q)_{t-1}}{t(1-r^t)} \frac{u^t}{(q)_t} \equiv \sum_{t=1}^{m} \frac{(q)_{t-1}}{t(1-r^t)} \frac{u^t}{(q)_t},$$

so

$$A(u;q) \equiv \exp\left(\sum_{t=1}^{m-1} \frac{(q)_{t-1}}{t(1-r^t)} u_t \right) \cdot \exp\left(\frac{(q)_{m-1}}{m(1-r^m)} u_m \right).$$

Using Lemma 4.1(i) we see that

$$\exp\left(\sum_{t=1}^{m-1} \frac{(q)_{t-1}}{t(1-r^t)} u_t \right) \equiv \sum_{i=0}^{m-1} B_i(q,r) u_i,$$

where $B_i(q,r)$ are polynomials in q whose coefficients are rational functions of r.

Now let ζ be a primitive mth root of unity. Setting $x = 1$ in

$$(1-\zeta x) \cdots (1-\zeta^{m-1} x) = (1-x^m)/(1-x) = 1 + x + \cdots + x^{m-1}$$

we see that

$$(1-\zeta) \cdots (1-\zeta^{m-1}) = m.$$

Thus $(q)_{m-1} \equiv m$, so with the terminology of Lemma 4.1 we have

$$\frac{(q)_{m-1}}{m(1-r^m)} u_m \equiv \frac{u_m}{1-r^m}$$

and

$$\exp\left(\frac{(q)_{m-1}}{m(1-r^m)} u_m \right) \equiv \exp\left(\frac{u_m}{1-r^m} \right) = \sum_{k=0}^{\infty} \frac{u_m^k}{k!\,(1-r^m)^k}.$$

It follows that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{A_n(q,r)}{(r)_n} \frac{u^n}{(q)_n} &\equiv \sum_{i=0}^{m-1} \sum_{k=0}^{\infty} B_i(q,r) \frac{u_i u_m^k}{k!\,(1-r^m)^k} \\ &\equiv \sum_{i=0}^{m-1} \sum_{k=0}^{\infty} \frac{B_i(q,r)}{(1-r^m)^k} \frac{u^{mk+i}}{(q)_{mk+i}}, \end{aligned}$$

by Lemma 4.1(ii). Thus, if $n = mk + i$ with $0 \le i < m$, then

$$\frac{A_n(q,r)}{(r)_n} \equiv \frac{B_i(q,r)}{(1-r^m)^k}$$

or equivalently

$$\frac{A_n(\zeta, r)}{(r)_n} = \frac{B_i(\zeta, r)}{(1 - r^m)^k}.$$

Letting $k = 0$ (so that $n = i$) we get

$$B_i(\zeta, r) = \frac{A_i(\zeta, r)}{(r)_i} \qquad (0 \le i < m)$$

and the result follows.

□

Second Proof of Theorem 1.3. Take $m = 2$ in Theorem 4.4 and simplify.

□

For some other results involving the evaluation of $A_n(q, r)$ at roots of unity, see [6] and [21].

5 A Signed Mahonian for B_n

Let B_n be the hyperoctahedral group. The *flag-major* of $\sigma \in B_n$ is defined as

$$\text{flag-major}(\sigma) := 2\,\text{maj}(\sigma) + \text{neg}(\sigma),$$

where

$$\text{neg}(\sigma) := \#\{\, i \mid \sigma(i) < 0 \,\}$$

and $\text{maj}(\sigma)$ is the major index of the sequence $(\sigma(1), \ldots, \sigma(n))$, with respect to the order

$$-1 < \cdots < -n < 1 < \cdots < n.$$

Recall that for every $\sigma \in B_n$ we define

$$\text{sign}(\sigma) = (-1)^{l(\sigma)},$$

where the length l (here and throughout this section) is taken with respect to the Coxeter generators of B_n.

Theorem 5.1

$$\sum_{\sigma \in B_n} \text{sign}(\sigma) q^{\text{flag-major}(\sigma)} = [2]_{-q}[4]_q \cdots [2n]_{(-1)^n q}.$$

Remark 5.2 *The above order appeared in [4]. In [1] we considered another natural order :*

$$-n < \cdots < -1 < 1 < \cdots < n.$$

The distribution of flag-major is the the same for both orders, but the joint distribution of flag-major and length is different, and Theorem 5.1 does not hold for flag-major *defined with respect to the latter order. It was shown in [4] that* flag-major *defined with respect to the first order satisfies some further remarkable properties (e.g., it is the length of a certain decomposition of the permutation). These properties do not hold for the second order.*

Proof. We use the decomposition

$$B_n = U_n \cdot S_n,$$

where

$$U_n := \{ \tau \in B_n \mid \tau(1) < \cdots < \tau(n) \}$$

with respect to the order

$$-1 < \cdots < -n < 1 < \cdots < n,$$

and

$$S_n := \{ \pi \in B_n \mid \text{neg}(\pi) = 0 \}.$$

This decomposition appeared in [1] (where it was taken with respect to the other order).

Note that every $\sigma \in B_n$ has a unique decomposition $\sigma = \tau\pi$, $\tau \in U_n, \pi \in S_n$. Then, by definition,

$$\text{flag-major}(\sigma) = 2 \cdot \text{maj}(\pi) + \text{neg}(\tau).$$

Thus

$$\begin{aligned} \sum_{\sigma \in B_n} \text{sign}(\sigma) q^{\text{flag-major}(\sigma)} &= \sum_{\tau \in U_n,\, \pi \in S_n} \text{sign}(\tau\pi) q^{2\cdot\text{maj}(\pi)+\text{neg}(\tau)} \\ &= \sum_{\tau \in U_n} \text{sign}(\tau) q^{\text{neg}(\tau)} \cdot \sum_{\pi \in S_n} \text{sign}(\pi) q^{2\cdot\text{maj}(\pi)}. \end{aligned}$$

By Corollary 3.3, the second factor is equal to

$$\sum_{\pi \in S_n} \text{sign}(\pi) q^{2\cdot\text{maj}(\pi)} = [1]_{q^2} [2]_{-q^2} \cdots [n]_{\pm q^2}.$$

We shall compute the first factor. Define

$$U_n(k) := \{\, \tau \in U_n \mid \operatorname{neg}(\tau) = k \,\} \qquad (0 \le k \le n).$$

Then

$$\sum_{\tau \in U_n} \operatorname{sign}(\tau) q^{\operatorname{neg}(\tau)} = \sum_{k=0}^{n} \sum_{\tau \in U_n(k)} \operatorname{sign}(\tau) \cdot q^k = \sum_{k=0}^{n} q^k \sum_{\tau \in U_n(k)} (-1)^{l(\tau)}.$$

Recall from [9, Proposition 3.1 and Corollary 3.2] [8, Propositions 8.1.1 and 8.1.2] that for every $\sigma \in B_n$

$$l(\sigma) = \operatorname{inv}(\sigma) + \sum_{\{\, 1 \le i \le n \mid \ \sigma(i) < 0 \,\}} |\sigma(i)|,$$

where $\operatorname{inv}(\sigma)$ is taken with respect to the order

$$-n < \cdots < -1 < 1 < \cdots < n.$$

Now U_n consists of all elements whose entries are increasing with respect to the order $-1 < \cdots < -n < 1 < \cdots < n$. Thus for every $\tau \in U_n(k)$

$$\operatorname{inv}(\tau) = \binom{k}{2}$$

and

$$l(\tau) = \binom{k}{2} + \sum_{i=1}^{k} |\tau(i)|.$$

It follows that

$$\begin{aligned} \sum_{\tau \in U_n(k)} (-1)^{l(\tau)} &= \sum_{\tau \in U_n(k)} (-1)^{\binom{k}{2} + \sum_{i=1}^{k} |\tau(i)|} \\ &= (-1)^{\binom{k}{2}} \sum_{1 \le i_1 < \cdots < i_k \le n} (-1)^{\sum_{j=1}^{k} i_j}. \end{aligned}$$

From the q-binomial theorem

$$\prod_{i=1}^{n} (1 + q^i x) = \sum_{k=0}^{n} q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k,$$

it follows that

$$\sum_{1 \le i_1 < \cdots < i_k \le n} q^{\sum_{j=1}^{k} i_j} = q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

We deduce that

$$\sum_{\tau \in U_n(k)} \operatorname{sign}(\tau) = (-1)^{\binom{k}{2}} (-1)^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{-1} = (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_{-1},$$

so

$$\sum_{\tau \in U_n} \operatorname{sign}(\tau) q^{\operatorname{neg}(\tau)} = \sum_{k=0}^{n} q^k \sum_{\tau \in U_n(k)} \operatorname{sign}(\tau) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{-1} (-q)^k.$$

From the case $m = 2$ of Corollary 4.2 we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_{-1} = \begin{cases} 0, & \text{if } k \text{ and } n-k \text{ are odd;} \\ \binom{\lfloor n/2 \rfloor}{\lfloor k/2 \rfloor}, & \text{otherwise.} \end{cases}$$

Thus, for n even ($n = 2m$):

$$\sum_{\tau \in U_n} \operatorname{sign}(\tau) q^{\operatorname{neg}(\tau)} = \sum_{t=0}^{m} \binom{m}{t} (-q)^{2t} = (1+q^2)^m,$$

and for n odd ($n = 2m+1$):

$$\sum_{\tau \in U_n} \operatorname{sign}(\tau) q^{\operatorname{neg}(\tau)} = (1-q) \sum_{t=0}^{m} \binom{m}{t} (-q)^{2t} = (1-q)(1+q^2)^m.$$

We conclude that, for n odd ($n = 2m+1$):

$$\begin{aligned} \sum_{\sigma \in B_n} \operatorname{sign}(\sigma) q^{\text{flag-major}(\sigma)} &= (1-q)(1+q^2)^m [1]_{q^2} [2]_{-q^2} \cdots [2m+1]_{q^2} \\ &= (1-q)(1+q^2)^m \frac{\prod_{t=1}^{2m+1} (1-q^{2t})}{(1-q^2)^{m+1}(1+q^2)^m} \\ &= \frac{(1-q) \prod_{t=1}^{2m+1} (1-q^{2t})}{(1-q^2)^{m+1}} = \frac{\prod_{t=1}^{2m+1} (1-q^{2t})}{(1+q)^{m+1}(1-q)^m} \\ &= [2]_{-q} [4]_q \cdots [2(2m+1)]_{-q}. \end{aligned}$$

The case of n even is similar.

□

6 Other One-Dimensional Characters of B_n

The group B_n has four one-dimensional characters: the trivial character; the sign character; $(-1)^{\mathrm{neg}(\sigma)}$; and the sign of $(|\sigma(1)|, \ldots, |\sigma(n)|) \in S_n$, denoted $\mathrm{sign}(|\sigma|)$. We now generalize the results of the previous section to the last two one-dimensional characters.

Theorem 6.1

$$\sum_{\sigma \in B_n} (-1)^{\mathrm{neg}(\sigma)} q^{\text{flag-major}(\sigma)} = [2]_{-q}[4]_{-q} \cdots [2n]_{-q}.$$

Proof. Replace q by $-q$ in Theorem 1.5, and use the fact that the parity of flag-major is equal to the parity of neg.

□

Theorem 6.2

$$\sum_{\sigma \in B_n} \mathrm{sign}(|\sigma|) q^{\text{flag-major}(\sigma)} = [2]_q[4]_{-q} \cdots [2n]_{(-1)^{n-1}q}.$$

Proof. Similarly, replace q by $-q$ in Theorem 5.1 and apply the identity $\mathrm{sign}(\sigma) = \mathrm{sign}(|\sigma|) \cdot (-1)^{\mathrm{neg}(\sigma)}$.

□

7 Major Index on Subgroups

Let A_n be the group of even permutations on n letters. Then

Corollary 7.1

$$\sum_{\pi \in A_n} q^{\mathrm{maj}(\pi)} = \frac{1}{2}([1]_q[2]_q \cdots [n]_q + [1]_q[2]_{-q} \cdots [n]_{(-1)^{n-1}q}).$$

Proof. Clearly,

$$\begin{aligned}
\sum_{\pi \in A_n} q^{\mathrm{maj}(\pi)} &= \sum_{\pi \in S_n} \frac{1 + \mathrm{sign}(\pi)}{2} q^{\mathrm{maj}(\pi)} \\
&= \frac{1}{2}\left(\sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} + \sum_{\pi \in S_n} \mathrm{sign}(\pi)\, q^{\mathrm{maj}(\pi)} \right).
\end{aligned}$$

Corollary 3.3 completes the proof.

□

Let B_n^+ be the subgroup of even elements in B_n, D_n the subgroup of elements with even neg (this is a classical Weyl group), and $C_2 \wr A_n$ the subgroup of elements $\sigma \in B_n$ with even sign$(|\sigma|)$. Then

Corollary 7.2

$$(1) \qquad \sum_{\sigma \in B_n^+} q^{\text{flag-major}(\sigma)} = \frac{1}{2}([2]_q[4]_q \cdots [2n]_q + [2]_{-q}[4]_q \cdots [2n]_{(-1)^n q}).$$

$$(2) \qquad \sum_{\sigma \in D_n} q^{\text{flag-major}(\sigma)} = \frac{1}{2}([2]_q[4]_q \cdots [2n]_q + [2]_{-q}[4]_{-q} \cdots [2n]_{-q}).$$

$$(3) \qquad \sum_{\sigma \in C_2 \wr A_n} q^{\text{flag-major}(\sigma)} = \frac{1}{2}([2]_q[4]_q \cdots [2n]_q + [2]_q[4]_{-q} \cdots [2n]_{(-1)^{n-1} q}).$$

Proof. Theorem 5.1 implies (1), Theorem 6.1 implies (2), and Theorem 6.2 implies (3).

□

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