The Method of Coefficients

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A simple example

Consider the identity

$$\sum_{k=0}^{m} \binom{n}{k} \left(\frac{3n - 2k}{m - k}\right)(-3)^k = \binom{n}{m/3}.$$ 

Note that in terms of hypergeometric series, this identity may be written

$$\begin{aligned}
\, _3F_2 \left( \begin{array}{c}
-n, -3n + m, -m \\
-\frac{3}{2}n, -\frac{3}{2}n + \frac{1}{2}
\end{array} \mid \begin{array}{c}
3 \\
\frac{3}{4}
\end{array} \right)
&= \begin{cases}
\binom{-n+\frac{1}{3}}{m} \binom{-n+\frac{2}{3}}{m} & \text{if } m = 3M \\
\binom{1}{M} \binom{2}{M} & \text{otherwise}
\end{cases} \\
&= \begin{cases}
0 & \text{otherwise}
\end{cases}
\end{aligned}$$

(a result of George Andrews).
We can prove this identity in the following way:

We start with

\[ 1 + y^3 = (1 + y)(1 - y + y^2) \]
\[ = (1 + y)((1 + y)^2 - 3y) \]

So

\[ (1 + y^3)^n = (1 + y)^n((1 + y)^2 - 3y)^n \]
\[ = (1 + y)^n \sum_k \binom{n}{k}(1 + y)^{2n-2k}(-3y)^k \]
\[ = \sum_k \binom{n}{k}(1 + y)^{3n-2k}(-3y)^k. \]

Equating coefficients of \( y^m \) on both sides gives the identity

\[ \sum_{k=0}^{m} \binom{n}{k}\binom{3n-2k}{m-k}(-3)^k = \binom{n}{m/3}. \]
But how could we find this proof?

We represent the binomial coefficients as coefficients of binomial expansions: \( \binom{n}{k} \) is the coefficient of \( x^k \) in \((1 + x)^n\) and \( \binom{3n-2k}{m-k} \) is the coefficient of \( y^{m-k} \) in \((1 + y)^{3n-2k}\). It is convenient to represent these coefficients as constant terms, so the sum is

\[
\sum_{k} \frac{(1 + x)^n (1 + y)^{3n-2k}}{x^k y^{m-k}} (-3)^k
\]

\[
= \sum_{k} \frac{(1 + x)^n}{x^k} \left( -3 \frac{y}{(1 + y)^2} \right)^k \frac{(1 + y)^{3n}}{y^m}
\]

Now we apply the variable elimination rule: If \( f(x) \) is any power series and \( \alpha \) is independent of \( x \) then

\[
\sum_{k=0}^{\infty} \frac{f(x)}{x^k} \alpha^k = \sum_{k=0}^{\infty} [x^k] f(x) \alpha^k = f(\alpha).
\]
We apply this rule to

\[
CT \sum_k \frac{(1 + x)^n}{x^k} \left( -3 \frac{y}{(1 + y)^2} \right)^k \frac{(1 + y)^{3n}}{y^m}
\]

\[
= CT_y \frac{(1 + y)^{3n}}{y^m} \sum_k \frac{(1 + x)^n}{x^k} \left( -3 \frac{y}{(1 + y)^2} \right)^k
\]

\[
= CT_y \frac{(1 + y)^{3n}}{y^m} \left( 1 - 3 \frac{y}{(1 + y)^2} \right)^n
\]

\[
= CT_y \frac{(1 + y)^{3n}}{y^m} \left( \frac{(1 + y)^2 - 3y}{(1 + y)^2} \right)^n
\]

\[
= CT_y \frac{(1 + y)^{3n}}{y^m} \left( 1 - y + y^2 \right)^n
\]

\[
= CT_y \frac{(1 + y)^{3n}}{y^m} \left( y^2 \right)^n
\]

\[
= CT_y \frac{(1 + y^3)^n}{y^m}
\]

\[
= \binom{n}{m/3}.
\]
The Method of Coefficients

Based on G. P. Egorychev’s book Интегральное представление и вычисление комбинаторных сумм

To evaluate a binomial coefficient sum:

1. First we represent the binomial coefficients as constant terms:

\[
\binom{n}{k} = \text{CT} \frac{(1 + x)^n}{x^k} = \text{CT} \frac{(1 + x)^n}{x^{n-k}}
\]

\[
= \text{CT} \frac{1}{x^k(1 - x)^{n-k+1}}
\]

\[
= \text{CT} \frac{1}{x^{n-k}(1 - x)^{k+1}}
\]

2. We apply the variable elimination rule

\[
\text{CT} \sum_{k=0}^{\infty} \frac{f(x)}{x^k} \alpha^k = \sum_{k=0}^{\infty} [x^k] f(x) \alpha^k = f(\alpha).
\]
Another example

Evaluate \( \sum_k (-2)^k \binom{n}{k} \binom{2n-k}{n-r-k} \).

We have

\[
\sum_k (-2)^k \binom{n}{k} \binom{2n-k}{n-r-k} = CT \sum_k (-2)^k \frac{(1 + x)^n}{x^k} \frac{(1 + y)^{2n-k}}{y^{n-r-k}} \\
= CT \sum_k \frac{(1 + x)^n}{x^k} \left( -\frac{2y}{1 + y} \right)^k \frac{(1 + y)^{2n}}{y^{n-r}} \\
= CT \left( 1 - \frac{2y}{1 + y} \right)^n \frac{(1 + y)^{2n}}{y^{n-r}} \\
= CT \left( \frac{1 - y}{1 + y} \right)^n \frac{(1 + y)^{2n}}{y^{n-r}} \\
= CT \frac{(1 - y^2)^n}{y^{n-r}} \\
= (-1)^{(n-r)/2} \binom{n}{(n-r)/2}
\]
Change of variables formulas

The simplest change of variables formula is

$$\text{CT} \ f(x) = \text{CT} \ f(\alpha x),$$

where $\alpha$ is independent of $x$.

More interesting is the linear change of variables formula

$$\text{CT} \ f(x) = \text{CT} \frac{1}{1 + \alpha x} f \left( \frac{x}{1 + \alpha x} \right)$$

To prove it, we need only consider the case $f(x) = x^k$, $k \in \mathbb{Z}$. The case $k = 0$ is clear. For $k > 0$,

$$\text{CT} \ \frac{x^k}{(1 + \alpha x)^{k+1}} = 0.$$

For $k = -j < 0$,

$$\text{CT} \ \frac{x^k}{(1 + \alpha x)^{k+1}} = \text{CT} \ \frac{(1 + x)^{j-1}}{x^j} = 0.$$
As a simple example of this change of variables formula, let

\[ S(a, b, n, \gamma) = \sum_k \binom{a}{k} \binom{b}{n-k} \gamma^k \]

\[ = \mathrm{CT} \frac{(1 + \gamma x)^a (1 + x)^b}{x^n} \]

\[ = \binom{b}{n} _2F_1 \left( \begin{array}{c} -a, -n \\ b - n + 1 \end{array} \mid \gamma \right). \]

Applying the change of variables formula with \( \alpha = -1 \) gives

\[ S(a, b, n, \gamma) = \frac{(1 + (\gamma - 1)x)^a (1 - x)^{n-a-b-1}}{x^n} \]

\[ = (-1)^n \mathrm{CT} \frac{(1 + (1 - \gamma)x)^a (1 + x)^{n-a-b-1}}{x^n} \]

\[ = (-1)^n S(a, n - a - b - 1, n, 1 - \gamma). \]

This is equivalent to a \(_2F_1\) linear transformation.
A 2-variable change of variables formula

(Gessel and Stanton) For any Laurent series \( f(x, y) \),

\[
\text{CT} \left( \frac{1}{1 - xy} \right) f(x, y) = \text{CT} f \left( \frac{x}{1 + y}, \frac{y}{1 + x} \right).
\]

To prove it we may assume that \( f(x, y) = x^l y^m \). The left side is 1 if \( l = m \leq 0 \) and 0 otherwise. The right side is

\[
\text{CT} \left( \frac{x^l}{(1 + y)^l} \right) \left( \frac{y^m}{(1 + x)^m} \right).
\]

This is clearly 0 if \( l \) or \( m \) is positive. If \( l = -r \) and \( m = -s \), with \( r, s \geq 0 \) then it is

\[
\text{CT} \frac{(1 + x)^s}{x^r} \frac{(1 + y)^r}{y^s} = \binom{s}{r} \binom{r}{s} = \delta_{r,s}.
\]
As an application we'll prove the identity

\[
\frac{(1 + x)^a(1 + y)^b}{(1 - xy)^{a+b+1}} = \sum_{l,m=0}^{\infty} \binom{a + m}{l} \binom{b + l}{m} x^l y^m.
\]

Applying the change of variables formula with

\[
f(x, y) = \frac{(1 + x)^a(1 + y)^b}{x^l y^m (1 - xy)^{a+b}}
\]

we get

\[
\text{CT} \frac{(1 + x)^a(1 + y)^b}{x^l y^m (1 - xy)^{a+b+1}} = \text{CT} \frac{(1 + x)^{a+m}(1 + y)^{b+l}}{x^l y^m} = \binom{a + m}{l} \binom{b + l}{m}
\]
General change of variables

For more general change of variables formulas, it is convenient to work with residues rather than constant terms. The residue of the Laurent series \( f(z) \), denoted \( \text{res} f(z) \), is the coefficient of \( z^{-1} \) in \( f(z) \).

We have the following change of variables formula: Let \( f(z) \) be a Laurent series and let \( g(y) \) be a power series, \( g(y) = g_1 y + g_2 y^2 + \cdots \), where \( g_1 \neq 0 \). Then

\[
\text{res} f(z) = \text{res} f(g(y)) g'(y).
\]

(Note: There is a multivariable generalization, due to Jacobi, that we will not discuss.)
The key fact in the proof is that the residue of a derivative is 0, and that a Laurent series with residue 0 is a derivative. Any Laurent series may be written as \( az^{-1} + F'(z) \) for some Laurent series \( F(z) \), so by linearity it is enough to prove the formula for \( f(z) = z^{-1} \) and \( f(z) = F'(z) \).

For \( f(z) = F'(z) \) we have \( \text{res } F'(z) = 0 \) and

\[
\text{res } F'(g(y))g'(y) = \text{res } F(g(y))' = 0.
\]

For \( f(z) = z^{-1} \) we have \( \text{res } z^{-1} = 1 \) and

\[
\text{res } \frac{1}{g(y)}g'(y) = \text{res } \frac{g_1 + 2g_2y + \cdots}{g_1y + g_2y^2 + \cdots}
\]

\[
= \text{res } \left( \frac{1}{y} + \frac{g_2}{g_1} + \cdots \right)
\]

\[
= 1.
\]
An application of the change of variables theorem

Prove the identity

\[
\sum_{k} \binom{a + k}{k} \binom{a + 2n + 1}{n - k} = 2^{2n} \binom{n + a/2}{n}.
\]

As a hypergeometric series identity, this is a form of Kummer’s theorem,

\[
\binom{1 + a}{2 + a + n} \binom{-n}{-1} = \frac{(a + 2)^n}{\left(\frac{3}{2} + \frac{a}{2}\right)^n}.
\]

The left side is

\[
\begin{align*}
\text{CT} \sum_k & \frac{1}{x^k(1 - x)^{a+1}} \frac{(1 + y)^{a+2n+1}}{y^{n-k}} \\
& = \text{CT} \frac{(1 + y)^{a+2n+1}}{y^n(1 - y)^{a+1}} = \text{CT} \left(\frac{(1 + y)^2}{y}\right)^n \frac{(1 + y)^{a+1}}{(1 - y)^{a+1}} \\
& = \text{res} \left(\frac{(1 + y)^2}{y}\right)^{n+1} \frac{(1 + y)^{a-1}}{(1 - y)^{a+1}}.
\end{align*}
\]
This suggests a change of variables \( z = y/(1 + y)^2 \). The computation is easiest if we start with the right side:

\[
2^{2n} \binom{n + a/2}{n} = \text{res} \frac{1}{z^{n+1}(1 - 4z)^{a/2+1}}
\]

Then we make the change of variables \( z = y/(1 + y)^2 \), so that \( 1 - 4z = (1 - y)^2/(1 + y)^2 \) and \( dz/dy = (1 - y)/(1 + y)^3 \).

We get

\[
\text{res} \left( \frac{(1 + y)^2}{y} \right)^{n+1} \left( \frac{1 + y}{1 - y} \right)^{a/2+1} \frac{1 - y}{(1 + y)^3} = \text{res} \left( \frac{(1 + y)^2}{y} \right)^{n+1} \frac{(1 + y)^{a-1}}{(1 - y)^{a+1}}
\]
A more complicated example

(Gessel and Stanton) Prove the hypergeometric series identity

\[ _2F_1 \left( \begin{array}{c} -n, \frac{1}{2} \\ 2n + \frac{3}{2} \end{array} \middle| \frac{1}{4} \right) = \frac{(\frac{1}{2})^n}{(2n + \frac{3}{2})^n} \left( \frac{27}{4} \right)^n \]

An equivalent binomial coefficient identity is

\[
\sum_k \binom{-\frac{1}{2}}{k} \binom{3n + \frac{1}{2}}{n - k} \left( \frac{1}{4} \right)^k = (-1)^n \left( \frac{27}{4} \right)^n \left( -\frac{1}{2} \right). 
\]

The left side is

\[
\text{CT} \sum_k \frac{(1 + x)^{-1/2}}{x^k} \frac{(1 + y)^{3n+1/2}}{y^{n-k}} \left( \frac{1}{4} \right)^k \\
= \text{res} \frac{(1 + \frac{1}{4}y)^{-1/2}(1 + y)^{3n+1/2}}{y^{n+1}}.
\]

This suggests the change of variables \( z = y/(1 + y)^3 \).
The right side is \( \text{res} \frac{1}{z^{n+1} \sqrt{1 - \frac{27}{4} z}} \). With the change of variables \( z = y/(1 + y)^3 \), we have \( dz/\, dy = (1 - 2y)/(1 + y)^4 \) and

\[
\sqrt{1 - \frac{27}{4} z} = \frac{(1 + \frac{1}{4} y)^{1/2}(1 - 2y)}{(1 + y)^{3/2}},
\]

so

\[
\text{res} \frac{1}{z^{n+1} \sqrt{1 - \frac{27}{4} z}} = \text{res} \frac{(1 + y)^{3n+3}}{y^{n+1}} \cdot \frac{(1 + y)^{3/2}}{(1 + \frac{1}{4} y)^{1/2}(1 - 2y)} \cdot \frac{1 - 2y}{(1 + y)^4}
\]

\[
= \text{res} \frac{(1 + \frac{1}{4} y)^{-1/2}(1 + y)^{3n+1/2}}{y^{n+1}}.
\]
Comparison with Wilf’s Snake Oil Method

To evaluate a sum $S_n = \sum_k S_{n,k}$, we find the generating function

$$\sum_n S_n x^n = \sum_k \sum_n S_{n,k} x^n.$$ 

Sometimes we need to adjust the parameters to get this to work.

Let us look at our second example,

$$\sum_k (-2)^k \binom{n}{k} \binom{2n - k}{n - r - k} = (-1)^{(n-r)/2} \binom{n}{(n-r)/2}.$$ 

To apply the Snake Oil Method, we replace $n - r$ with $m$, so the identity becomes

$$\sum_k (-2)^k \binom{n}{k} \binom{2n - k}{m - k} = (-1)^{m/2} \binom{n}{m/2}.$$
Now we can multiply the left side by $x^m$ and sum on $m$ to get

$$\sum_m \sum_k (-2)^k \binom{n}{k} \binom{2n - k}{m - k} x^m$$

$$= \sum_{j,k} (-2)^k \binom{n}{k} \binom{2n - k}{j} x^{k+j} \quad (m = k + j)$$

$$= \sum_k (-2)^k \binom{n}{k} x^k (1 + x)^{2n-k}$$

$$= (1 + x)^n ((1 + x) - 2x)^n = (1 + x)^n (1 - x)^n$$

$$= (1 - x^2)^n = \sum_m (-1)^{m/2} \binom{n}{m/2}.$$
A Lattice Path Example

We want to count lattice paths in the plane with unit steps north and east that stay below the line $y = 2x$ and that start at the point $(1, 1)$. If $P(m, n)$ is the number of such of such paths that end at $(m, n)$ then $P(m, n)$ is easily computed for $n \leq 2m$ by the recurrence

$$P(m, n) = \begin{cases} 
1, & \text{if } (m, n) = (1, 1) \\
0, & \text{if } n = 2m \text{ or } n = 2m + 1 \text{ or } n < 0 \\
P(m - 1, n) + P(m, n - 1), & \text{otherwise.}
\end{cases}$$

Note that the red numbers are the “ternary tree numbers” $\frac{1}{2n+1} \binom{3n}{n}$ with generating function $t(x)$ satisfying $t(x) = 1 + xt(x)^3$. 

\begin{tabular}{cccc}
0 & 12 & & \\
0 & 12 & & \\
0 & 3 & 12 & \\
0 & 3 & 9 & \\
0 & 1 & 3 & 6 \\
0 & 1 & 2 & 3 \\
1 & 1 & 1 & 1
\end{tabular}
We can extend the definition of $P(m, n)$ above the line $y = 2x$ so that the recurrence still holds:

\[
\begin{array}{cccccccc}
0 & 12 \\
-14 & -9 & -6 & -4 & -3 & -3 & -3 & 0 & 12 \\
5 & 3 & 2 & 1 & 0 & 0 & 3 & 12 \\
-2 & -1 & -1 & -1 & 0 & 3 & 9 \\
1 & 0 & 0 & 1 & 3 & 6 \\
-1 & 0 & 1 & 2 & 3 \\
1 & 1 & 1 & 1 & 1
\end{array}
\]

Note that the blue numbers seem to be Catalan numbers!

To prove this, we need to prove the identity,

\[
\sum_j \frac{1}{2j+1} \binom{3j}{j} \binom{n+j-1}{3j} = C_{n-1}
\]

where $C_m$ is the Catalan number $\frac{1}{m+1} \binom{2m}{m}$. 
We first apply the method of coefficients. We rewrite the sum as

\[
\frac{1}{n} \sum_j \binom{n+j-1}{j} \binom{n}{n-2j-1} = \frac{1}{n} \sum_j \frac{1}{x^j(1-x)^n y^{n-2j-1}} (1+y)^n
\]

\[
= \frac{1}{n} CT \frac{1}{(1-y^2)^n} \frac{(1+y)^n}{y^{n-1}}
\]

\[
= \frac{1}{n} CT \frac{1}{y^{n-1}(1-y)^n} = \frac{1}{n} \binom{2n-2}{n-1}.
\]

But we can also try using the Snake Oil Method. We need to evaluate

\[
\sum_{j,n} \frac{1}{2j+1} \binom{3j}{j} \binom{n+j-1}{3j} x^n.
\]
The sum on $n$ is easy to evaluate, and we get

$$\frac{x}{1-x} \sum_j \frac{1}{2j+1} \binom{3j}{j} \left( \frac{x^2}{(1-x)^3} \right)^j$$

$$= \frac{x}{1-x} t \left( \frac{x^2}{(1-x)^3} \right),$$

Putting these two results we get,

$$\frac{1}{1-x} t \left( \frac{x^2}{(1-x)^3} \right) = c(x),$$

where

$$t(x) = 1 + xt(x)^3 \text{ and } c(x) = 1 + xc(x)^2.$$