The Method of Coefficients

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A simple example

Consider the identity

$$\sum_{k=0}^{m} \binom{n}{k} \binom{3n-2k}{m-k} (-3)^k = \binom{n}{m/3}.$$

Note that in terms of hypergeometric series, this identity may be written

$$
{}_3F_2\left(\left.\begin{matrix} -n, -3n+m, -m \\ -\frac{3}{2}n, -\frac{3}{2}n+\frac{1}{2} \end{matrix} \right| \frac{3}{4}\right)
= \begin{cases} \dfrac{\left(-n+\frac{1}{3}\right)_M \left(-n+\frac{2}{3}\right)_M}{\left(\frac{1}{3}\right)_M \left(\frac{2}{3}\right)_M} & \text{if } m = 3M \\ 0 & \text{otherwise} \end{cases}
$$

(a result of George Andrews).

We can prove this identity in the following way:

We start with

$$\begin{aligned} 1 + y^3 &= (1+y)(1-y+y^2) \\ &= (1+y)\big((1+y)^2 - 3y\big) \end{aligned}$$

So

$$\begin{aligned} (1+y^3)^n &= (1+y)^n\big((1+y)^2 - 3y\big)^n \\ &= (1+y)^n \sum_k \binom{n}{k} (1+y)^{2n-2k}(-3y)^k \\ &= \sum_k \binom{n}{k} (1+y)^{3n-2k}(-3y)^k. \end{aligned}$$

Equating coefficients of y^m on both sides gives the identity

$$\sum_{k=0}^{m} \binom{n}{k}\binom{3n-2k}{m-k}(-3)^k = \binom{n}{m/3}.$$

But how could we find this proof?

We represent the binomial coefficients as coefficients of binomial expansions: $\binom{n}{k}$ is the coefficient of x^k in $(1+x)^n$ and $\binom{3n-2k}{m-k}$ is the coefficient of y^{m-k} in $(1+y)^{3n-2k}$. It is convenient to represent these coefficients as constant terms, so the sum is

$$\mathrm{CT}\sum_k \frac{(1+x)^n}{x^k}\frac{(1+y)^{3n-2k}}{y^{m-k}}(-3)^k$$

$$= \mathrm{CT}\sum_k \frac{(1+x)^n}{x^k}\left(-3\frac{y}{(1+y)^2}\right)^k \frac{(1+y)^{3n}}{y^m}$$

Now we apply the variable elimination rule: If $f(x)$ is any power series and α is independent of x then

$$\mathrm{CT}_x \sum_{k=0}^{\infty} \frac{f(x)}{x^k}\alpha^k = \sum_{k=0}^{\infty}[x^k]f(x)\alpha^k = f(\alpha).$$

We apply this rule to

$$\begin{aligned}
&\mathrm{CT} \sum_k \frac{(1+x)^n}{x^k} \left(-3\frac{y}{(1+y)^2}\right)^k \frac{(1+y)^{3n}}{y^m} \\
&= \mathrm{CT}_y \frac{(1+y)^{3n}}{y^m} \mathrm{CT}_x \sum_k \frac{(1+x)^n}{x^k} \left(-3\frac{y}{(1+y)^2}\right)^k \\
&= \mathrm{CT}_y \frac{(1+y)^{3n}}{y^m} \left(1-3\frac{y}{(1+y)^2}\right)^n \\
&= \mathrm{CT}_y \frac{(1+y)^{3n}}{y^m} \left(\frac{(1+y)^2-3y}{(1+y)^2}\right)^n \\
&= \mathrm{CT}_y \frac{(1+y)^{3n}}{y^m} \left(\frac{1-y+y^2}{(1+y)^2}\right)^n \\
&= \mathrm{CT}_y \frac{\big((1+y)(1-y+y^2)\big)^n}{y^m} \\
&= \mathrm{CT}_y \frac{(1+y^3)^n}{y^m} \\
&= \binom{n}{m/3}.
\end{aligned}$$

The Method of Coefficients

Based on G. P. Egorychev's book Интегралное представление и вычисление комбинаторных сумм

To evaluate a binomial coefficient sum:

1. First we represent the binomial coefficients as constant terms:

$$\binom{n}{k} = \mathrm{CT}\,\frac{(1+x)^n}{x^k} = \mathrm{CT}\,\frac{(1+x)^n}{x^{n-k}}$$
$$= \mathrm{CT}\,\frac{1}{x^k(1-x)^{n-k+1}}$$
$$= \mathrm{CT}\,\frac{1}{x^{n-k}(1-x)^{k+1}}$$

2. We apply the variable elimination rule

$$\mathrm{CT}\sum_{k=0}^{\infty}\frac{f(x)}{x^k}\alpha^k = \sum_{k=0}^{\infty}[x^k]f(x)\alpha^k = f(\alpha).$$

Another example

Evaluate $\sum_k (-2)^k \binom{n}{k}\binom{2n-k}{n-r-k}$.

We have

$$
\begin{aligned}
\sum_k (-2)^k \binom{n}{k}\binom{2n-k}{n-r-k} \\
&= \mathrm{CT} \sum_k (-2)^k \frac{(1+x)^n}{x^k} \frac{(1+y)^{2n-k}}{y^{n-r-k}} \\
&= \mathrm{CT} \sum_k \frac{(1+x)^n}{x^k} \left(-\frac{2y}{1+y}\right)^k \frac{(1+y)^{2n}}{y^{n-r}} \\
&= \mathrm{CT} \left(1-\frac{2y}{1+y}\right)^n \frac{(1+y)^{2n}}{y^{n-r}} \\
&= \mathrm{CT} \left(\frac{1-y}{1+y}\right)^n \frac{(1+y)^{2n}}{y^{n-r}} \\
&= \mathrm{CT} \frac{(1-y^2)^n}{y^{n-r}} \\
&= (-1)^{(n-r)/2}\binom{n}{(n-r)/2}
\end{aligned}
$$

Change of variables formulas

The simplest change of variables formula is

$$\mathrm{CT}\, f(x) = \mathrm{CT}\, f(\alpha x),$$

where α is independent of x.

More interesting is the linear change of variables formula

$$\mathrm{CT}\, f(x) = \mathrm{CT}\, \frac{1}{1+\alpha x} f\left(\frac{x}{1+\alpha x}\right)$$

To prove it, we need only consider the case $f(x) = x^k$, $k \in \mathbb{Z}$. The case $k = 0$ is clear. For $k > 0$,

$$\mathrm{CT}\, \frac{x^k}{(1+\alpha x)^{k+1}} = 0.$$

For $k = -j < 0$,

$$\mathrm{CT}\, \frac{x^k}{(1+\alpha x)^{k+1}} = \mathrm{CT}\, \frac{(1+x)^{j-1}}{x^j} = 0.$$

As a simple example of this change of variables formula, let

$$\begin{aligned} S(a,b,n,\gamma) &= \sum_k \binom{a}{k}\binom{b}{n-k}\gamma^k \\ &= \operatorname{CT} \frac{(1+\gamma x)^a (1+x)^b}{x^n} \\ &= \binom{b}{n} {}_2F_1\left(\begin{matrix} -a, & -n \\ & b-n+1 \end{matrix} \,\middle|\, \gamma\right). \end{aligned}$$

Applying the change of variables formula with $\alpha = -1$ gives

$$\begin{aligned} S(a,b,n,\gamma) &= \frac{\big(1+(\gamma-1)x\big)^a (1-x)^{n-a-b-1}}{x^n} \\ &= (-1)^n \operatorname{CT} \frac{\big(1+(1-\gamma)x\big)^a (1+x)^{n-a-b-1}}{x^n} \\ &= (-1)^n S(a, n-a-b-1, n, 1-\gamma). \end{aligned}$$

This is equivalent to a ${}_2F_1$ linear transformation.

A 2-variable change of variables formula

(Gessel and Stanton) For any Laurent series $f(x, y)$,

$$\mathrm{CT}\,\frac{1}{1-xy}f(x,y) = \mathrm{CT}\,f\left(\frac{x}{1+y}, \frac{y}{1+x}\right).$$

To prove it we may assume that $f(x, y) = x^l y^m$. The left side is 1 if $l = m \leq 0$ and 0 otherwise. The right side is

$$\mathrm{CT}\,\frac{x^l}{(1+y)^l}\frac{y^m}{(1+x)^m}.$$

This is clearly 0 if l or m is positive. If $l = -r$ and $m = -s$, with $r, s \geq 0$ then it is

$$\mathrm{CT}\,\frac{(1+x)^s}{x^r}\frac{(1+y)^r}{y^s} = \binom{s}{r}\binom{r}{s} = \delta_{r,s}.$$

As an application we'll prove the identity

$$\frac{(1+x)^a(1+y)^b}{(1-xy)^{a+b+1}} = \sum_{l,m=0}^{\infty} \binom{a+m}{l}\binom{b+l}{m} x^l y^m.$$

Applying the change of variables formula with

$$f(x,y) = \frac{(1+x)^a(1+y)^b}{x^l y^m (1-xy)^{a+b}}$$

we get

$$\begin{aligned} \mathrm{CT}\,\frac{(1+x)^a(1+y)^b}{x^l y^m (1-xy)^{a+b+1}} &= \mathrm{CT}\,\frac{(1+x)^{a+m}(1+y)^{b+l}}{x^l y^m} \\ &= \binom{a+m}{l}\binom{b+l}{m} \end{aligned}$$

General change of variables

For more general change of variables formulas, it is convenient to work with residues rather than constant terms. The residue of the Laurent series $f(z)$, denoted $\operatorname{res} f(z)$, is the coefficient of z^{-1} in $f(z)$.

We have the following change of variables formula: Let $f(z)$ be a Laurent series and let $g(y)$ be a power series, $g(y) = g_1 y + g_2 y^2 + \cdots$, where $g_1 \neq 0$. Then

$$\operatorname{res} f(z) = \operatorname{res} f(g(y))g'(y)).$$

(Note: There is a multivariable generalization, due to Jacobi, that we will not discuss.)

The key fact in the proof is that the residue of a derivative is 0, and that a Laurent series with residue 0 is a derivative. Any Laurent series may be written as $az^{-1} + F'(z)$ for some Laurent series $F(z)$, so by linearity it is enough to prove the formula for $f(z) = z^{-1}$ and $f(z) = F'(z)$.

For $f(z) = F'(z)$ we have $\operatorname{res} F'(z) = 0$ and

$$\operatorname{res} F'(g(y))g'(y) = \operatorname{res} F(g(y))' = 0.$$

For $f(z) = z^{-1}$ we have $\operatorname{res} z^{-1} = 1$ and

$$\begin{aligned}
\operatorname{res} \frac{1}{g(y)} g'(y) &= \operatorname{res} \frac{g_1 + 2g_2 y + \cdots}{g_1 y + g_2 y^2 + \cdots} \\
&= \operatorname{res} \left(\frac{1}{y} + \frac{g_2}{g_1} + \cdots \right) \\
&= 1.
\end{aligned}$$

An application of the change of variables theorem

Prove the identity

$$\sum_k \binom{a+k}{k}\binom{a+2n+1}{n-k} = 2^{2n}\binom{n+a/2}{n}.$$

As a hypergeometric series identity, this is a form of Kummer's theorem,

$${}_2F_1\left(\begin{matrix}1+a, & -n \\ & 2+a+n\end{matrix}\,\middle|\, -1\right) = \frac{(a+2)_n}{(\frac{3}{2}+\frac{a}{2})_n}.$$

The left side is

$$\begin{aligned}
&\operatorname{CT}\sum_k \frac{1}{x^k(1-x)^{a+1}}\frac{(1+y)^{a+2n+1}}{y^{n-k}} \\
&= \operatorname{CT}\frac{(1+y)^{a+2n+1}}{y^n(1-y)^{a+1}} = \operatorname{CT}\left(\frac{(1+y)^2}{y}\right)^n\frac{(1+y)^{a+1}}{(1-y)^{a+1}} \\
&= \operatorname{res}\left(\frac{(1+y)^2}{y}\right)^{n+1}\frac{(1+y)^{a-1}}{(1-y)^{a+1}}.
\end{aligned}$$

This suggests a change of variables $z = y/(1+y)^2$. The computation is easiest if we start with the right side:

$$2^{2n}\binom{n+a/2}{n} = \operatorname{res} \frac{1}{z^{n+1}(1-4z)^{a/2+1}}$$

Then we make the change of variables $z = y/(1+y)^2$, so that $1-4z = (1-y)^2/(1+y)^2$ and $dz/dy = (1-y)/(1+y)^3$.

We get

$$\begin{aligned}\operatorname{res}\left(\frac{(1+y)^2}{y}\right)^{n+1}\left(\frac{(1+y)^2}{(1-y)^2}\right)^{a/2+1}\frac{1-y}{(1+y)^3}\\ = \operatorname{res}\left(\frac{(1+y)^2}{y}\right)^{n+1}\frac{(1+y)^{a-1}}{(1-y)^{a+1}}\end{aligned}$$

A more complicated example

(Gessel and Stanton) Prove the hypergeometric series identity

$$ {}_2F_1\left(\begin{matrix} -n, \frac{1}{2} \\ 2n+\frac{3}{2} \end{matrix} \,\middle|\, \frac{1}{4}\right) = \frac{(\frac{1}{2})_n}{(2n+\frac{3}{2})_n}\left(\frac{27}{4}\right)^n $$

An equivalent binomial coefficient identity is

$$ \sum_k \binom{-\frac{1}{2}}{k}\binom{3n+\frac{1}{2}}{n-k}\left(\frac{1}{4}\right)^k = (-1)^n\left(\frac{27}{4}\right)^n\binom{-\frac{1}{2}}{n}. $$

The left side is

$$ \mathrm{CT}\sum_k \frac{(1+x)^{-1/2}}{x^k}\frac{(1+y)^{3n+1/2}}{y^{n-k}}\left(\frac{1}{4}\right)^k $$

$$ = \operatorname{res}\frac{(1+\frac{1}{4}y)^{-1/2}(1+y)^{3n+1/2}}{y^{n+1}}. $$

This suggests the change of variables $z = y/(1+y)^3$.

The right side is $\operatorname{res} \dfrac{1}{z^{n+1}\sqrt{1-\frac{27}{4}z}}$. With the change of variables $z = y/(1+y)^3$, we have $dz/dy = (1-2y)/(1+y)^4$ and

$$\sqrt{1-\tfrac{27}{4}z} = \frac{(1+\frac{1}{4}y)^{1/2}(1-2y)}{(1+y)^{3/2}},$$

so

$$\begin{aligned}
&\operatorname{res} \frac{1}{z^{n+1}\sqrt{1-\frac{27}{4}z}} \\
&\quad = \operatorname{res} \frac{(1+y)^{3n+3}}{y^{n+1}} \cdot \frac{(1+y)^{3/2}}{(1+\frac{1}{4}y)^{1/2}(1-2y)} \cdot \frac{1-2y}{(1+y)^4} \\
&\quad = \operatorname{res} \frac{(1+\frac{1}{4}y)^{-1/2}(1+y)^{3n+1/2}}{y^{n+1}}.
\end{aligned}$$

Comparison with Wilf's Snake Oil Method

To evaluate a sum $S_n = \sum_k S_{n,k}$, we find the generating function

$$\sum_n S_n x^n = \sum_k \sum_n S_{n,k} x^n.$$

Sometimes we need to adjust the parameters to get this to work.

Let us look at our second example,

$$\sum_k (-2)^k \binom{n}{k} \binom{2n-k}{n-r-k} = (-1)^{(n-r)/2} \binom{n}{(n-r)/2}.$$

To apply the Snake Oil Method, we replace $n-r$ with m, so the identity becomes

$$\sum_k (-2)^k \binom{n}{k} \binom{2n-k}{m-k} = (-1)^{m/2} \binom{n}{m/2}.$$

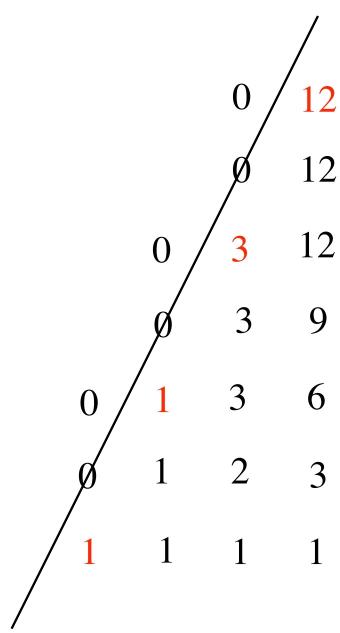
Now we can multiply the left side by x^m and sum on m to get

$$\begin{aligned}\sum_m \sum_k (-2)^k \binom{n}{k}\binom{2n-k}{m-k} x^m \\ &= \sum_{j,k} (-2)^k \binom{n}{k}\binom{2n-k}{j} x^{k+j} \qquad (m=k+j) \\ &= \sum_k (-2)^k \binom{n}{k} x^k (1+x)^{2n-k} \\ &= (1+x)^n \big((1+x)-2x\big)^n = (1+x)^n(1-x)^n \\ &= (1-x^2)^n = \sum_m (-1)^{m/2}\binom{n}{m/2}.\end{aligned}$$

A Lattice Path Example

We want to count lattice paths in the plane with unit steps north and east that stay below the line $y = 2x$ and that start at the point $(1, 1)$. If $P(m, n)$ is the number of such of such paths that end at (m, n) then $P(m, n)$ is easily computed for $n \leq 2m$ by the recurrence

$$P(m,n) = \begin{cases} 1, & \text{if } (m,n) = (1,1) \\ 0, & \text{if } n = 2m \text{ or } n = 2m+1 \text{ or } n < 0 \\ P(m-1,n) + P(m,n-1), & \text{otherwise.} \end{cases}$$



Note that the red numbers are the "ternary tree numbers" $\frac{1}{2n+1}\binom{3n}{n}$ with generating function $t(x)$ satisfying $t(x) = 1 + xt(x)^3$.

We can extend the definition of $P(m, n)$ above the line $y = 2x$ so that the recurrence still holds:

							0	12
-14	-9	-6	-4	-3	-3	-3	0	12
	5	3	2	1	0	0	3	12
		-2	-1	-1	-1	0	3	9
			1	0	0	1	3	6
				-1	0	1	2	3
					1	1	1	1

Note that the blue numbers seem to be Catalan numbers!

To prove this, we need to prove the identity,

$$\sum_j \frac{1}{2j+1}\binom{3j}{j}\binom{n+j-1}{3j} = C_{n-1}$$

where C_m is the Catalan number $\frac{1}{m+1}\binom{2m}{m}$.

We first apply the method of coefficients. We rewrite the sum as

$$\frac{1}{n}\sum_j \binom{n+j-1}{j}\binom{n}{n-2j-1}$$

$$= \frac{1}{n}\,\mathrm{CT}\sum_j \frac{1}{x^j(1-x)^n}\frac{(1+y)^n}{y^{n-2j-1}}$$

$$= \frac{1}{n}\,\mathrm{CT}\,\frac{1}{(1-y^2)^n}\frac{(1+y)^n}{y^{n-1}}$$

$$= \frac{1}{n}\,\mathrm{CT}\,\frac{1}{y^{n-1}(1-y)^n} = \frac{1}{n}\binom{2n-2}{n-1}.$$

But we can also try using the Snake Oil Method. We need to evaluate

$$\sum_{j,n} \frac{1}{2j+1}\binom{3j}{j}\binom{n+j-1}{3j}x^n.$$

The sum on n is easy to evaluate, and we get

$$\frac{x}{1-x}\sum_j \frac{1}{2j+1}\binom{3j}{j}\left(\frac{x^2}{(1-x)^3}\right)^j$$
$$= \frac{x}{1-x} t\left(\frac{x^2}{(1-x)^3}\right),$$

Putting these two results we get,

$$\frac{1}{1-x} t\left(\frac{x^2}{(1-x)^3}\right) = c(x),$$

where

$$t(x) = 1 + xt(x)^3 \text{ and } c(x) = 1 + xc(x)^2.$$