Quotients of the Malvenuto-Reutenauer algebra and permutation enumeration

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Exponential generating functions and permutations

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We multiply exponential generating functions by

$$\frac{x^m}{m!} \cdot \frac{x^n}{n!} = \binom{m+n}{m} \frac{x^{m+n}}{(m+n)!}$$

and distributivity so

$$\sum_{i=0}^{\infty} u_i \frac{x^i}{i!} \cdot \sum_{j=0}^{\infty} v_j \frac{x^j}{j!} = \sum_{n=0}^{\infty} \left( \sum_{i+j=n} \binom{n}{i} u_i v_j \right) \frac{x^n}{n!},$$

where

$$w_n = \sum_{i+j=n} \binom{n}{i} u_i v_j.$$
There is a well-known way of interpreting exponential generating functions as counting “labeled objects”. Here is an informal description.

Suppose that for each $i$, $u_i$ is the number of objects that we are interested in with the label set $[i] = \{1, 2, \ldots, i\}$. For example, we might want to count directed graphs in which every vertex is reachable from vertex 1, and then $u_3 = 32$.

Then for any totally ordered set $S$ of size $i$ (e.g., a set of integers), we can construct a set of $w_i$ objects with label set $S$ by taking each of the objects with label set $[i]$, and replacing each $j$ with the $j$th smallest element of $S$. 
For example if $S = \{3, 6, 8\}$ we might have
Suppose also that \( v_j \) counts some kind of labeled object with label set \([j]\). Then the interpretation of the term \( \binom{i+j}{i} u_i v_j \) that appears in the coefficient of \( x^{i+j} / (i + j)! \) in

\[
\sum_{i=0}^{\infty} u_i \frac{x^i}{i!} \cdot \sum_{j=0}^{\infty} v_j \frac{x^j}{j!}
\]

is that we take an object \( U \) counted by \( u_i \) and an object \( V \) counted by \( v_j \). We take the set \([i+j]\) and partition it into two sets \( A \) and \( B \) of sizes \( i \) and \( j \), and we replace the labels in \( U \) with the elements of \( A \), and replace the labels in \( V \) with the elements of \( B \), preserving their order, obtaining an ordered pair \((U', V')\).
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We have a similar interpretation to multiplying more than two exponential generating functions.
An example: Eulerian polynomials

Let’s take \( u(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \), so all coefficients are 1. The coefficient of \( \frac{x^n}{n!} \) in \( e^x \) is the number of increasing permutations of \([n]\), which is just 1.
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Therefore the coefficient of $x^n/n!$ in $u(x)^2 = (e^x)^2$ is the number of permutations of $[n]$ split into two increasing permutations:

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$|12345|$

Similarly $(e^x)^k$ counts permutations of $[n]$ split into $k$ increasing permutations:

$134|25|6|$

I call this a barred permutation.
Let us weight a barred permutation with \( k \) bars (and thus \( k \) parts) by \( t^k \). So the exponential generating function for all barred permutations is

\[
\sum_{k=0}^{\infty} t^k (e^x)^k = \frac{1}{1 - te^x}.
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We want to count barred permutations in another way. We define the descent set of a permutation $\pi = a_1 a_2 \cdots a_n$ to be the set $D(\pi) = \{ i : a_i > a_{i+1} \}$ and we define the descent number to be $\text{des}(\pi) = |D(\pi)|$. 
For any barred permutation we can get an underlying permutation by removing the bars:

\[ 134|25||6| \rightarrow 134256 \]

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For any barred permutation we can get an underlying permutation by removing the bars:

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What are the barred permutations corresponding to 134256? Since the numbers in each part must increase, we need a bar between 4 and 2, and we need a bar at the end:

\[ 134\overline{256} \]

After we’ve inserted these required bars, we can put an arbitrary number of bars in each of the other 7 spaces (between the numbers and at the beginning and end). So the contribution from the permutation 134256 is

\[
t^2 (1 + t + t^2 + \cdots)^7 = \frac{t^2}{(1 - t)^7}.
\]
In general, the contribution from a permutation $\pi$ of length $n$ is

$$\frac{t^{\text{des}(\pi)+1}}{(1 - t)^{n+1}}.$$
In general, the contribution from a permutation $\pi$ of length $n$ is

$$t^{\text{des}(\pi)+1} \over (1 - t)^{n+1}.$$ 

So we have the identity

$$1 \over 1 - t e^x = 1 + \sum_{n=1} A_n(t) \over (1 - t)^{n+1} n! \over n!,$$

where the Eulerian polynomials $A_n(t)$ are defined by

$$A_n(t) = \sum_{\pi \in S_n} t^{\text{des}(\pi)+1},$$

and $S_n$ is the set of permutations of $[n]$.

$$A_1(t) = t, A_2(t) = t + t^2, A_3(t) = t + 4t^2 + t^3, \ldots$$
We can get more information from this analysis by looking more closely at the operation on permutations that exponential generating functions reflect.

The standardization of a sequence of numbers is obtained by replacing the smallest entry by 1, the next smallest by 2, and so on. So $s_t(436) = 213$. (For now, we assume that the entries are distinct.)

Then permutations $\pi = a_1 \cdots a_m \in S_m$ and $\sigma = b_1 \cdots b_n \in S_n$ combine to give all permutations $c_1 \cdots c_{m+n}$ of $[m+n]$ such that $s_t(c_1 \cdots c_m) = \pi$ and $s_t(c_m+1 \cdots c_{m+n}) = \sigma$. We define the product $\pi \cdot \sigma$ to be the sum of all these permutations (in the vector space spanned by permutations).

For example $12 \cdot 1 = 123 + 132 + 231$. (For simplicity, I’ll also use $\pi \cdot \sigma$ to denote the set of these permutations.)
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Then multiplication of exponential generating functions, when applied to permutations can be stated in the following way: For a permutation $\pi$, let

$$\Phi(\pi) = \frac{x^{\vert \pi \vert}}{|\pi|!},$$

where $\vert \pi \vert$ is the length of $\pi$, and extend $\Phi$ by linearity.
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where $|\pi|$ is the length of $\pi$, and extend $\Phi$ by linearity.

Then for any two permutations $\pi$ and $\sigma$, $\Phi(\pi \cdot \sigma) = \Phi(\pi)\Phi(\sigma)$; i.e., $\Phi$ is a homomorphism from $\text{MR}$ to the algebra of formal power series in $x$. 
The computation that we did for the Eulerian polynomials works in exactly the same way in MR, and it gives the following formula:

\[
\left(1 - t \sum_{n=0}^{\infty} 12 \cdots n\right)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{\sum_{\pi \in S_n} t^{\text{des}(\pi) + 1} \pi}{(1 - t)^{n+1}},
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Our formula for the exponential generating function for Eulerian polynomials is obtained by applying the homomorphism \( \Phi \) to this formula.

But there are other homomorphisms that we can apply that give us more information.
Inversions of permutations

An *inversion* of the permutation \( \pi = a_1 \cdots a_n \) is a pair \((i, j)\) with \( i < j \) such that \( a_i > a_j \). Let \( \text{inv}(\pi) \) be the number of inversions of the permutation \( \pi \).
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An inversion of the permutation $\pi = a_1 \cdots a_n$ is a pair $(i, j)$ with $i < j$ such that $a_i > a_j$. Let $\text{inv}(\pi)$ be the number of inversions of the permutation $\pi$.

Inversions are compatible with the Malvenuto-Reutenauer algebra in the following way:
We define the $q$-factorial $n!_q$ to be

$$1 \cdot (1 + q) \cdot (1 + q + q^2) \cdots (1 + q + \cdots q^{n-1}).$$

There is also a variation

$$(q)_n = (1 - q)^n n!_q = (1 - q)(1 - q^2) \cdots (1 - q^n).$$

We define the $q$-binomial coefficient $\left[ \begin{array}{c} m \\ n \end{array} \right]$ by

$$\left[ \begin{array}{c} m \\ n \end{array} \right] = \frac{m!_q}{n!_q(m - n)!_q} = \frac{(q)_m}{(q)_n(q)_{m-n}}.$$
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**Theorem.** For any two permutations $\pi$ and $\sigma$ of lengths $m$ and $n$ we have

$$\sum_{\tau \in \pi \cdot \sigma} q^{\text{inv}(\tau)} = q^{\text{inv}(\pi) + \text{inv}(\sigma)} \left[ \begin{array}{c} m + n \\ m \end{array} \right].$$
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Eulerian generating functions multiply just like exponential generating functions but with $q$-binomial coefficients instead of binomial coefficients, i.e.,

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$$

Then the Eulerian generating function for $\pi \cdot \sigma$, where $|\pi| = m$ and $|\sigma| = n$, and inversions are weighted by $q$, is

$$
q^{\text{inv}(\pi)+\text{inv}(\sigma)} \begin{bmatrix} m + n \\ m \end{bmatrix} \frac{x^{m+n}}{(m+n)!_q} = q^{\text{inv}(\pi)} \frac{x^m}{m!_q} \cdot q^{\text{inv}(\sigma)} \frac{x^n}{n!_q}.
$$
In other words the linear map from MR to the ring of Eulerian generating functions given by

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In other words the linear map from \( \text{MR} \) to the ring of Eulerian generating functions given by

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is a homomorphism.

So if we define the \( q \)-Eulerian polynomials \( A_n(t, q) \) by

\[
A_n(t, q) = \sum_{\pi \in S_n} t^\text{des}(\pi) + 1 q^{\text{inv}(\pi)}
\]

then

\[
\frac{1}{1 - te_q(x)} = 1 + \sum_{n=1} \frac{A_n(t, q)}{(1 - t)^{n+1}} \frac{x^n}{n!} q,
\]

where

\[
e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} q.
\]
Are there any other permutation statistics with the same property?
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A trivial example is the number of non-inversions.

A more interesting example is the inverse major index (the major index of $\pi - 1$).

The descent set of a permutation $\pi = a_1 \cdots a_n$ is the set $\{i : a_i > a_{i+1}\}$. The major index $\text{maj}(\pi)$ is the sum of the descents.

So the same $q$-Eulerian polynomials $A_n(t, q)$ count permutations by descents and inverse major index.
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So the same $q$-Eulerian polynomials $A_n(t, q)$ count permutations by descents and inverse major index.
Up-down permutations

Another example of a simple exponential generating function formula for permutations is

\[
\left( \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \right)^{-1} = \sum_{n=0}^{\infty} E_{2n} \frac{x^{2n}}{(2n)!},
\]

where \( E_{2n} \) is the number of up-down permutations \( a_1 \cdots a_{2n} \) of \([n]\) satisfying \( a_1 < a_2 > a_3 < a_4 > \cdots < a_{2n} \).
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where \( E_{2n} \) is the number of up-down permutations \( a_1 \cdots a_{2n} \) of \([n]\) satisfying \( a_1 < a_2 > a_3 < a_4 > \cdots < a_{2n} \). It’s not hard to show that there is an analogue of this formula in the Malvenuto-Reutenauer algebra:

\[
\left( \sum_{n=0}^{\infty} (-1)^n 12 \cdots (2n) \right)^{-1} = \sum_{n=0}^{\infty} \sum_{\pi \in \mathcal{U}_{2n}} \pi,
\]

where \( \mathcal{U}_{2n} \) is the set of up-down permutations of \([2n]\).
So

\[
\left( \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!q} \right)^{-1} = \sum_{n=0}^{\infty} E_{2n}(q) \frac{x^{2n}}{(2n)!},
\]

where \( E_{2n}(q) \) counts up-down permutations of \([2n]\) by inversions, or by inverse major index.
MR-compatible permutation statistics

We might also look for statistics with a more general property. We will call a permutation statistic \( \text{stat} \) (i.e., any function defined on the set of permutations) \( MR\-compatible \) if it has the following property:

Any permutation statistic gives an equivalence relation on permutations by \( \pi \equiv \sigma \) if \( \text{stat}(\pi) = \text{stat}(\sigma) \) and \( |\pi| = |\sigma| \).

Equivalently, we can define an equivalence relation on permutations to be compatible if it comes from a compatible permutation statistic. (Different permutation statistics could give the same equivalence relation, but in practice the permutation statistics are easier to describe.)
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For any two permutations \( \pi \) and \( \sigma \), the multiset \( \{ \text{stat}(\tau) : \tau \in \pi \cdot \sigma \} \) depends only on \( \text{stat}(\pi) \), \( \text{stat}(\sigma) \), \( |\pi| \), and \( |\sigma| \).
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Unfortunately I have nothing to say about this question. I will discuss only some MR-compatible statistics related to the descents of permutations.
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For any permutation $\pi \in \text{MR}$, let $W(\pi)$ be the sum of all words whose standardization is $\pi$, in the algebra generated by the free monoid on $\{1, 2, 3 \ldots \}$. 
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Theorem.

\[ W(\pi) W(\sigma) = \sum_{\tau \in \pi \cdot \sigma} W(\tau) \]

Thus MR is isomorphic to the algebra generated by the \( W(\pi) \) (with \( \pi \mapsto W(\pi) \)).
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Now let’s map letters to commuting variables: \( i \mapsto x_i \). The image of MR is called the algebra of *quasi-symmetric functions*, denoted \( \text{QSym} \).

**Example:** What are the words whose standardization is \( 312 \)? They are the words \( i_3i_1i_2 \) such that \( i_1 \leq i_2 < i_3 \). So the image of \( W(312) \) is

\[
\sum_{i_1 \leq i_2 < i_3} x_{i_1}x_{i_2}x_{i_3}
\]

For any permutation \( \pi \), the strict inequalities occur when \( i + 1 \) is to the left of \( i \), and these are the descents of \( \pi^{-1} \).
It’s convenient to work with descents of permutations rather than descents of their inverses, so we will denote the commutative image of $W(\pi)$ by $F_{\pi^{-1}}$. So $F_{\pi}$ depends only on the descent set of $\pi$ (and the length of $\pi$).
It’s convenient to work with descents of permutations rather than descents of their inverses, so we will denote the commutative image of $W(\pi)$ by $F_{\pi^{-1}}$. So $F_\pi$ depends only on the descent set of $\pi$ (and the length of $\pi$).

For example, if $\pi$ has length 8 and descent set $\{3, 5\}$ then

$$F_\pi = \sum x_{i_1} x_{i_2} \cdots x_{i_8}$$

where

$$i_1 \leq i_2 \leq i_3 < i_4 \leq i_5 < i_6 \leq i_7 \leq i_8;$$

the strict inequalities come after the positions of the descents.
It’s convenient to work with descents of permutations rather than descents of their inverses, so we will denote the commutative image of $W(\pi)$ by $F_{\pi^{-1}}$. So $F_{\pi}$ depends only on the descent set of $\pi$ (and the length of $\pi$).

For example, if $\pi$ has length 8 and descent set $\{3, 5\}$ then

$$F_{\pi} = \sum x_{i_1} x_{i_2} \cdots x_{i_8}$$

where

$$i_1 \leq i_2 \leq i_3 < i_4 \leq i_5 < i_6 \leq i_7 \leq i_8;$$

the strict inequalities come after the positions of the descents.

The distinct $F_{\pi}$ are linearly independent, so if we know the quasi-symmetric generating function for a set of permutations, we know how many permutations are in the set with each descent set.
In working with descent sets rather than inverse descent sets, it’s convenient to look at the Malvenuto-Reutenauer algebra a little differently. We define another operation on permutations by

$$\pi \ast \sigma = \sum_{\tau^{-1} \in \pi^{-1} \cdot \sigma^{-1}} \tau$$

The span of $\bigcup_n \mathfrak{S}_n$ with the operation $\ast$ is isomorphic to $\text{MR}$, since it is just a relabeling of the basis elements. The operation $\ast$ can be described most easily in terms of shuffles.
Shuffles

Let us now redefine a permutation to be any sequence of distinct integers. Two permutations are \textit{disjoint} if they have no elements in common.
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If $\pi$ and $\sigma$ are disjoint permutations, let $\pi \uplus \sigma$ be the set of all shuffles of $\pi$ and $\sigma$.

Example:

$$\pi = 1 \ 4 \ 2 \quad \sigma = 3 \ 7 \ 5 \ 8$$
Shuffles

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Example:

\[
\begin{align*}
\pi &= 1 \ 4 \ 2 \\
\sigma &= 3 \ 7 \ 5 \ 8
\end{align*}
\]

\[
\begin{array}{cccc}
1 & 4 & & 2 \\
\ & 3 & 7 & 5 & 8
\end{array}
\]

\[
\begin{array}{cccccc}
1 & 4 & 3 & 7 & 5 & 2 & 8
\end{array}
\]
To compute $\pi * \sigma$, we first add $|\pi|$ to each entry of $\sigma$, obtaining a permutation $\sigma'$ of $\{|\pi| + 1, \ldots, |\pi| + |\sigma|\}$. Then we sum the elements of $\pi \upharpoonright \sigma'$. For example, if $\pi = 12$ and $\sigma = 1$ then $\sigma' = 3$ and $\pi \upharpoonright \sigma' = 12 \upharpoonright 3 = \{123, 132, 312\}$, so $\pi * \sigma = 123 + 132 + 312$. 
We have seen that \( \pi \ast \sigma \) depends only on the descent sets of \( \pi \) and \( \sigma \). However a stronger property holds.

**Theorem.** (Stanley). Let \( \pi \) and \( \sigma \) be disjoint permutations. Then the number of permutations in \( \pi \sqcup \sigma \) with descent set \( A \) depends only on \( D(\pi) \), \( D(\sigma) \), and \( A \).
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Note that we do not require that all the entries of $\sigma$ be greater than the entries of $\pi$, only that $\pi$ and $\sigma$ be disjoint. (More generally $\pi$ and $\sigma$ could be words with repetitions, as long as they are disjoint.)
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Note that we do not require that all the entries of \( \sigma \) be greater than the entries of \( \pi \), only that \( \pi \) and \( \sigma \) be disjoint. (More generally \( \pi \) and \( \sigma \) could be words with repetitions, as long as they are disjoint.)
Shuffle-compatible permutation statistics

With our new definition of permutations, let us call two permutations equivalent if they have the same standardization:

\[132 \equiv 253 \equiv 174.\]

A permutation statistic is a function defined on permutations that takes the same value on equivalent permutations. For example if \( f \) is a permutation statistic then

\[f(132) = f(253) = f(174).\]

A permutation statistic \( \text{stat} \) is shuffle-compatible if it has the property that the multiset \( \{ \text{stat}(\tau) : \tau \in \pi \sqcup \sigma \} \) depends only on \( \text{stat}(\pi) \) and \( \text{stat}(\sigma) \) (and the lengths of \( \pi \) and \( \sigma \)).
A permutation statistic is a **descent statistic** if it depends only on the descent set. Some important descent statistics:

- the descent set $D(\pi)$
- the descent number $\text{des}(\pi) = \# D(\pi)$
- the major index $\text{maj}(\pi) = \sum_{i \in D(\pi)} i$
- the comajor index $\text{comaj}(\pi) = \sum_{i \in D(\pi)} (n - i)$, where $|\pi| = n$.
- the peak set $P(\pi) = \{ i : \pi(i - 1) < \pi(i) > \pi(i + 1) \}$
- the peak number $\text{pk}(\pi) = \# P(\pi)$
- the ordered pair $(\text{des}, \text{maj})$
- the ordered pair $(\text{pk}, \text{des})$

An important permutation statistic that is not a descent statistic is the number of inversions.
All of the above descent statistics are shuffle-compatible. This was proved by Richard Stanley, using P-partitions for des, maj, and (des, maj), and by John Stembridge, using enriched P-partitions, for the peak set and the peak number.
Question: Can we describe all shuffle-compatible permutation statistics and their algebras?
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To make the problem a little easier, we consider only shuffle-compatible descent statistics. (I don’t know of any other kind of shuffle compatible statistics.)
Note that for any shuffle-compatible permutation statistic $\text{stat}$ we get an algebra $A_{\text{stat}}$:

First we define an equivalence relation $\equiv_{\text{stat}}$ on permutations by $\pi \equiv_{\text{stat}} \sigma$ if $\pi$ and $\sigma$ have the same length and $\text{stat}(\pi) = \text{stat}(\sigma)$. We define $A_{\text{stat}}$ by taking as a basis all equivalence classes of permutations, with multiplication defined as follows: To multiply two equivalence classes, choose disjoint representatives $\pi$ and $\sigma$ of the equivalence classes $[\pi]$ and $[\sigma]$ and define their product to be

$$[\pi][\sigma] = \sum_{\tau \in \pi \shuffle \sigma} [\tau].$$

By the definition of a shuffle-compatible permutation statistic, this product is well-defined.
So $\mathcal{A}_D$ is $\text{QSym}$ and each of the other algebras is a quotient of $\text{QSym}$. We would like to describe them “explicitly” if possible. In each case we can describe this quotient in terms of evaluations of the variables (though it’s not always straightforward, especially when peaks are involved).
Given a quotient algebra of $\texttt{QSym}$, in order for it to be associated with a shuffle-compatible equivalence relation, the distinct images of the $F_\pi$ must be linearly independent (over the rationals).
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As a first example, we’ll look at the comajor index. If we set $x_i = q^i z$ for $i \geq 0$ in $F_\pi$, where $|\pi| = n$, we get

$$q^{\text{comaj}(\pi)} \frac{z^n}{(q)_n}$$

So we get the algebra of Eulerian generating functions directly as a quotient of $\mathbb{QSym}$, and the quotient algebra $\mathcal{A}_{\text{comaj}}$ is essentially the algebra of Eulerian generating functions. (We could have gotten $\text{maj}$ instead of $\text{comaj}$ by reversing left-to-right.)
To get the descent number we need to do something a little more complicated.
If we evaluate $F_\pi$ at

$$x_0 = x_1 = \cdots = x_\lambda = z$$
$$x_{\lambda+1} = x_{\lambda+2} = \cdots = 0$$

we get $\left(\begin{array}{c}
\lambda + n - \text{des}(\pi) \\
n
\end{array}\right) z^n$. We think of this as a function of $\lambda$. 
To get the descent number we need to do something a little more complicated. If we evaluate $F_{\pi}$ at

$$x_0 = x_1 = \cdots = x_\lambda = z$$
$$x_{\lambda+1} = x_{\lambda+2} = \cdots = 0$$

we get $(\lambda+n-\text{des}(\pi))_n z^n$. We think of this as a function of $\lambda$.

As polynomials in $\lambda$, the polynomials $(\lambda+n-i)_n$ for $i = 0, \ldots, n-1$ are linearly independent, and they form a basis for the polynomials of degree at most $n$ that vanish at $-1$ (for $n > 0$). So the algebra $A_{\text{des}}$ is isomorphic to the algebra spanned by $p(\lambda)z^n$, where $p(\lambda)$ is a polynomial of degree at most $n$ that vanishes at $-1$. 
Another way to look at this algebra is to take the generating functions on $\lambda$ of such polynomials:

$$\sum_{\lambda=0}^{\infty} \binom{\lambda + n - i}{n} t^{\lambda} = \frac{t^i}{(1 - t)^{n+1}}$$

So instead of multiplying polynomials in $\lambda$, we can take Hadamard products of rational functions of $t$:

$$\sum_{m} a_{m} t^{m} \ast \sum_{n} b_{n} t^{n} = \sum_{n} a_{n} b_{n} t^{n}$$
Similarly for the ordered pair \((\text{des}, \text{comaj})\) we take the homomorphism from \(\text{QSym}\) in which we evaluate quasi-symmetric functions at

\[
    x_i \mapsto \begin{cases} 
        q^i z & \text{for } 0 \leq i \leq \lambda \\
        0 & \text{for } i > \lambda 
    \end{cases}
\]

Here the image of \(F_{\pi}\), where \(|\pi| = n\), is

\[
    q^{\text{comaj}(\pi)} \left[ \begin{array}{c} \lambda + n - \text{des}(\pi) \\ n \end{array} \right] z^n.
\]

If we apply this homomorphism to our earlier example of Eulerian polynomials, we count permutations \(\pi\) according to the descent number of \(\pi\) and the descent number and comajor index of \(\pi^{-1}\),
The peak number algebra

We can describe the peak number algebra in a similar, but somewhat more complicated way. Instead of the rational functions $t^k/(1 - t)^{n+1}$, we have the rational functions

$$P_{n,j}(t) = 2^{2j-1} \frac{t^j(1 + t)^{n-2j+1}}{(1 - t)^{n+1}}$$

$$= \frac{1}{2} \frac{(1 + t)^{n+1}}{(1 - t)^{n+1}} \left(\frac{4t}{(1 + t)^2}\right)^j,$$

for $1 \leq j \leq \lfloor (n + 1)/2 \rfloor$, corresponding to permutation of length $n$ with $j - 1$ peaks.
Then coefficients in the expansion of $P_{m,i}(t) \ast P_{n,j}(t)$ as a linear combination of the $P_{m+n,k}(t)$ are the structure constants for the peak number algebra (John Stembridge).
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More precisely, suppose that $|\pi| = m$, $|\sigma| = n$, $pk(\pi) = i$, and $pk(\sigma) = j$. Then the number of permutations in $\pi \uplus \sigma$ with $k$ peaks is the coefficient $c_k$ in the expansion

$$P_{m,i}(t) \ast P_{n,j}(t) = \sum_k c_k P_{m+n,k}(t).$$
Peaks and descents

It is possible to generalize both the descent number algebra and the peak number algebra:
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The ordered pair \((\text{des}, \text{pk})\) is shuffle-compatible.
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The ordered pair \((\text{des}, \text{pk})\) is shuffle-compatible.

There exists a permutation of length \(n > 0\) with \(j - 1\) peaks and \(k - 1\) descents if and only if \(1 \leq j \leq \frac{(n + 1)}{2}\) and \(j \leq k \leq n + 1 - j\). For such \(n, j, k\) let

\[
\text{PD}_{n,j,k}(t, y) = \frac{t^j(y + t)^{k-j}(1 + yt)^{n-j-k+1}(1 + y)^{2j-1}}{(1 - t)^{n+1}}
\]

Then the structure constants for the \((\text{des}, \text{pk})\) algebra are the same as the structure constants for the rational functions \(\text{PD}_{n,j,k}\) under the operation of Hadamard product in \(t\).
Let’s see how this result specializes to the known results for descents and peaks separately.

\[ PD_{n,j,k}(t, y) = \frac{t^j(y + t)^k - j(1 + yt)^{n-j-k+1}(1 + y)^{2j-1}}{(1 - t)^{n+1}} \]

If we set \( y = 0 \), we get

\[ PD_{n,j,k}(t, 0) = \frac{t^k}{(1 - t)^{n+1}} \]

and if we set \( y = 1 \), we get

\[ PD_{n,j,k}(t, 1) = \frac{t^j(1 + t)^{n-2j+1}2^{2j-1}}{(1 - t)^{n+1}} = P_{n,j}(t). \]
The simplest example:

\[ PD_{1,1,1} = \frac{t(1 + y)}{(1 - t)^2} \]

\[ PD_{2,1,1} = \frac{t(1 + y)(1 + yt)}{(1 - t)^3}, \quad PD_{2,1,2} = \frac{t(1 + y)(y + t)}{(1 - t)^3} \]
The simplest example:

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\begin{align*}
PD_{1,1,1} &= \frac{t(1 + y)}{(1 - t)^2} \\
PD_{2,1,1} &= \frac{t(1 + y)(1 + yt)}{(1 - t)^3}, \\
PD_{2,1,2} &= \frac{t(1 + y)(y + t)}{(1 - t)^3}
\end{align*}
\]

\[
PD_{1,1,1} \ast PD_{1,1,1} = (1 + y)^2 \left( \sum_{\lambda=0}^{\infty} \lambda t^\lambda \right) \ast \left( \sum_{\lambda=0}^{\infty} \lambda t^\lambda \right)
\]

\[
= (1 + y)^2 \left( \sum_{\lambda=0}^{\infty} \lambda^2 t^\lambda \right) = (1 + y)^2 \frac{t + t^2}{(1 - t)^3}
\]

\[
= t(1 + y) \frac{(1 + y)(1 + t)}{(1 - t)^3}
\]

\[
= t(1 + y) \frac{(1 + yt) + (y + t)}{(1 - t)^3}
\]

\[
= PD_{2,1,1} + PD_{2,1,2}
\]
How could we prove this? We can also do this by evaluating quasi-symmetric functions, but we need to use a different representation for them, based on John Stembridge’s enriched P-partitions (“superized quasi-symmetric functions”).
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My original derivation used noncommutative symmetric functions.
Signed permutations and type B quasi-symmetric functions

A signed permutation is a permutation in which each entry has a plus or minus sign:

\[ -2 \ 1 \ 6 \ -4 \ 5 \]

This looks much better if we write \( \bar{a} \) for \( -a \):

\[ \bar{2} \ 1 \ 6 \ \bar{4} \ 5 \]

We define the descent set of a signed permutation in the same way as for an ordinary permutation (with the usual ordering \( \cdots -2 < -1 < 0 < 1 < 2 \cdots \) except that if the permutation starts with a negative number we also count 0 as a descent. (Thus we might think of the permutation as starting with a 0.) So the descent set of \( \bar{2} \ 1 \ 6 \ \bar{4} \ 5 \) is \( \{0, 3\} \).
We can represent descent sets for signed permutations of length $n$ by generalized compositions of $n$ in which we allow the first part to be 0, corresponding to an initial descent. So the descent composition of $\bar{2} 1 6 \bar{4} 5$ is $(0, 3, 2)$. Just as with ordinary permutations, the descent set is shuffle compatible: the multiset $\{D(\tau) : \tau \in \pi \sigma\}$ depends only on $D(\pi)$ and $D(\sigma)$ (and the lengths of $\pi$ and $\sigma$), and the algebra of descent sets is isomorphic to the algebra of "type B quasi-symmetric functions". These were studied by Chak-On Chow, but a closely related algebra, which keeps track of signs, was studied earlier by Stéphane Poirier (related to the Mantaci-Reutenauer algebra).
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These were studied by Chak-On Chow, but a closely related algebra, which keeps track of signs, was studied earlier by Stéphane Poirier (related to the Mantaci-Reutenauer algebra).
These are very much like ordinary quasi-symmetric functions, but the variable $x_0$ that behaves somewhat differently from the other variables. For example

$$F_{(3,2,3)} = \sum x_{i_1} x_{i_2} \cdots x_{i_8}$$

where

$$0 \leq i_1 \leq i_2 \leq i_3 < i_4 \leq i_5 < i_6 \leq i_7 \leq i_8$$

and

$$F_{(0,3,2,3)} = \sum x_{i_1} x_{i_2} \cdots x_{i_8}$$

where

$$0 < i_1 \leq i_2 \leq i_3 < i_4 \leq i_5 < i_6 \leq i_7 \leq i_8$$
Just as with ordinary permutations, we have for disjoint signed permutations $\pi$ and $\sigma$

$$F_{\pi} F_{\sigma} = \sum_{\tau \in \pi \cup \sigma} F_{\tau}.$$  

Similarly we get a quotient algebra corresponding to the number of descents in which the image of $F_{\pi}$, where $|\pi| = n$, is the polynomial in $\lambda$

$$\binom{\lambda + n - \text{des}(\pi)}{n} z^n$$

but here $\text{des}(\pi)$ can range from 0 to $n$. 
There is an interesting variation of the descent number of a signed permutation. Adin, Brenti, and Roichman defined the flag descent number $f_{\text{des}}(\pi)$ by

$$f_{\text{des}}(\pi) = \begin{cases} 
2 \text{ des}(\pi), & \text{if } \pi(1) > 0 \\
2 \text{ des}(\pi) - 1, & \text{if } \pi(1) < 0
\end{cases}$$

In other words, a descent at the beginning of a signed permutation contributes 1 to the flag descent number, but a descent anywhere else contributes 2.
The flag descent number is shuffle-compatible, and in the corresponding quotient algebra the image of $F_\pi$, where $|\pi| = n$, is the quasi-polynomial function of $\lambda$

$$\left(n + \left\lfloor \frac{1}{2} (\lambda - \text{fdes}(\pi)) \right\rfloor \right) z^n$$

Equivalently, the image of $F_\pi$ is

$$\frac{t^{\text{fdes}(\pi)}}{(1 - t)(1 - t^2)^n} z^n$$

with the operation of Hadamard product in $t$. 
We can transform the problem of counting signed permutations by descents into a problem of counting ordinary permutations.
We can transform the problem of counting signed permutations by descents into a problem of counting ordinary permutations.

Suppose we want to count permutations of the set \([-3, -2, 1]\) by descents. We think of these permutations as prefixed with a 0, so if we start with \(-3\) or \(-2\) we will have an initial descent.

But we could just add 4 to everything to get the set \(\{1, 2, 5\}\) and think of the permutations as prefixed with a 4, so that if we start with a 1 or 2 we have an initial descent.

So in our algebra, we’re shuffling permutations with a fixed 4 at the beginning that doesn’t move.
Augmented quasi-symmetric functions

We could also have a fixed last element for our permutations, so that we might have descents at the beginning and end. For example, if we consider only permutations that start with 4 and end with 1, we might shuffle the two permutations of length 1 (the fixed beginning and end don’t move):

\[
4 \cdot 2 \cdot 1 \text{ with descent set } \{0, 1\}
\]

and

\[
4 \ 5 \cdot 1 \text{ with descent set } \{1\}
\]

to get \(4 \cdot 2 \ 5 \cdot 1\) with descent set \(\{0, 2\}\) and \(4 \ 5 \cdot 2 \cdot 1\) with descent set \(\{1, 2\}\).
We can represent these descent sets (subsets of \( \{0, 1, \ldots, n\} \)) by compositions in which both the first and last parts may be 0, corresponding to descents at the beginning and end. For our generalized quasi-symmetric functions, we need in addition to the variable \( x_0 \) a new variable \( x_\infty \). So for the generalized compositions of \( 1 \) we have

\[
F_{(0,1,0)} = \sum_{0<i<\infty} x_i \\
F_{(0,1)} = \sum_{0<i\leq\infty} x_i \\
F_{(1,0)} = \sum_{0\leq i<\infty} x_i \\
F_{(1)} = \sum_{0\leq i\leq\infty} x_i
\]

But this isn't quite right. These aren't linearly independent.
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\]

\[
F_{(1)} = \sum_{0 \leq i \leq \infty} x_i
\]

But this isn’t quite right. These aren’t linearly independent.
They’re spanned by $x_0$, $x_\infty$, and $\sum_{0<i<\infty} x_i$, so

$$F_{(0,1)} + F_{(1,0)} = F_{(0,1,0)} + F_{(1)}.$$
They’re spanned by \( x_0, x_\infty \), and \( \sum_{0<i<\infty} x_i \), so

\[
F_{(0,1)} + F_{(1,0)} = F_{(0,1,0)} + F_{(1)}.
\]

What’s wrong?
They’re spanned by $x_0$, $x_\infty$, and $\sum_{0<i<\infty} x_i$, so

$$F_{(0,1)} + F_{(1,0)} = F_{(0,1,0)} + F_{(1)}.$$  

What’s wrong? We should decompose $0 \leq i \leq \infty$ as

- $0 < i < \infty$
- $0 = i < \infty$
- $0 < i = \infty$
- $0 = i = \infty$

We need to add a new variable $x_{0=\infty}$ with

$$x_{0=\infty} x_j = x_{0=\infty} x_0 = x_{0=\infty} x_\infty = 0.$$  

Then everything works: the descent set is shuffle-compatible and the corresponding algebra is described by these generalized quasi-symmetric functions.
There is one hitch: we can’t shuffle a permutation having all possible descents with a permutation having no descents.
There is one hitch: we can’t shuffle a permutation having all possible descents with a permutation having no descents.

Combinatorially, they can’t both exist, and algebraically, $F(m)F_{(0,1,1\ldots,1,0)}$ has some negative coefficients when expanded into the $F_L$.

So in order to be combinatorially meaningful, we need to work in one of two subalgebras: one in which all of the $F(m)$ are excluded and one in which all of the $F_{(0,1,1\ldots,1,0)}$ are excluded.
As before the descent number is shuffle-compatible. The quotient algebra is obtained, just as for type $B$ quasi-symmetric functions, by

$$F_\pi \mapsto \binom{\lambda + n - \text{des}(\pi)}{n} z^n,$$

where $|\pi| = n$, but now $\text{des}(\pi)$ can range from 0 to $n + 1$. 
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where $|\pi| = n$, but now $\text{des}(\pi)$ can range from 0 to $n + 1$.

As polynomials, these are not linearly independent, but we’re not allowed to use both $\binom{\lambda + n}{n}$ and $\binom{\lambda - 1}{n}$ at the same time.
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As polynomials, these are not linearly independent, but we’re not allowed to use both $\binom{\lambda + n}{n}$ and $\binom{\lambda - 1}{n}$ at the same time.

When we look at the corresponding generating functions,

$$\sum_{\lambda} \binom{\lambda + n - i}{n} t^\lambda = \frac{t^i}{(1 - t)^{n+1}},$$

$\binom{\lambda + n}{n}$ must start with $\lambda = 0$ and $\binom{\lambda - 1}{n}$ must start with $\lambda = 1$. 
Cyclic permutations and cyclic shuffles

We consider cycles as equivalence classes of ordinary permutations, so \((152)\) is the equivalence class of the permutations \(152, 521\) and \(215\).
Cyclic permutations and cyclic shuffles

We consider cycles as equivalence classes of ordinary permutations, so \((152)\) is the equivalence class of the permutations 152, 521 and 215.

If \(\alpha\) and \(\beta\) are cycles then we define the set of cyclic shuffles \(\alpha \bowtie \beta\) to be the set of equivalence classes of shuffles of the elements of \(\alpha\) and of \(\beta\).
We consider cycles as equivalence classes of ordinary permutations, so \((152)\) is the equivalence class of the permutations 152, 521 and 215.

If \(\alpha\) and \(\beta\) are cycles then we define the set of cyclic shuffles \(\alpha \shuffle \beta\) to be the set of equivalence classes of shuffles of the elements of \(\alpha\) and of \(\beta\).

For examples, the shuffles of \((12)\) and \((34)\) are

\[(1234), (1324), (1432), \ldots\]

How many shuffles of \(\alpha\) and \(\beta\) are there?
Theorem. Let \( \alpha \) and \( \beta \) be disjoint cycles of lengths \( m \) and \( n \). Then the number of different shuffles of \( \alpha \) and \( \beta \) is
\[
\frac{mn}{m+n} \binom{m+n}{m} = \frac{(m+n-1)!}{(m-1)! (n-1)!}
\]

Proof. By shuffling the \( m \) representatives of \( \alpha \) with the \( n \) representatives of \( \beta \), we get \( mn \binom{m+n}{m} \) as the total number of representatives of all the shuffles of \( \alpha \) and \( \beta \). We then divide by \( m+n \) to get the number of distinct cycles.
Note that \( \frac{(m + n - 1)!}{(m - 1)! (n - 1)!} \) is the coefficient of \( \frac{x^{m+n}}{(m + n - 1)!} \) in \( \frac{x^m}{(m - 1)!} \frac{x^n}{(n - 1)!} \).

So the algebra of shuffles of cycles is the algebra of “shifted exponential generating functions”:

\[
\sum_{n=1}^{\infty} a_n \frac{x^n}{(n - 1)!}
\]
We can define a “descent composition” of a cycle $\alpha$. Suppose that $|\alpha| \geq 2$. Then choose a representative $a$ of $\alpha$ whose last entry is greater than its first entry. Then we define $C(\alpha)$ to be the equivalence class under rotation of $C(a)$. For example, the descent composition of $(25134)$ is $(23)$. 
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We can define a “descent composition” of a cycle \( \alpha \). Suppose that \(|\alpha| \geq 2\). Then choose a representative \( a \) of \( \alpha \) whose last entry is greater than its first entry. Then we define \( C(\alpha) \) to be the equivalence class under rotation of \( C(a) \). For example, the descent composition of \((25134)\) is \((23)\).

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**Theorem.** The multiset \( \{ C(\gamma) : \gamma \in \alpha \circ \beta \} \) depends only on \( C(\alpha) \) and \( C(\beta) \).
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**Theorem.** The multiset $\{ C(\gamma) : \gamma \in \alpha \ominus \beta \}$ depends only on $C(\alpha)$ and $C(\beta)$.

So there is an algebra of cyclic quasi-symmetric functions though it’s not so easy to describe explicitly.
However, there’s a simple description of the quotient algebra for the number of descents (the number of parts in the descent composition). The image of $F_{\alpha}^\circ$, where $|\alpha| = n$ and $\alpha$ has $i$ cyclic descents, is the polynomial in $\lambda$

$$\left(\frac{n + \lambda - i - 1}{n - 1}\right)z^n,$$

where we multiply polynomials in $\lambda$ by

$$p(\lambda) \star q(\lambda) = \lambda p(\lambda)q(\lambda).$$
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\binom{n + \lambda - i - 1}{n - 1} z^n,
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where we multiply polynomials in $\lambda$ by

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p(\lambda) \ast q(\lambda) = \lambda p(\lambda)q(\lambda).
$$

In other words, if $\alpha$ and $\beta$ are cycles of lengths $m$ and $n$ with $i$ and $j$ cyclic descents, then the number of elements of $\alpha \sqcup^0 \beta$ with $l$ cyclic descents is the coefficient of $t^l$ in $A(t)$, where

$$
\sum_{\lambda=0}^{\infty} \lambda \binom{m + \lambda - i - 1}{m - 1} \binom{n + \lambda - j - 1}{n - 1} t^\lambda = \frac{A(t)}{(1 - t)^{m+n}}.
$$