Rook theory and simplicial complexes

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In Celebration of the Mathematics of

Michelle Wachs
Rook numbers

We start with \([n] \times [n]\), where \([n] = \{1, 2, \ldots, n\}\):
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We use Cartesian numbering.
A board is a subset of these $n^2$ squares:
The **rook number** $r_k$ is the number of ways to put $k$ non-attacking rooks on the board, that is, the number of ways to choose $k$ squares from the board with no two in the same row or column.
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In our example, $r_0 = 1$, $r_1 = 5$, $r_2 = 6$, $r_3 = 1$, $r_4 = r_5 = 0$. 
Hit numbers

We can identify a permutation $\pi$ of $[n] = \{1, 2, \ldots, n\}$ with the set of ordered pairs $\{(i, \pi(i)) : i \in [n]\} \subseteq [n] \times [n]$, and we can represent such a set of ordered pairs as a set of $n$ squares from $[n] \times [n]$, no two in the same row or column.

This is the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 3 & 2 \end{pmatrix}$. (The rows are $i$ and the columns are $\pi(i)$.)
The squares of a permutation that are on the board are called hits of the permutation. So this permutation has just one hit:

The hit number $h_k$ is the number of permutations of $[n]$ with $k$ hits.
Examples

For the board

$h_k$ is the number of permutations with $k$ fixed points, and in particular, $h_0$ is the number of derangements.
For the upper triangular board

$h_k$ is the number of permutations with $k$ excedances, an Eulerian number.
The fundamental identity

\[ \sum_i h_i \binom{j}{i} = r_j (n - j)! . \]

**Proof:** Count pairs \((\pi, H)\) where \(H\) is a \(j\)-subset of the set of hits of \(\pi\). Picking \(\pi\) first gives the left side. Picking \(H\) first gives the right side, since a choice of \(j\) nonattacking rooks can be extended to a permutation of \([n]\) in \((n - j)!\) ways.
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Multiplying by \(t^j\) and summing on \(j\) gives

\[ \sum_i h_i(1 + t)^i = \sum_j t^j r_j(n - j)! . \]
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So replacing \(t\) with \(t - 1\) shows that the hit numbers are determined by the rook numbers.
Matching numbers

Let $G$ be a graph with vertex set $[2n]$.

Let $m_k$ be the number of $k$-matchings in $G$, that is, the number of sets of $k$ vertex-disjoint edges in $G$ (analogous to rook numbers).

For any complete matching $M$ of $[2n]$, a hit of $M$ is an edge of $M$ that is in $G$. The hit number $h_i$ of $G$ is the number of complete matchings of $[2n]$ with $i$ hits.
For example, this complete matching has two hits:
The number of complete matchings of the complete graph $K_{2l}$, i.e., the number of partitions of $[2l]$ into blocks of size 2, is $(2l - 1)!! = 1 \cdot 3 \cdots (2l - 1) = (2l)!/2^l l!$. Then by the same reasoning as for ordinary rook numbers we have

$$\sum_i h_i \binom{i}{j} = m_j(2n - 2j - 1)!!.$$
Rook complexes

Let us define a rook complex to be a simplicial complex $\Delta$ on a set $A$ (every subset of an element of $\Delta$ is an element of $\Delta$) with the property that every facet (maximal face) has size $n$, and there are integers $c_0, c_1, \ldots, c_n$ such that if $U \in \Delta$ with $|U| = j$ then the number of facets containing $U$ is $c_{n-j}$. We will call $c_0, \ldots, c_n$ the factorial sequence for $\Delta$. (The faces of $\Delta$ correspond to placements of nonattacking rooks.)
Let us define a **rook complex** to be a simplicial complex $\Delta$ on a set $A$ (every subset of an element of $\Delta$ is an element of $\Delta$) with the property that every facet (maximal face) has size $n$, and there are integers $c_0, c_1, \ldots, c_n$ such that if $U \in \Delta$ with $|U| = j$ then the number of facets containing $U$ is $c_{n-j}$. We will call $c_0, \ldots, c_n$ the **factorial sequence** for $\Delta$. (The faces of $\Delta$ correspond to placements of nonattacking rooks.)

A sufficient condition for $\Delta$ to be a rook complex is that every face of size $k$ is covered by the same number of faces of size $k + 1$. 
So for **rook numbers**, 

- $A$ is $[n] \times [n]$ 
- $\Delta$ is the set of subsets of $A$ in which two different ordered pairs must differ in both coordinates (i.e., sets of nonattacking rooks) 
- $c_j = j!$
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So for rook numbers,

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Note that in both cases, we have a sequence of rook complexes indexed by a nonnegative integer \( n \), and the factorial sequence is independent of \( n \). In our next example, the factorial sequence depends on \( n \).
Forest complexes

Let $A = [n + 1] \times [n + 1]$ which we think of as directed edges, and let $\Delta$ be the set of subsets of $A$ that are forests of rooted trees (with all edges directed towards the roots). Then $\Delta$ is a rook complex with $c_j = (n + 1)^j$. 
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In other words, given any rooted forest on $[n+1]$ with $j$ edges (and therefore $n+1 - j$ trees), the number of rooted trees containing it is $(n+1)^{n-j}$.
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This can be proved by Jim Pitman’s method for counting trees.
Let $\Delta$ be rook complex on the set $|A|$ with factorial sequence $c_0, \ldots, c_n$. For any $B \subseteq A$ we define the rook numbers $r_i$ to be the number of elements of $\Delta$ of size $i$ contained in $B$, and we define the hit number $h_i$ of $B$ to be the number of facets of $\Delta$ that intersect $B$ in $i$ points.
Let $\Delta$ be rook complex on the set $|A|$ with factorial sequence $c_0, \ldots, c_n$. For any $B \subseteq A$ we define the rook numbers $r_i$ to be the number of elements of $\Delta$ of size $i$ contained in $B$, and we define the hit number $h_i$ of $B$ to be the number of facets of $\Delta$ that intersect $B$ in $i$ points.

Then by exactly the same reasoning as in the previous two cases, we have

$$\sum_i h_i \binom{i}{j} = r_j c_{n-j}.$$
For example, in the forest complex with $n = 2$ (forests on 3 points), let $B$ be the directed graph (in this case a rooted tree)

Then $r_0 = 1$, $r_1 = 2$, and $r_2 = 1$ and also $h_0 = 4$, $h_1 = 4$, and $h_2 = 1$. 
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Goldman, Joichi, and White (1975) defined the factorial rook polynomial of $B$ to be

$$F_B(x) = \sum_{j} r_j x(x-1) \cdots (x-(n-j)+1) = \sum_{j} r_j x_{\downarrow n-j}.$$  

Why is it useful?
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Why is it useful?

1. **Factorization theorems.** Goldman, Joichi, and White proved that for Ferrers boards the factorial rook polynomial factors into linear factors.

2. **Reciprocity theorems.** There is a simple relation between the factorial rook polynomial of $B$ and of its complement.
Given hit numbers $h_0, \ldots, h_n$ for a board $B$ in $[n] \times [n]$, the hit numbers $\overline{h}_i$ for the complementary board $\overline{B}$ in $[n] \times [n]$ are given by $\overline{h}_i = h_{n-i}$. Since the rook numbers and hit numbers determine each other, there must be a relation between the rook numbers of a board and its complement. The factorial rook polynomial enables us to express this relation in an elegant way.
In the fundamental identity \( \sum_i h_i \binom{j}{i} = r_j (n - j)! \), we multiply both sides by \( \binom{x}{n-j} \) and sum on \( j \). Applying Vandermonde’s theorem gives

\[
F_B(x) = \sum_j r_j x \downarrow_{n-j} = \sum_i h_i \binom{x+i}{n}.
\]

So the coefficients of \( F_B(x) \) in the basis \( \{ x \downarrow_j \} \) for polynomials are the rook numbers for \( B \), and the coefficients of \( F_B(x) \) in the basis \( \{ \binom{x+i}{n} \}_{0 \leq i \leq n} \) for polynomials of degree at most \( n \) are the hit numbers for \( B \).
So the factorial rook polynomial for the complementary board $\overline{B}$ is

$$F_{\overline{B}}(x) = \sum_{i} h_{n-i} \binom{x + i}{n}$$

$$= \sum_{i} h_{i} \binom{x + n - i}{n}$$

$$= (-1)^{n} \sum_{i} h_{i} \binom{-x - 1 + i}{n}$$

$$= (-1)^{n} F_{B}(-x - 1).$$

This is the reciprocity theorem for factorial rook polynomials, due to Timothy Chow (proved by a different method).
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Chow’s reciprocity theorem was actually for the more general “cover polynomial”.
An example of the factorial rook polynomial

Let’s consider the upper triangular board

Here $h_k$ is the number of permutations with $k$ excedances (the Eulerian number $A_{n,k+1}$), where an excedance of a permutation $\pi$ is an $i$ for which $\pi(i) > i$. The rook number $r_k$ is the Stirling number of the second kind $S(n, n - k)$. 
The factorial rook polynomial is

\[ \sum_j S(n, n - j)x_{\downarrow n-j} = \sum_j S(n, j)x_j = x^n \]

so we have the formula

\[ x^n = \sum_i A_{n,i+1} \binom{x + i}{n}. \]

The reciprocity theorem gives

\[ F_B(x) = (-1)^n(-x - 1)^n = (x + 1)^n = \sum_i A_{n,i} \binom{x + i}{n}. \]
Generalized factorial rook polynomials

We can construct a similar “factorial rook polynomial” for any rook complex. Given the fundamental identity

\[ \sum_i h_i \binom{i}{j} = r_j c_{n-j}, \]

we multiply both sides by some polynomial \( u_{n-j}(x) \) (to be determined) and sum on \( j \). We get

\[ \sum_j r_j c_{n-j} u_{n-j}(x) = \sum_i h_i \sum_j \binom{i}{j} u_{n-j}(x). \]
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To get a nice reciprocity theorem, we want \( P_i(x) \) to have the property that \( P_{n-i}(x) = (-1)^n P_i(-x - \gamma) \) for some \( \gamma \).
We could just take $u_{n-j}(x)$ to be $\binom{x}{n-j}$, so that $P_i(x) = \binom{x+i}{n}$, as we did for the factorial rook polynomial, and we would get a reciprocity theorem. But it doesn’t seem that this gives the nicest results (e.g., factorization theorems).
It turns out that if the factorial sequence is of the form

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which gives

\[ P_i(x) = \frac{\prod_{j=0}^{n-i-1} (x - j\beta) \prod_{j=0}^{i-1} (x + \alpha + j\beta)}{\prod_{j=0}^{n-1} \alpha + j\beta} \]

and

\[ P_{n-i}(x) = (-1)^n P_i(-x - \alpha). \]
To recap, we have \( c_j = \prod_{i=0}^{j-1}(\alpha + \beta i) \) and

\[
F_B(x) := \sum_{j} r_j \prod_{i=0}^{n-j-1} (x - \beta i) = \sum_{i} h_i P_i(x),
\]

where

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and we have the reciprocity theorem

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To recap, we have $c_j = \prod_{i=0}^{j-1}(\alpha + \beta i)$ and

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$$F_B(x) = (-1)^n F_B(-x - \alpha).$$

Note that $F_B(x)$ has integer coefficients as a polynomial in $x$, but $P_i(x)$ does not.
The special case $\alpha = \beta = 1$ corresponds to the ordinary factorial rook polynomial. The case $\alpha = 1, \beta = 2$ corresponds to a “factorial matching polynomial” introduced (with a different normalization) by Reiner and White, and further studied by Haglund and Remmel.
A canonical example: Colored permutations

We can think of a nonattacking rook placement in \([n] \times [n]\) as the set of edges of a digraph on \([n]\) in which every vertex has indegree at most one and outdegree at most one. So the facets (permutations) are digraphs in which every vertex has indegree one and outdegree one:
Now we generalize this by coloring each edge with an integer modulo $\beta$, but we require that the sum of the colors around any cycle must be congruent to one of $0, 1, \ldots, \alpha - 1$ modulo $\beta$. (So $1 \leq \alpha \leq \beta$.) This gives a rook complex with

$$c_k = \alpha(\alpha + \beta)(\alpha + 2\beta) \cdots (\alpha + (k - 1)\beta)$$

The basic idea behind this formula is that when we add an edge that does not close a cycle, there are $\beta$ ways to color it, but an edge that closes a cycle can be colored in $\alpha$ ways. (The case $\alpha = \beta$ was considered by Briggs and Remmel.)
Recall that the forest complex consists of subsets of $[n + 1] \times [n + 1]$ (directed edges) that are forests of rooted trees (with all edges directed towards the roots). This is a rook complex with $c_j = (n + 1)^j$. (Note that $c_j$ depends on $n$, unlike our other examples.)
Recall that the forest complex consists of subsets of $[n + 1] \times [n + 1]$ (directed edges) that are forests of rooted trees (with all edges directed towards the roots). This is a rook complex with $c_j = (n + 1)^j$. (Note that $c_j$ depends on $n$, unlike our other examples.)

This fits into our general formula with $\alpha = n + 1$ and $\beta = 0$. 
Forests

Recall that the forest complex consists of subsets of $[n + 1] \times [n + 1]$ (directed edges) that are forests of rooted trees (with all edges directed towards the roots). This is a rook complex with $c_j = (n + 1)^i$. (Note that $c_j$ depends on $n$, unlike our other examples.)

This fits into our general formula with $\alpha = n + 1$ and $\beta = 0$.

In this case, the product $\prod_{i=0}^{n-j-1} (x - \beta i)$ is just $x^{n-j}$ so our generalized factorial rook polynomial, reduces to an “ordinary” rook polynomial $\sum_j r_j x^{n-j}$. 
We have

\[ F_B(x) = \sum_j r_j x^{n-j} = \sum_i h_i \frac{x^{n-i}(x + n + 1)^i}{(n+1)^n} \]

and the reciprocity theorem

\[ F_B(x) = (-1)^n F_B(-x - n - 1), \]

due to Bedrosian and Kelmans (further generalized by Pak and Postnikov).
In the example we looked at before, with \( n = 2 \) (forests on 3 points), where \( B \) is the directed graph

\[
\begin{align*}
F_B(x) &= x^2 + 2x + 1 = (x + 1)^2 \\
&= 4 \frac{x^2}{9} + 4 \frac{x(x + 3)}{9} + 1 \frac{(x + 3)^2}{9}
\end{align*}
\]

and

\[
F_B(x) = F_B(-x - 3) = (x + 2)^2 = x^2 + 4x + 4.
\]
\[ F_B(x) = x^2 + 4x + 4. \]

is the factorial rook polynomial for the complementary digraph:
A weighted rook complex is a rook complex $\Delta$ in which every face is assigned a weight, and there are quantities $c_0, c_1, \ldots, c_n$ such that if $U \in \Delta$ with $|U| = j$ then the sum of the weights of the facets containing $U$ is $c_{n-j}$. 
A **weighted rook complex** is a rook complex $\Delta$ in which every face is assigned a weight, and there are quantities $c_0, c_1, \ldots, c_n$ such that if $U \in \Delta$ with $|U| = j$ then the sum of the weights of the facets containing $U$ is $c_{n-j}$.

We define the rook “numbers” and hit “numbers” as before, and we still have the fundamental identity

$$\sum_i h_i \binom{i}{j} = r_j c_{n-j},$$

so we can define the factorial rook polynomials as before (but $\alpha$ can be a weight).
One of the most interesting examples (and the inspiration for this theory) consists of ordinary rook placements (viewed as digraphs in which every vertex has indegree and outdegree at most 1) in which the weight of a digraph with $i$ cycles is $\alpha^i$. 
Here

\[ c_j = \alpha(\alpha + 1) \cdots (\alpha + j - 1) = \alpha^j. \]

and

\[ F_B(x) = \sum_{k} r_k x_{\downarrow n-k}. \]

This polynomial was introduced by Chung and Graham in 1995 under the name **cover polynomial**.
The hit numbers (which are now polynomials in $\alpha$) are related to the cover polynomial by

$$F_B(x) = \sum_i h_i \frac{(x + \alpha)^i}{\alpha^n} x^{n-i}.$$
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We have the generating function

$$\sum_{m=0}^{\infty} \binom{m + \alpha - 1}{m} F_B(m) t^m = \frac{\sum_i h_{n-i} t^i}{(1 - t)^{n+\alpha}}$$

and the reciprocity theorem

$$F_B(x) = (-1)^n F_B(-x - \alpha).$$

For example, in the upper triangular board, there are no cycles within the board, so the factorial rook polynomial is $F_B(m) = x^n$ as before, but now the hit polynomial counts permutations by excedances and cycles.
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For ordinary rook polynomials, permuting the rows or columns doesn’t change the rook numbers or hit numbers. But they do change when we keep track of cycles.

There is a beautiful result of Morris Dworkin giving a sufficient condition for the cover polynomial of a permuted Ferrers board to factor.
Let $T_n$ be the staircase board $\{\{i, j\} : 1 \leq i \leq j \leq n\}$. Its cover polynomial is $F_{T_n}(x, \alpha) = (x + \alpha)^n$. 
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For a permutation $\sigma$, let $\sigma(T_n)$ be $T_n$ with its rows permuted by $\sigma$, so there are $\sigma(i)$ squares in row $i$. 
A special case of Dworkin’s theorem: If $\sigma$ is a noncrossing permutation with $c$ cycles, then $F_{\sigma}(T_n) = (x + \alpha)^c(x + 1)^{n-c}$.

As a consequence, the generating polynomial $A_{n,c}(t, \alpha)$ for permutations $\pi$ of $[n]$ according to the cycles of $\pi$ and excedances of $\sigma \circ \pi$ is given by

$$
\frac{A_{n,c}(t, \alpha)}{(1 - t)^{n+\alpha}} = \sum_{m=0}^{\infty} \binom{m + \alpha - 1}{m} (m + \alpha)^c(m + 1)^{n-c} t^m.
$$
What is a noncrossing permutation?
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A noncrossing permutation with one cycle looks like this:
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In generally, a noncrossing permutation is made from a noncrossing partition by making each block into a cycle of this type:
So the number of noncrossing permutations of \([n]\) is the Catalan number
\[ C_n = \frac{1}{n+1} \binom{2n}{n} \]
and the number of noncrossing permutations of \([n]\) with \(c\) cycles is the Narayana number
\[ \frac{1}{n} \binom{n}{c} \binom{n}{c-1}. \]
Weighted Forests

We can make the forest complex into a weighted rook complex by defining the weight of a rooted forest on \([n + 1]\) to be 
\[ u_1^{j_1} \cdots u_{n+1}^{j_{n+1}}, \]
where \(j_i\) is the indegree of vertex \(i\). Here 
\[ c_j = (u_1 + \cdots + u_{n+1})^j, \]
and all of our formulas apply with 
\[ \alpha = u_1 + \cdots + u_{n+1} \]
and \(\beta = 0\).
We can make the forest complex into a weighted rook complex by defining the weight of a rooted forest on \([n + 1]\) to be 
\[u_1^{j_1} \cdots u_{n+1}^{j_{n+1}},\]
where \(j_i\) is the indegree of vertex \(i\). Here 
\[c_j = (u_1 + \cdots + u_{n+1})^j,\]
and all of our formulas apply with 
\[\alpha = u_1 + \cdots + u_{n+1}\] and \(\beta = 0\).

In particular, we have 
\[
F_B(x) = \sum_j r_j x^{n-j} = \sum_i h_i \frac{x^{n-i}(x + u_1 + \cdots + u_{n+1})^i}{(u_1 + \cdots + u_{n+1})^n}
\]
and the reciprocity theorem is 
\[
F_{\overline{B}}(x) = (-1)^n F_B(-x - u_1 - \cdots - u_{n+1})
\]
which is Pak and Postnikov’s generalization of Bedrosian and Kelmans’s reciprocity theorem.