The WZ method and zeta function identities

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There are several interesting formulas that give quickly converging series for generating functions for the zeta function, due to Almkvist, Borwein, Bradley, Granville, Koecher, Leshchiner, and Rivoal.
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For example, Koecher’s formula is

$$\sum_{k=0}^{\infty} \zeta(2k+3)x^{2k} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3 \binom{2n}{n}} \left( \frac{1}{2} + \frac{2}{1 - x^2/n^2} \right) \prod_{k=1}^{n-1} \left( 1 - \frac{x^2}{k^2} \right).$$
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I will give a brief explanation of the WZ method and I will explain how the WZ proofs of these zeta function identities are closely related to classical hypergeometric series identities, and how this connection allows us to generalize them.
The WZ method

We consider a grid graph with a weight function defined on every directed edge with the following property:

▶ For all vertices $A$ and $B$, all paths from $A$ to $B$ have the same weight. (Path-invariance)
This property is equivalent to the existence of a potential function defined on the vertices: the weight of an edge is the difference in the potential of its endpoints:
Some observations

To check that a weighted grid graph has the path invariance property, it is sufficient to check it on each rectangle:

\[ a + b = c + d \]
Some observations

▶ The weights of a directed edge and its reversal are negatives of each other.
Some observations

- We can add arbitrary other edges to the graph, and they will have uniquely determined weights that satisfy the path-invariance property.
In particular, from one path-invariant weighted grid graph, we can get many more by taking a different grid on the same set of points:
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The WZ method

Suppose that we have a path-invariant weighted grid graph, where the vertices are in $\mathbb{Z} \times \mathbb{Z}$. Let $f(i, j)$ be the weight on the edge from $(i, j)$ to $(i + 1, j)$ and let $g(i, j)$ be the weight on the edge from $(i, j)$ to $(i, j + 1)$:
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\[
\begin{array}{c}
(i, j) & f(i, j) & g(i, j) \\
& f(i, j + 1) & g(i + 1, j)
\end{array}
\]

The path invariance property is

\[
f(i, j) + g(i + 1, j) = g(i, j) + f(i, j + 1)
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The path invariance property is

$$f(i, j) + g(i + 1, j) = g(i, j) + f(i, j + 1)$$

Note: Traditionally the order of the parameters is switched.
A pair of functions \((f, g)\) satisfying this identity is called a WZ-pair. For our purposes, we want to assume that \(f\) and \(g\) have a particular form: we want each to be of the form

\[
\frac{\Gamma(a_1 i + b_1 j + u_1) \cdots \Gamma(a_k i + b_k j + u_k)}{\Gamma(c_1 i + d_1 j + v_1) \cdots \Gamma(c_l i + d_l j + v_l)} z^i w^j
\]

where the \(a_n, b_n, c_n,\) and \(d_n\) are integers, and the \(u_n, v_n, z,\) and \(w\) are complex numbers.
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\frac{\Gamma(a_1i + b_1j + u_1) \cdots \Gamma(a_ki + b_kj + u_k)}{\Gamma(c_1i + d_1j + v_1) \cdots \Gamma(c_li + d_lj + v_l)} z^i w^j
\]

where the \(a_n, b_n, c_n,\) and \(d_n\) are integers, and the \(u_n, v_n, z,\) and \(w\) are complex numbers.

This implies that \(f(i + 1, j)/f(i, j), f(i, j + 1)/f(i, j),\) and \(f(i, j)/g(i, j)\) are rational functions.
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This implies that \(f(i + 1, j)/f(i, j), f(i, j + 1)/f(i, j),\) and \(f(i, j)/g(i, j)\) are rational functions.

Note: We will want to allow \(i\) and \(j\) to be complex numbers.
We illustrate our approach with the simplest interesting WZ pair, which is associated with the binomial theorem. One form of this WZ pair is

\[ f(i, j) = \frac{\Gamma(i + j)}{\Gamma(i + 1)\Gamma(j)} x^i (1 - x)^j = \binom{i + j - 1}{i} x^i (1 - x)^j \]

\[ g(i, j) = -\frac{\Gamma(i + j)}{\Gamma(i - 1)\Gamma(j + 1)} x^i (1 - x)^j = -\binom{i + j - 1}{i - 1} x^i (1 - x)^j \]

Let’s first see the connection between this WZ pair and the binomial theorem.
We sum along the two paths $P_1$ and $P_2$:

Summing along $P_1$ gives

$$
\sum_{i=0}^{m-1} f(i, 0) + \sum_{j=0}^{n-1} g(m, j) = 1 - \sum_{j=0}^{n-1} \binom{m+j-1}{i-1} x^m (1 - x)^j.
$$

and summing along $P_2$ gives

$$
\sum_{j=0}^{n-1} g(0, j) + \sum_{i=0}^{m-1} f(i, n) = 0 + \sum_{i=0}^{m-1} \binom{i+n-1}{i} x^i (1 - x)^n.
$$
So we have the identity

\[ 1 - \sum_{j=0}^{n-1} \binom{m+j-1}{i-1} x^m (1-x)^j = \sum_{i=0}^{m-1} \binom{i+n-1}{i} x^i (1-x)^n. \]

Taking the limit as \( m \to \infty \) we get the binomial theorem in the form

\[ 1 = \sum_{i=0}^{\infty} \binom{i+n-1}{i} x^i (1-x)^n. \]
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Conversely, starting with the binomial theorem in this form, it’s easy to find $f$ and $g$, using Gosper’s algorithm.
We can use this WZ pair to obtain identities for the sum

\[ L(x, \alpha) = \sum_{i=0}^{\infty} \frac{\alpha^i}{\alpha + i} x^i, \]

which can be expressed in terms of the Lerch transcendent as

\[ \alpha \Phi(x, 1, \alpha). \]
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which can be expressed in terms of the Lerch transcendent as \( \alpha \Phi(x, 1, \alpha) \).

Note that

\[ L(x, 1) = -x^{-1} \log(1 - x) \]

and

\[ L(x, 1/2) = x^{-1/2} \tanh^{-1} \sqrt{x}. \]
The sum $L(x, \alpha)$ is a generating function for the polylogarithm

$$\text{Li}_s(x) = \sum_{i=1}^{\infty} \frac{x^i}{i^s}$$

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The sum $L(x, \alpha)$ is a generating function for the polyalgorithm

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We have

$$L(x, \alpha) = \sum_{i=0}^{\infty} \frac{\alpha}{\alpha + i} x^i = 1 + \alpha \sum_{i=1}^{\infty} \frac{1}{i} \frac{x^i}{1 + \alpha/i}$$

$$= 1 + \sum_{m=0}^{\infty} (-1)^m \alpha^{m+1} \sum_{i=1}^{\infty} \frac{x^i}{i^{m+1}}$$

$$= 1 + \frac{1}{x} \sum_{m=1}^{\infty} (-1)^{m-1} \alpha^m Li_m(x).$$
Let’s look at
\[ f(i, j) = \frac{\Gamma(i + j)}{\Gamma(i + 1)\Gamma(j)} x^i (1 - x)^j. \]

We want to modify \( f \) and \( g \) so that \( \sum_{i=0}^{\infty} f(i, 0) \) becomes something like \( \sum_{i=0}^{\infty} \alpha x^i / (\alpha + i) \).
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\[ f(\alpha + i, j) = \frac{\Gamma(\alpha + i + j)}{\Gamma(\alpha + i + 1) \Gamma(j)} x^i \cdot \frac{x^{\alpha}(1 - x)^j}{\Gamma(j)} \]
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Now if we set \( j = 0 \), we get

\[ f(\alpha + i, 0) = \frac{\alpha}{\alpha + i} x^i \cdot \frac{x^\alpha}{\alpha \Gamma(0)} \]

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\[ f(\alpha + i, 0) = \frac{\alpha}{\alpha + i} x^i \cdot \frac{x^\alpha}{\alpha \Gamma(0)} \]

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Can we replace $i$ with $\alpha + i$? What about the $\Gamma(0)$?
The first question is easy:

**Lemma.** If $(f(i, j), g(i, j))$ is a WZ pair then so is $(f(\alpha + i, \beta + j), g(\alpha + i, \beta + j))$ for any $\alpha$ and $\beta$. 
To deal with the $\Gamma(0)$ we need to do a little more. We observe that $(f(i, j), g(i, j))$ is a WZ pair, then so is

$$(p(j)f(i, j), p(j)g(i, j)),$$

where $p(j)$ is a function that is periodic with period 1. We take $p(j)$ to be

$$(-1)^j \Gamma(j) \Gamma(1 - j) = (-1)^j \frac{\pi}{\sin \pi j}.$$
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$$(-1)^j \Gamma(j) \Gamma(1 - j) = (-1)^j \frac{\pi}{\sin \pi j}.$$

This is what Wilf and Zeilberger call shadowing.
So (after multiplying $f$ and $g$ by $\alpha x^{-\alpha}$) we get a modified WZ pair

$$f(i, j) = \alpha \frac{\Gamma(\alpha + i + j)\Gamma(1 - j)}{\Gamma(\alpha + i + 1)} x^i(x - 1)^j$$

$$g(i, j) = \alpha \frac{\Gamma(\alpha + i + j)\Gamma(-j)}{\Gamma(\alpha + i)} x^i(x - 1)^j$$

and we have

$$\sum_{i=0}^{\infty} f(i, 0) = \sum_{i=0}^{\infty} \frac{\alpha}{\alpha + i} x^i.$$
To transform this sum, we consider three paths:

$$\text{(0, 0)} \rightarrow \text{(m, 0)}$$

The sum along the black path is equal to the sum along the red path plus the sum along the green path. We take the limit as $m \rightarrow \infty$ and we hope that the green path goes to 0 and the red path gives us something interesting.
More precisely, we pick positive integers $s$ and $t$ and define $h(k)$ to be the weight of a path from $(sk, -tk)$ to $(s(k + 1), -t(k + 1))$. 

\[ (0,0) \rightarrow (s, -t) \rightarrow (2s, -2t) \rightarrow (3s, -3t) \]

It's not hard to see that as $m \to \infty$ the sum along the green path goes to 0, so we have the identity

\[ \sum_{i=0}^{\infty} \alpha_i x^i = \sum_{k=0}^{\infty} h(k) \]
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$$\sum_{i=0}^{\infty} \frac{\alpha}{\alpha + i} x^i = \sum_{k=0}^{\infty} h(k)$$
What is $h(k)$? It’s the weight of any path from $(sk, tk)$ to $(s(k + 1), t(k + 1))$. So if we take the path that goes down first and then to the right

we get

$$h(k) = - \sum_{j=-t(k+1)}^{-tk-1} g(sk, j) + \sum_{i=sk}^{s(k+1)-1} f(i, -t(k + 1))$$
It’s not hard to see that $h(k)$ will be a rational function of $\alpha$, $x$, and $k$ times

$$
\frac{(tk)! (\alpha)_{(s-t)k}}{(\alpha + 1)_{sk}} \left( \frac{(-1)^t x^s}{(1 - x)^t} \right)^k,
$$

where $(u)_m$ is the rising factorial,

$$(u)_m = \begin{cases} 
  u(u+1) \cdots (u+m-1) & \text{if } m \geq 0 \\
  \frac{(-1)^m}{(1 - u)_m} & \text{if } m < 0.
\end{cases}$$


The simplest cases are

\[
\sum_{i=0}^{\infty} \frac{\alpha}{\alpha + i} x^i = \sum_{k=0}^{\infty} \frac{k!}{(\alpha + 1)_k} \frac{(-x)^k}{(1 - x)^{k+1}} \quad (s = 1, t = 1)
\]

\[
= \sum_{k=0}^{\infty} (1 + k - \alpha + xk + x\alpha)
\]

\[
\times \frac{(2k)!}{(\alpha + 1)_k(1 - \alpha)_{k+1}} \frac{(-x)^k}{(1 - x)^{2k+2}} \quad (s = 1, t = 2)
\]

\[
= \sum_{k=0}^{\infty} (1 + 2k + \alpha - x - xk)
\]

\[
\times \frac{k!(\alpha)_k}{(\alpha + 1)_{2k+1}} \frac{(-x^2)^k}{(1 - x)^{k+1}} \quad (s = 2, t = 1)
\]
The simplest cases are

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\times \frac{k!(\alpha)_k}{(\alpha + 1)_{2k+1}} \frac{(-x^2)^k}{(1 - x)^{k+1}} \quad (s = 2, t = 1)
\]

Note: The first formula is a \( \binom{2}{1} \) linear transformation.
For $\alpha = 1$ the first formula gives

$$- \log(1 - x) = \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{x}{1 - x} \right)^k = \log \left( 1 + \frac{x}{1 - x} \right)$$
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In the second formula, we can’t directly set $\alpha = 1$ because of the $1 - \alpha$ in the denominator. But by taking an appropriate limit, we can get

$$- \log(1 - x) = \sum_{k=1}^{\infty} (k^2 x + k^2 - 3k + 1) \frac{(2k - 2)!}{k!^2} \left( -\frac{x}{(1 - x)^2} \right)^k.$$
The third formula gives

$$- \log(1 - x) = (x - 2) \sum_{k=1}^{\infty} \frac{(k - 1)! k!}{(2k)!} (-1)^k \frac{x^{2k-1}}{(1 - x)^k}.$$
The third formula gives

\[- \log(1 - x) = (x - 2) \sum_{k=1}^{\infty} \frac{(k - 1)! \, k!}{(2k)!} (-1)^k \frac{x^{2k-1}}{(1 - x)^k}.\]

This is equivalent to the known formula

\[\frac{\sin^{-1} x}{\sqrt{1 - x^2}} = x \sum_{k=0}^{\infty} \frac{k!^2}{(2k + 1)!} (2x)^{2k}.\]
For $\alpha = 1/2$ the first formula gives

$$\tanh^{-1} x = \frac{2x}{1 - x} \sum_{k=0}^{\infty} \frac{k! (k + 1)!}{(2k + 2)!} \left( -\frac{4x^2}{1 - x^2} \right)^k,$$

which is also equivalent to

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$$

which is also equivalent to

$$
\frac{\sin^{-1} x}{\sqrt{1 - x^2}} = x \sum_{k=0}^{\infty} \frac{k!^2}{(2k + 1)! (2k)!} (2x)^{2k}.
$$

The second formula gives another formula equivalent to this.
The third formula gives

\[ \tanh^{-1} x = \frac{4x}{1 - x^2} \sum_{k=0}^{\infty} \left( 3 + 4k - 2(1 + k)x^2 \right) \times \frac{(2k)! (2k + 2)!}{(4k + 4)!} \left( -\frac{4x^4}{1 - x^2} \right)^k. \]
The simplest interesting polylogarithm formula is the dilogarithm formula obtained from the second transformation formula

\[ Li_2(x) = \sum_{i=1}^{\infty} \frac{x^i}{i^2} = \frac{x}{1 - x} \]

\[ + \sum_{k=1}^{\infty} (-1)^k ((1 + 2k - (k + 1)x)(H_{2k+1} - H_k) - 1) \]

\[ \times \frac{(k - 1)! k!}{(2k + 1)!} \frac{x^{2k}}{(1 - x)^{k+1}} \]

where \( H_n \) is the harmonic number \( 1 + 1/2 + \cdots + 1/n \).
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where \( H_n \) is the harmonic number \( 1 + 1/2 + \cdots + 1/n \).

We can’t set \( x = 1 \), but we can set \( x = -1 \), to get a formula for \( \zeta(2) \).
\[
Li_2(-1) = -\frac{1}{2} \zeta(2) = -\frac{\pi^2}{12}
\]
\[
= -\frac{1}{2} + \sum_{k=1}^{\infty} (-1)^k ((2 + 3k)(H_{2k+1} - H_{k-1}) - 1)
\]
\[
\times \frac{(k - 1)! k!}{(2k + 1)!} \frac{1}{2^{k+1}}
\]
What do we do with a more complicated identity? Let’s look at the Bailey-Borwein-Bradley formula

\[ \sum_{n=0}^{\infty} \zeta(2n+1)x^{2n} = \sum_{k=1}^{\infty} \frac{1}{k^2 - x^2} \]
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\[
\sum_{n=0}^{\infty} \zeta(2n+1)x^{2n} = \sum_{k=1}^{\infty} \frac{1}{k^2 - x^2} = 3 \sum_{k=1}^{\infty} \frac{1}{(2k)\binom{2k}{k}(k^2 - x^2)} \prod_{m=1}^{k-1} \frac{4x^2 - m^2}{x^2 - m^2}
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\]

\[
= 3 \sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}(k^2 - x^2)} \prod_{m=1}^{k-1} \frac{4x^2 - m^2}{x^2 - m^2}
\]

\[
= \frac{3/2}{1 - x^2} \sum_{k=0}^{\infty} \frac{(2)_k(1-2x)_k(1+2x)_k}{(\frac{3}{2})_k(2-x)_k(2+x)_k} \left(\frac{1}{4}\right)^k.
\]

This identity is associated with a WZ pair that corresponds to Gauss’s theorem (the nonterminating form of Vandermonde’s theorem).
Gauss’s theorem is
\[ \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{k! (c)_k} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \]

We want to transform the summand of Gauss’s theorem into a constant times
\[ \frac{1}{(k+1)^2 - x^2} = \frac{(1+x)_k(1-x)_k}{(1-x^2)(2+x)_k(2-x)_k}. \]

If we replace \( k \) by \( 1 + x + k \) in \( (a)_k(b)_k/k! (c)_k \) and then set \( a = 0, b = -2x, \) and \( c = 1 - 2x, \) we get
\[ -\frac{2x}{\Gamma(0)(1-x^2)} \cdot \frac{(1+x)_k(1-x)_k}{(2+x)_k(2-x)_k}. \]
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\sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{k!(c)_k} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.
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We want to transform the summand of Gauss’s theorem into a constant times

\[
\frac{1}{(k+1)^2 - x^2} = \frac{(1+x)_k(1-x)_k}{(1-x^2)(2+x)_k(2-x)_k}.
\]

If we replace \( k \) by \( 1 + x + k \) in \( (a)_k(b)_k/k!(c)_k \) and then set \( a = 0, b = -2x, \) and \( c = 1 - 2x, \) we get

\[
-\frac{2x}{\Gamma(0)(1-x^2)} \cdot \frac{(1+x)_k(1-x)_k}{(2+x)_k(2-x)_k}
\]

By shadowing as before we can get rid of the \( \Gamma(0) \) in the denominator.
But in trying to find a WZ pair we have a problem: Gauss’s theorem has four parameters (the summation index and three others) but to get a WZ pair all we need is the summation index and one parameter. How do we know which one to use?
But in trying to find a WZ pair we have a problem: Gauss’s theorem has four parameters (the summation index and three others) but to get a WZ pair all we need is the summation index and one parameter. How do we know which one to use?

We use all of them, and instead of a WZ pair we get a WZ 4-tuple (or WZ form). This gives a path-invariant grid graph in $\mathbb{Z}^4$ rather than $\mathbb{Z}^2$. Everything works as before, except that we have more choices.
The WZ 4-tuple \( f = (f_1(i, j, k, l), \ldots, f_4(i, j, k, l)) \) corresponding to Gauss’s theorem is
\[
\mathbf{f} = \left( \frac{G}{i}, \frac{G}{j}, \frac{G}{k}, \frac{G}{l} \right),
\]
where
\[
G = (-1)^k \frac{\Gamma(i + k) \Gamma(i + l) \Gamma(j + k) \Gamma(j + l) \Gamma(1 - k)}{\Gamma(i) \Gamma(j) \Gamma(k) \Gamma(l) \Gamma(i + j + k + l)}.
\]

What this means is that if we put a weight of \( f_t(i, j, k, l) \) on the edge from \((i, j, k, l)\) to the adjacent point \((i, j, k, l) + e_t\), where \( e_t = (0, \ldots, 1, \ldots, 0) \), then we get a path-invariant grid graph.
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Note that \( i, j, k, \) and \( l \) don’t need to be integers.
Now if we sum $f$ in the direction $(1, 0, 0, 0)$ (i.e., parallel to the $x_1$-axis) starting at the point $(1 + x, 1, 0, -2x)$ we get

$$-2x \sum_{k=0}^{\infty} \frac{1}{(1 + k)^2 - x^2} = -2x \sum_{k=1}^{\infty} \frac{1}{k^2 - x^2}.$$
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We find that the direction $(1, 2, -1, -1)$ gives us what we want: the sum starting at $(1 + x, 1, 0, -2x)$ gives

$$- \frac{3x}{1 - x^2} \sum_{n=0}^{\infty} \frac{(2)_n(1 - 2x)_n(1 + 2x)_n}{(\frac{3}{2})_n(2 - x)_n(2 + x)_n} \left( \frac{1}{4} \right)^n.$$
An advantage of this approach is that it gives us something more; if we start at the point \((x, 1, 0, y - x)\) we get the more general formula

\[
\sum_{n=0}^{\infty} \frac{1}{(n+x)(n+y)}
\]

\[
= \frac{1}{2xy} \sum_{n=0}^{\infty} (1 + x + y + 3n)
\]

\[
\times \frac{n!(1 + x - y)_n(1 + y - x)_n}{\left(\frac{3}{2}\right)_n(1 + x)_n(1 + y)_n} \left(\frac{1}{4}\right)^n.
\]