On the WZ Method

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In Honor of Herbert Wilf’s 80th Birthday
The WZ (Wilf-Zeilberger) method

The way the WZ method is usually described:

We have two functions $f$ and $g$ satisfying the **WZ equation**

$$f(i, j + 1) - f(i, j) = g(i + 1, j) - g(i, j)$$  \[(WZ)\]
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If we sum on $i$ the right side telescopes, so

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If $g(N + 1, j) = g(0, j) = 0$ then $\sum_{i=0}^{N} f(i, j + 1)$ is independent of $j$ so we can evaluate $\sum_{i} f(i, j)$. 

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An example—Vandermonde’s theorem

Let’s use the WZ method to prove the Chu-Vandermonde theorem:

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It’s convenient to replace the upper limit of summation with \( \infty \). We divide the left side by the right side to get a sum which is independent of the parameters:

\[ \sum_{i=0}^{\infty} t_i = 1, \text{ where } t_i = \frac{\binom{p}{i} \binom{q}{r-i}}{\binom{p+q}{r}}. \]

To apply the WZ method, we need a second parameter \( j \). But we have three parameters, \( p, q, \) and \( r \). Which one should we take?
Let’s try taking $p$ to be $j$. So our identity becomes
\[ \sum_{i=0}^{\infty} f(i, j) = 1, \]
where
\[ f(i, j) = \binom{j}{i} \binom{q}{r-i} / \binom{j+q}{r}. \]
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$$f(i, j + 1) - f(i, j) = g(i + 1, j) - g(i, j).$$

We can do this using Gosper's algorithm, which gives

$$g(i, j) = \frac{i(q-r+i)}{(i-1-j)(j+1+q)} f(i, j)$$

$$= -\frac{i+q-r}{j+q+1} \binom{j}{i-1} \binom{q}{r-i} / \binom{j+q}{r}.$$
Let’s try taking $p$ to be $j$. So our identity becomes
$$\sum_{i=0}^{\infty} f(i, j) = 1,$$
where
$$f(i, j) = \binom{j}{i} \left( \begin{array}{c} q \\ r - i \end{array} \right) / \left( \begin{array}{c} j + q \\ r \end{array} \right).$$

We would like to find a WZ-mate for $f$: a solution of the WZ equation $f(i, j + 1) - f(i, j) = g(i + 1, j) - g(i, j)$. We can do this using Gosper’s algorithm, which gives
$$g(i, j) = \frac{i(q - r + i)}{(i - 1 - j)(j + 1 + q)} f(i, j)$$
$$= -\frac{i + q - r}{j + q + 1} \left( \begin{array}{c} j \\ i - 1 \end{array} \right) \left( \begin{array}{c} q \\ r - i \end{array} \right) / \left( \begin{array}{c} j + q \\ r \end{array} \right).$$

Since $g(0, j) = 0$ and $g(i, j) = 0$ for $i > j + 1$, we have
$$\sum_{i=0}^{\infty} (f(i, j + 1) - f(i, j)) = 0.$$
So

\[ \sum_i f(i, j) \]

is independent of \( j \) and is therefore equal to

\[ \sum_i f(i, 0) = 1 \]

(at least as long as \( j \) is a nonnegative integer).
What if we take $r$ to be $j$? So

$$f(i, j) = \binom{p}{i} \binom{q}{j-i} / \binom{p+q}{j}.$$ 

We apply Gosper’s algorithm and it is again successful: we find the WZ-mate

$$g(i, j) = \frac{i(q-j+i)}{(i-1-j)(p+q-j)} f(i, j)$$

$$= -\binom{j}{i-1} \binom{p+q-j-1}{p-i} / \binom{p+q}{p}.$$
We find that any way of making the parameters linear functions of $j$ works.
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For example, if we replace $i \rightarrow i + j$, $p \rightarrow p + j$, $q \rightarrow q - j$, so that

$$f(i, j) = \binom{p + j}{i + j} \binom{q - j}{r - i - j} / \binom{p + q + j}{r}.$$  

then Gosper’s algorithm finds a WZ-mate for $f$,

$$g(i, j) = \frac{q - r + i}{q - j} f(i, j).$$
So we have many (infinitely many, in fact) slightly different WZ proofs of Vandermonde’s theorem.
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- Why does Gosper’s algorithm always work?
- Why should we care?
Path Invariance

(Gosper, Kadell, Zeilberger)

Another way to look at WZ pairs: We consider a grid graph with a weight function defined on every directed edge with the following property:

▶ For all vertices $A$ and $B$, all paths from $A$ to $B$ have the same weight. (path invariance)
This property is equivalent to the existence of a *potential function* defined on the vertices: the weight of an edge is the difference in the potential of its endpoints:
Some observations

To check that a weighted grid graph has the path invariance property, it is sufficient to check it on each rectangle:

\[
\begin{align*}
a + b &= c + d \\
&= b + a
\end{align*}
\]
Some observations

- We can add arbitrary other edges to the graph, and they will have uniquely determined weights that satisfy the path invariance property.
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▶ The weights of a directed edge and its reversal are negatives of each other.
In particular, from one path-invariant weighted grid graph, we can get another by taking a different grid on the same set of points:
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Suppose that we have a path-invariant weighted grid graph, where the vertices are in $\mathbb{Z} \times \mathbb{Z}$. Let $f(i, j)$ be the weight on the edge from $(i, j)$ to $(i + 1, j)$ and let $g(i, j)$ be the weight on the edge from $(i, j)$ to $(i, j + 1)$:
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The path invariance property is

$$f(i, j) + g(i + 1, j) = g(i, j) + f(i, j + 1)$$

This is the same as the WZ identity.
We can think of our previous application of the WZ identity in terms of path invariance: We rewrite the identity

$$\sum_{i=0}^{N} f(i, j+1) - \sum_{i=0}^{N} f(i, j) = g(N+1, j) - g(0, j).$$

as

$$g(0, j) + \sum_{i=0}^{N} f(i, j+1) = \sum_{i=0}^{N} f(i, j) + g(N+1, j).$$

This comes from path invariance along these paths:
Using path invariance, can find other kinds of identities by summing along different paths.
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Rather than proving the telescoping formula

\[ \sum_{i=0}^{N} f(i, j + 1) - \sum_{i=0}^{N} f(i, j) = 0, \]

we can prove directly that

\[ \sum_{i=0}^{N} f(i, j) = 1. \]

using the following two paths:
\[ \sum_{i=0}^{N} f(i, j) \]
Another important example of path invariance:

If the weight of the green connecting path goes to 0 as $M \to \infty$ then

$$\sum_{i=0}^{\infty} f(i, 0) = \sum_{j=0}^{\infty} g(0, j).$$
Change of Variables

Path invariance allows us to assign a weight to any step, thereby giving us change of variables formulas for WZ pairs.
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The weight of an edge from \((i, j)\) to \((i + 1, j)\) is \(f(i, j)\).

Let \(h(i, j)\) be the weight of a step from \((i, j)\) to \((i + 1, j + 1)\), so

\[
h(i, j) = f(i, j) + g(i + 1, j) = g(i, j) + f(i, j + 1).
\]
Then we have the path invariance formula

\[ h(i, j) + f(i + 1, j + 1) = f(i, j) + h(i + 1, j) \]  

(\ast)

To get a WZ pair we must represent the values of \( f \) and \( h \) in the coordinates \((1, 0)\) and \((1, 1)\).
Then we have the path invariance formula

\[ h(i, j) + f(i + 1, j + 1) = f(i, j) + h(i + 1, j) \]  \( (*) \)

To get a WZ pair we must represent the values of \( f \) and \( h \) in the coordinates \((1, 0)\) and \((1, 1)\).

Since \( i(1, 0) + j(1, 1) = (i + j, j) \), we set \( F(i, j) = f(i + j, j) \) and \( H(i, j) = h(i + j, j) \). Then replacing \( i \) with \( i + j \) in \((*)\) gives

\[ H(i, j) + F(i, j + 1) = F(i, j) + H(i + 1, j) \]

so \((F(i, j), H(i, j))\) is a WZ pair.
Path invariance in several variables

In $n$ variables we may define a **WZ $n$-tuple** to be an $n$-tuple

$$(f_1(i_1, \ldots, i_n), \ldots, f_n(i_1, \ldots, i_n))$$

such that for $j \neq k$, the pair $(f_j, f_k)$ is a WZ-pair with respect to the variables $i_j$ and $i_k$. 
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This implies that if we put a weight of $f_j(i_1, \ldots, i_n)$ on the edge from $(i_1, \ldots, i_n)$ to the adjacent point $(i_1, \ldots, i_j + 1, \ldots, i_n)$ then we get a path-invariant grid graph in $\mathbb{Z}^n$ (or $\mathbb{C}^n$ or $\mathbb{R}^n$).
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Since there are ${n \choose 2}$ conditions, it would seem to be difficult to satisfy them all. But it’s really not so hard:

**Lemma.** Suppose that for $j = 2, \ldots, n$, $(f_1, f_j)$ is a WZ pair with respect to the variables $i_1$ and $i_j$. Then $(f_1, \ldots, f_n)$ is a WZ $n$-tuple in the variables $i_1, \ldots, i_n$. 
Zeilberger introduced **WZ forms** (in *Closed Form (Pun Intended!)*) as a way to associate the variables to the functions in a WZ $n$-tuple. To the WZ $n$-tuple $(f_1, \ldots, f_n)$ we associate the WZ form

$$\sum_{j=1}^{n} f_j(i_1, \ldots, i_n) \delta i_j,$$

by analogy with differential forms, where $\delta i_j$ is a “formal symbol”. So summing a WZ form along a path is analogous to integrating a differential form along a path.
Going back to the Chu-Vandermonde theorem,

\[ \sum_{i=0}^{r} \binom{p}{i} \binom{q}{r-i} = \binom{p+q}{r}, \]

by applying Gosper's algorithm we can find a corresponding WZ form in four variables, \( i, p, q, \) and \( r \):

\[
\omega_V = \frac{\binom{p}{i} \binom{q}{r-i}}{\binom{p+q}{r}} \left( \delta i - \frac{i(q-r+i)}{(p+1-i)(p+1+q)} \delta p ight) \\
+ \frac{i}{p+q+1} \delta q - \frac{i(q-r+i)}{(r-i+1)(p+q-r)} \delta r \]

As in our change of variables formula, we can get all the different WZ pairs (or more generally, WZ forms) corresponding to parametrizations of Vandermonde’s theorem by taking different basis steps and using $\omega_V$ to evaluate the weights of these steps.
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But we don’t need to find these WZ pairs; all the results that we might have found from them we can get directly from path invariance.
The central fact developed is that identities are both inexhaustible and unpredictable; the age-old dream of putting order in this chaos is doomed to failure.

John Riordan, *Combinatorial Identities*
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WZ forms bring order to this chaos.
What WZ forms are there?

All WZ forms are associated with one of four standard hypergeometric series summation formulas:
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- Dougall’s very-well-poised \( 7F_6 \) summation theorem
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\(^1\)almost
Shifting and shadowing

In addition to change of variables, which comes from path independence, there are two other simple operations on WZ forms that give forms we will consider to be equivalent.

It is convenient to assume that all of our functions are “closed form” or “proper hypergeometric terms” or gamma quotients: in two variables \((i\text{ and } j)\) for simplicity, we want each to be of the form

\[
f(i, j) = \frac{\Gamma(a_1 i + b_1 j + u_1) \cdots \Gamma(a_k i + b_k j + u_k)}{\Gamma(c_1 i + d_1 j + v_1) \cdots \Gamma(c_l i + d_l j + v_l)} z^i w^j
\]

where the \(a_n, b_n, c_n,\) and \(d_n\) are integers, and the \(u_n, v_n, z,\) and \(w\) are complex numbers. It is also convenient to allow \(i\) and \(j\) to be complex numbers, so that \(f\) is a meromorphic function. (Similarly for more than two variables.)
Shifting
We replace a variable $i$ in a WZ form with $i + c$ for some $c \in \mathbb{C}$. 

Shadowing
We can multiply a WZ form by any periodic function with period 1 in each variable. In particular, we can take as our multiplier $\left(\frac{-1}{\Gamma(i)\Gamma(1-i)}\right)$.

So we can replace $\frac{1}{\Gamma(i)}$ with $\left(\frac{-1}{\Gamma(i)\Gamma(1-i)}\right)$ or vice versa.

This allows us to get rid of inconvenient zeros or poles.
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Rising factorials

It is convenient to use the notation for rising factorials,

\[(a)_k = a(a + 1) \cdots (a + k - 1)\]

\[= \frac{\Gamma(a + k)}{\Gamma(a)}.\]

So, for example,

\[n! = (1)_n\]

\[(2n)! = 2^{2n}(\frac{1}{2})_n(1)_n\]

\[n + 1 = (2)_n/(1)_n\]
A zeta function example

A well-known formula for $\zeta(2) = \pi^2/6$ is

$$\zeta(2) = \sum_{i=1}^{\infty} \frac{1}{i^2} = 3 \sum_{i=1}^{\infty} \frac{1}{i^2 \binom{2i}{i}}$$

$$= 3 \sum_{i=0}^{\infty} \frac{(1)_i^2}{\left(\frac{3}{2}\right)_i (2)_i} \left(\frac{1}{4}\right)^i .$$

A generalization was proved by Herb Wilf using (a variation of) the WZ method.
This identity is a special case \((x = 0)\) of the Bailey-Borwein-Bradley formula

\[
\sum_{n=0}^{\infty} \zeta(2n + 2)x^{2n} = \sum_{i=1}^{\infty} \frac{1}{i^2 - x^2}
\]
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\]

\[
= 3 \sum_{i=1}^{\infty} \frac{1}{\binom{2i}{i}(i^2 - x^2)} \prod_{m=1}^{i-1} \frac{4x^2 - m^2}{x^2 - m^2}
\]

Proved using a WZ pair by Khodabakhsh and Tatiana Hessami Pilehrood.
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$$= \frac{3/2}{1 - x^2} \sum_{i=0}^{\infty} \frac{(2)_i(1 - 2x)_i(1 + 2x)_i}{(\frac{3}{2})_i(2 - x)_i(2 + x)_i} \left(\frac{1}{4}\right)^i.$$
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= \frac{3/2}{1 - x^2} \sum_{i=0}^{\infty} \frac{(2)_i(1 - 2x)_i(1 + 2x)_i}{(3/2)_i(2 - x)_i(2 + x)_i} \left(\frac{1}{4}\right)^i.
\]

Proved using a WZ pair by Khodabakhsh and Tatiana Hessami Pilehrood.
We will see that their WZ pair comes from the WZ form for the Chu-Vandermonde theorem, or equivalently Gauss’s theorem, which is

\[
\sum_{i=0}^{\infty} \frac{(a)_i(b)_i}{(1)_i(c)_i} = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}.
\]

It is convenient to write this in the form

\[
\sum_{i=0}^{\infty} \frac{(k)_i(l)_i}{(1)_i(j + k + l)_i} = \frac{\Gamma(j)\Gamma(j + k + l)}{\Gamma(j + k)\Gamma(j + l)}
\]

which corresponds to the nicely symmetric WZ form

\[
\frac{\Gamma(i + k)\Gamma(i + l)\Gamma(j + k)\Gamma(j + l)}{\Gamma(i)\Gamma(j)\Gamma(k)\Gamma(l)\Gamma(i + j + k + l)} \left( \frac{\delta i}{i} + \frac{\delta j}{j} + \frac{\delta k}{k} + \frac{\delta l}{l} \right)
\]
We have

\[
\sum_{n=0}^{\infty} \zeta(2n + 2)x^{2n} = \sum_{i=1}^{\infty} \frac{1}{i^2 - x^2} = \sum_{i=0}^{\infty} \frac{1}{(i + 1)^2 - x^2} = \sum_{i=0}^{\infty} \frac{1}{(1 + x + i)(1 - x + i)} = \frac{1}{1 - x^2} \sum_{i=0}^{\infty} \frac{(1 + x)_i(1 - x)_i}{(2 + x)_i(2 - x)_i}.
\]
We want to transform the summand of Gauss’s theorem,

\[
\frac{(k)_i(l)_i}{(1)_i(j + k + l)_i}
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into a constant times

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\]

If we replace \( i \) by \( 1 + x + i \) in \( (k)_i(l)_i/(1)_i(j + k + l)_i \) (shifting) and then set \( j = 1, k = 0, l = -2 \) we get

\[
- \frac{2x}{\Gamma(0)(1 - x^2)} \cdot \frac{(1 + x)_i(1 - x)_i}{(2 + x)_i(2 - x)_i}
\]
We want to transform the summand of Gauss’s theorem,

\[
\frac{(k)_i(l)_i}{(1)_i(j + k + l)_i}
\]

into a constant times

\[
\frac{(1 + x)_i(1 - x)_i}{(2 + x)_i(2 - x)_i}.
\]

If we replace \( i \) by \( 1 + x + i \) in \( (k)_i(l)_i/(1)_i(j + k + l))_i \) (shifting) and then set \( j = 1, k = 0, l = -2 \) we get

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- \frac{2x}{\Gamma(0)(1 - x^2)} \cdot \frac{(1 + x)_i(1 - x)_i}{(2 + x)_i(2 - x)_i}
\]

We can get rid of the \( \Gamma(0) \) in the denominator by shadowing.
The shadowed WZ form is

$$\omega = (-1)^k \frac{\Gamma(i + k) \Gamma(i + l) \Gamma(j + k) \Gamma(j + l) \Gamma(1 - k)}{\Gamma(i) \Gamma(j) \Gamma(l) \Gamma(i + j + k + l)} \times \left( \frac{\delta i}{i} + \frac{\delta j}{j} + \frac{\delta k}{k} + \frac{\delta l}{l} \right)$$

If we sum $\omega$ in the direction $(1, 0, 0, 0)$ starting at the point $(1 + x, 1, 0, -2x)$ we get

$$-2x \sum_{i=0}^{\infty} \frac{1}{(1 + i)^2 - x^2} = -2x \sum_{i=1}^{\infty} \frac{1}{i^2 - x^2}.$$
The shadowed WZ form is

\[
\omega = (-1)^k \frac{\Gamma(i + k) \Gamma(i + l) \Gamma(j + k) \Gamma(j + l) \Gamma(1 - k)}{\Gamma(i) \Gamma(j) \Gamma(l) \Gamma(i + j + k + l)}
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If we sum \( \omega \) in the direction \((1, 0, 0, 0)\) starting at the point \((1 + x, 1, 0, -2x)\) we get

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\]

To transform the sum, we compute the value of the \( i \)th step starting at the same point and going in another direction. We look for something interesting and we have to check that the sum along the “connecting path" goes to 0.
We find that the direction $(1, 2, -1, -1)$ gives us what we want: the sum starting at $(1 + x, 1, 0, -2x)$ gives

$$-\frac{3x}{1 - x^2} \sum_{n=0}^{\infty} \frac{(2)_n(1 - 2x)_n(1 + 2x)_n}{(\frac{3}{2})_n(2 - x)_n(2 + x)_n} \left(\frac{1}{4}\right)^n.$$
We find that the direction \((1, 2, -1, -1)\) gives us what we want: the sum starting at \((1 + x, 1, 0, -2x)\) gives

\[- \frac{3x}{1 - x^2} \sum_{n=0}^{\infty} \frac{(2)_n(1 - 2x)_n(1 + 2x)_n}{(\frac{3}{2})_n(2 - x)_n(2 + x)_n} \left( \frac{1}{4} \right)^n.\]

An advantage of this approach is that it gives us something more; if we start at the point \((x, 1, 0, y - x)\) we get the more general formula

\[\sum_{i=0}^{\infty} \frac{1}{(i + x)(i + y)} = \frac{1}{2xy} \sum_{i=0}^{\infty} (1 + x + y + 3i) \times \frac{i! (1 + x - y)_i(1 + y - x)_i}{(\frac{3}{2})_i(1 + x)_i(1 + y)_i} \left( \frac{1}{4} \right)^i.\]
Amalgamated forms of identities

Recall that the coefficients of the variables in a WZ form must be integers.

We have the identity

\[(d)_{2i} = 2^{2i}(\frac{1}{2} d)_i(\frac{1}{2} d + \frac{1}{2})_i\]
Amalgamated forms of identities

Recall that the coefficients of the variables in a WZ form must be integers.

We have the identity

\[(d)_{2i} = 2^{2i}(\frac{1}{2}d)_i(\frac{1}{2}d + \frac{1}{2})_i\]

So in Gauss’s theorem

\[
\sum_{i=0}^{\infty} \frac{(a)_i(b)_i}{(1)_i(c)_i} = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}.
\]

we can set \(a = d/2, \ b = d/2 + 1/2\) and simplify to get

\[
\sum_{i=0}^{\infty} \frac{(d)_{2i}}{(1)_i(c)_i} 2^{-2i} = \frac{2^{2c-d-2}\Gamma(c)\Gamma(c - d - \frac{1}{2})}{\sqrt{\pi}\Gamma(2c - d - 1)}
\]
We can get a corresponding WZ form

\[
\frac{2^{1-2i-2j-k}\sqrt{\pi} \Gamma(2i+k)\Gamma(2j+k)}{\Gamma(i)\Gamma(j)\Gamma(k)\Gamma(i+j+k+\frac{1}{2})} \left( \frac{\delta i}{i} + \frac{\delta j}{j} - 2\frac{\delta k}{k} \right)
\]
We can get a corresponding WZ form

\[
2^{1-2i-2j-k}\sqrt{\pi} \frac{\Gamma(2i+k)\Gamma(2j+k)}{\Gamma(i)\Gamma(j)\Gamma(k)\Gamma(i+j+k+\frac{1}{2})} \left( \frac{\delta i}{i} + \frac{\delta j}{j} - 2\frac{\delta k}{k} \right)
\]

Is it possible to derive this directly from the WZ form for the general Gauss’s theorem without a separate application of Gosper’s algorithm?
Sporadic WZ forms

There are many evaluations of \( {}_2F_1 \)'s with just one free parameter. They all have WZ pairs, but these seem to be sporadic.