Noncommutative Symmetric Functions of Type B

by

Chak-On Chow

B.Sc., The Hong Kong University of Science & Technology (1994)
S.M., Massachusetts Institute of Technology (1999)

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Author . . . .

Department of Mathematics
May 4, 2001

Certified by . . . .

Ira M. Gessel
Professor of Mathematics
Thesis Supervisor

Accepted by . . . .

Daniel J. Kleitman
Chairman, Applied Mathematics Committee

Accepted by . . . .

Tomasz S. Mrowka
Chairman, Department Committee on Graduate Students

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Abstract

The noncommutation symmetric functions **Sym** of Gelfand *et al.* give not only a lifting of the well-developed commutative theory of symmetric functions to the noncommutative level, but also relate the descent algebras of Solomon and the quasi-symmetric functions, where the latter are dual to the noncommutative symmetric functions equipped with the internal product, which are anti-isomorphic to the descent algebras. Using this anti-isomorphism, properties of both noncommutative symmetric functions and of descent algebras can be studied.

Generalizations of the above theory are made in the present work. The starting point is the quasi-symmetric functions of type B, BQSym, which are shown to have an algebra, a comodule, and a coalgebra structures.

The noncommutative symmetric functions **BSym** are then introduced as a module over **Sym** dual to the comodule structure of BQSym. It is then made into a coalgebra dual to the algebra structure of BQSym, and into an algebra dual to the coalgebra structure of BQSym. The latter duality defines the internal product $*_B$ on **BSym**, which makes (**BSym**, $*_B$) anti-isomorphic to the descent algebra ΣB_n of the hyperoctahedral groups B_n, studied by Bergeron and Bergeron.

Lie idempotents of both **BSym** and ΣB_n are then studied via the anti-isomorphism. In particular, a one-parameter family of Lie idempotents, which is a q-analog of a known idempotent, is found. A specialization of this family gives, in the descent algebra ΣB_n, a Dynkin-like idempotent whose action on words is a signed left bracketing.

Natural noncommutative generalizations of the Eulerian numbers and of the Euler numbers of type B are given. By a specialization, formulas for some refinements of the Euler numbers of type B are also derived.

Thesis Supervisor: Ira M. Gessel
Title: Professor of Mathematics

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Chapter 1

Introduction

The theory of symmetric functions is central to the study of modern algebra and representation theory. The commutative theory of symmetric functions is well-developed, not only algebraically but also combinatorially. Expositions on the commutative theory can be found in the book by Macdonald [23], and in Chapter 7 of Stanley [33].

On the other hand, the emergence of noncommutative theory is rather recent. The first few available expositions in this line of development began with Gelfand *et al.* [13], and the papers sequel to it [20], [8], [21], [22]. The noncommutative theory initiated by Gelfand *et al.* makes use of the theory of quasi-determinants, developed by Gelfand and Retakh [15], [14]. It turns out that the noncommutative theory so obtained has the advantage that most of the identities satisfied by symmetric functions of commuting indeterminates continue to hold in the noncommutative case.

Another generalization of the commutative symmetric functions is the so-called quasi-symmetric functions, which were introduced by Gessel [16] in counting permutations of the symmetric group $\mathfrak{S}_n$ by descent sets. Gessel remarked on a duality between the quasi-symmetric functions and the descent algebra of symmetric groups, which relates the internal coproduct of the former to the product in the latter. Malvenuto and Reutenauer [24] pursued the study of this duality further and established that the quasi-symmetric functions are dual to the noncommutative concatenation Hopf algebra, which can be identified with the descent algebra of the symmetric groups.

These dualities were later unified by the noncommutative symmetric functions of Gelfand *et al.* It turns out that the noncommutative symmetric functions are more natural object to deal with, given that much of the commutative theory remains valid at the noncommutative level. And these form a powerful tool to study the descent algebras of $\mathfrak{S}_n$, of which the multiplicative structure was decomposed by Garsia and Reutenauer [11].

Another related development is the decomposition of the descent algebra ΣB_n of the hyperoctahedral group B_n, by Bergeron and Bergeron [4], and Bergeron [3].

The present dissertation is an attempt to obtain the analogous type B theory. The theory of quasi-symmetric functions of type B, BQSym, which is missing in the literature, is our starting point. The algebraic structures of BQSym are dissected in chapter 2. With the noncommutative symmetric functions of Gelfand *et al.* and the quasi-symmetric functions of type B at hand, the noncommutative symmetric functions of type B, **BSym**, are then introduced in chapter 3. Discussed there include the dualities between **BSym** and BQSym, as well as the connection between **BSym** and the descent algebras of type B, ΣB_n. Chapter 4 is devoted to studying idempotents in **BSym** and in ΣB_n. Amongst other results, a one-parameter family of Lie idempotents, which is a q-analog of a known idempotent, is given. A specialization of this family gives, in the descent algebra ΣB_n, a Dynkin-like idempotent whose action on words is a signed left bracketing. In the final chapter, natural noncommutative generalizations of the Eulerian numbers and of the Euler numbers of type B are given. By a specialization, formulas for the some refinements of the Euler numbers of type B are also derived.

In the remaining sections, we give a brief account of the objects of which generalizations will be made. In passing, this account will also introduce to the readers the history of development of the subject.

From now on, K denotes a field of characteristic zero, e.g., $K = \mathbb{C}$, $\mathbb{R}$, or $\mathbb{Q}$. We do not bother making explicit what K is but leave to the context to imply which is the most appropriate one.

1.1 QSym

We first give a brief review of quasi-symmetric functions. Let $\{x_1, x_2, \dots\}$ be an infinite alphabet of commuting indeterminates. A formal power series $F \in K[[x_1, x_2, \dots]]$ is said to be *quasi-symmetric* if, for any positive integers $c_1, c_2, \dots, c_k$, the coefficient of $x_{i_1}^{c_1} x_{i_2}^{c_2} \cdots x_{i_k}^{c_k}$ is equal to that of $x_{j_1}^{c_1} x_{j_2}^{c_2} \cdots x_{j_k}^{c_k}$ in F, whenever $0 < i_1 < i_2 < \cdots < i_k$ and $0 < j_1 < j_2 < \cdots < j_k$. Let $C = (c_1, c_2, \dots, c_k)$ be a composition of n. We shall use the notation $|C| = n$ or $C \models n$ to denote that C is a composition of n. Let $S(C)$ be the subset $\{c_1, c_1 + c_2, \dots, c_1 + \cdots + c_{k-1}\}$ of $[n-1]$. It is easy to see that the correspondence $C \leftrightarrow S(C)$ is bijective. Now introduce a partial order on the collection of compositions of n by

$$C \preccurlyeq D \iff S(C) \subseteq S(D).$$

There are two bases for QSym, namely the monomial quasi-symmetric function, defined by

$$M_C = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{c_1} x_{i_2}^{c_2} \cdots x_{i_k}^{c_k},$$

and the fundamental quasi-symmetric function, defined by

$$F_C = \sum_{\substack{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n \\ j \in S(C) \Rightarrow i_j < i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

It is easy to see that F_C and M_C are related by inclusion-exclusion:

$$F_C = \sum_{C \preccurlyeq D} M_D, \quad M_C = \sum_{C \preccurlyeq D} (-1)^{l(D)-l(C)} F_D.$$

Let $\pi = \pi_1 \pi_2 \cdots \pi_n \in \mathfrak{S}_n$. The descent set $\mathrm{Des}(\pi)$ of π is defined as $\{i \in [n-1] \colon \pi_i > \pi_{i+1}\}$, and the descent composition $C(\pi)$ of π is then defined as the composition

$C(\mathrm{Des}(\pi))$ of n associated to $\mathrm{Des}(\pi)$. Let S be a set of positive integers. Denote by $\mathfrak{S}_S$ the set of permutations of S. The collection QSym of quasi-symmetric functions is a graded commutative associative algebra, of which the multiplication rule is given as follows. Let $\tau \in \mathfrak{S}_m$, and $\sigma \in \mathfrak{S}_{[m+1,m+n]}$. Then

$$F_{C(\tau)} F_{C(\sigma)} = \sum F_{C(\pi)},$$

where the sum ranges over all shuffles π of the two words τ and σ.

The QSym is equipped with two naturally defined coproducts.

Let $X = \{x_1, x_2, \dots\}$ and $Y = \{y_1, y_2, \dots\}$ be two infinite alphabets of commuting indeterminates, totally ordered by $x_i < x_j$ if and only if $i < j$, and similarly for Y. Denote by $X + Y$ the ordered sum of X and Y in which $x_i < y_j$ for any $x_i \in X$, and $y_j \in Y$, with x_i, y_j observing their respective order in X and in Y. The external coproduct $\gamma\colon$ QSym $\to$ QSym $\otimes$ QSym is defined by $\gamma(F) = F(X + Y)$, for all $F \in$ QSym. The coproduct γ has a counit $\varepsilon\colon$ QSym $\to$ QSym, defined by $\varepsilon(F) =$ constant term of F, for all $F \in$ QSym.

Denote by XY the alphabet $\{x_i y_j \colon x_i \in X, y_j \in Y\}$ totally ordered by lexicographic order on the indices (i, j). The internal coproduct $\delta\colon$ QSym $\to$ QSym $\otimes$ QSym is defined by $\delta(F) = F(XY)$, for any $F \in$ QSym. The coproduct δ has a counit $\varepsilon'\colon$ QSym $\to$ QSym, defined by $\varepsilon'(F_I) = F_I$ if $l(I) = 1$ and 0 otherwise.

In the definition of both coproducts, we have used the identification $\sum g(X)h(Y) \leftrightarrow \sum g \otimes h$.

1.2 Sym

Let $A = \{a_1, a_2, \dots\}$ be an infinite alphabet of noncommuting letters, and t an indeterminate commuting with a_k, $k \geqslant 1$. Define the complete symmetric functions

$S_k(A)$ of alphabet A by

$$\sigma(A;t) = \sum_{n\geqslant 0} S_n(A)t^n = (1-ta_1)^{-1}(1-ta_2)^{-1}\cdots\,, \tag{1.1}$$

where $S_0(A) = 1$. Define the algebra $\mathbf{Sym}(A)$ of formal noncommutative symmetric functions of alphabet A to be the free associative algebra $K\langle S_1(A), S_2(A), \ldots\rangle$ generated by $\{S_n(A)\}$ over the field K. This algebra is graded by the weight function $\mathrm{wt}(S_k(A)) = k$. The homogeneous component of degree n is denoted by $\mathbf{Sym}_n(A)$. The definition of $\mathbf{Sym}(A)$ given here is equivalent to that given in [13].

As the algebra *Sym* of commutative symmetric functions is equipped with several well-known sets of basis, the same is true of its noncommutative counterpart.

DEFINITION 1.2.1. The elementary symmetric functions $\Lambda_k(A)$, the power sum symmetric functions of the first kind $\Psi_k(A)$, and of the second kind $\Phi_n(A)$, of alphabet A, are defined by their generating functions

$$\begin{aligned} \lambda(A;t) &= \sum_{k\geqslant 0} t^k \Lambda_k(A), \\ \psi(A;t) &= \sum_{k\geqslant 1} t^{k-1}\Psi_k(A), \\ \phi(A;t) &= \sum_{k\geqslant 1} t^k \frac{\Phi_k(A)}{k}, \end{aligned}$$

which are related to $\sigma(A;t)$ by

$$\begin{aligned} \sigma(A;t) &= \lambda(A;-t)^{-1}, \\ \sigma'(A;t) &= \sigma(A;t)\psi(A;t), \\ \sigma(A;t) &= \exp(\phi(A;t)), \end{aligned} \tag{1.2}$$

where $'$ denotes differentiation with respect to t.

It is clear that (1.1) and (1.2) are also satisfied by the commutative symmetric functions, and when a_k commute, the noncommutative symmetric functions reduce to their commutative counterparts, e.g., the complete symmetric function $S_n(A)$ reduces to $h_n(A)$; the elementary symmetric function $\Lambda_n(A)$ reduces to $e_n(A)$; and the two power sum symmetric functions $\Psi_n(A)$ and $\Phi_n(A)$ degenerate to $p_n(A)$.

Other relations involving families of bases of **Sym** can be obtained by manipulated those relations as in (1.2). For example, multiplying the second line of (1.2) on the left by $\lambda(A;-t)$ and taking into account the first line, we have that $\psi(A;t) = \lambda(A;-t)\sigma'(A;t)$. By equating the coefficient of t^{n-1} on both sides, we obtain

$$\Psi_n(A) = \sum_{i=0}^{n-1}(-1)^i(n-i)\Lambda_i(A)S_{n-i}(A). \tag{1.3}$$

In the sequel, we shall drop the specification of the underlying alphabets unless the different sets of alphabets involved make a distinction.

The commutative symmetric functions are indexed by partitions. The noncommutative ones are indexed by compositions.

Definition 1.2.2. Let $I = (i_1, \dots, i_k)$ be a composition of n. Define

$$S^I = S_{i_1}S_{i_2}\cdots S_{i_k},$$

and similarly for Λ^I, Ψ^I and Φ^I.

There is one more basis for **Sym**, namely the ribbon Schur functions R_I, which are defined in terms of the homogeneous functions S^I by

$$S^I = \sum_{J \preccurlyeq I} R_J.$$

Although the noncommutative symmetric functions of Gelfand *et al.* [13] come out as an application of the theory of quasi-determinants, developed by Gelfand and Retakh [14], [15], we shall have no occasion to use the machinery of quasi-determinants in the sequel. Instead, those expressions given in [13] and [20] are sufficient for our purposes. More specifically, we shall need expressions for bases of **Sym** expanded in terms of one and another. Here is a list sufficient for later uses. The ribbon Schur function R_I can be expressed in terms of S^I as

$$R_I = \sum_{J \preccurlyeq I} (-1)^{l(I)-l(J)} S^J;$$

the power sum of the second kind Φ_n in terms of $\tilde{S}^I$ and $\tilde{R}_I$ as

$$\frac{\Phi_n}{n} = \sum_{|I|=n} \frac{(-1)^{l(I)-1}}{l(I)} S^I = \frac{1}{n} \sum_{|I|=n} \frac{(-1)^{l(I)-1}}{\binom{n-1}{l(I)-1}} R_I, \tag{1.4}$$

with sums over compositions of n; the power sum of the first kind Ψ_n in terms of $\tilde{S}^I$ and $\tilde{R}_I$ as

$$\frac{\Psi_n}{n} = \frac{1}{n} \sum_{|I|=n} (-1)^{l(I)} \operatorname{lp}(I) S^I = \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k R_{(1^k, n-k)}, \tag{1.5}$$

with sums over compositions of n and $\operatorname{lp}(I)$ denotes the last part of I.

There are several natural involutions defined on **Sym**. We mention here one that will appear very frequently later. Define the reversal $F \to \bar{F}$ as the anti-automorphism sending $S_n \to S_n$. Equivalently, the reversal sends Λ_n to Λ_n.

EXAMPLE 1.2.3. An inspection of (1.4) reveals that $\bar{\Phi}_n = \Phi_n$. However, when $n \geqslant 2$, Ψ_n does not enjoy this property since (1.5) is not invariant under reversal.

For S^I, $\overline{S^I} = S^{\bar{I}}$, where $\bar{I}$ is the composition obtained by reversing the parts of I. Similarly, $\overline{R_I} = R_{\bar{I}}$. For the power sums, in view of Example 1.2.3, $\overline{\Phi^I} = \Phi^{\bar{I}}$ holds, but $\overline{\Psi^I}$ is not, in general, equal to $\Psi^{\bar{I}}$.

The algebra **Sym** has a coalgebra structure. More precisely, let Δ: **Sym** $\to$

$\mathbf{Sym} \otimes \mathbf{Sym}$ be the coproduct defined by

$$\Delta(\Psi_k) = \Psi_k \otimes 1 + 1 \otimes \Psi_k, \quad k \geqslant 1,$$

or equivalently by $\Delta\Phi_k = \Phi_k \otimes 1 + 1 \otimes \Phi_k$. The S_k, and Λ_k form infinite sequences of divided powers with respect to Δ, i.e.,

$$\Delta(S_k) = \sum_{i=0}^{k} S_i \otimes S_{k-i}, \qquad \Delta(\Lambda_k) = \sum_{i=0}^{k} \Lambda_i \otimes \Lambda_{k-i}.$$

Let $C = (C, \Delta, \varepsilon)$ be a coalgebra with coproduct Δ, and counit ε. An element g of C is group-like if $\Delta g = g \otimes g$. A pleasant property of $\sigma(t)$ is the following

LEMMA 1.2.4. *The element $\sigma(t)$ of* **Sym** *is group-like.*

Proof. The lemma follows from

$$\begin{aligned}
\Delta\sigma(t) = & \sum_{n\geqslant 0} t^n \Delta S_n = \sum_{n\geqslant 0} t^n \sum_{0\leqslant k\leqslant n} S_k \otimes S_{n-k} = \sum_{n\geqslant 0} t^n \sum_{0\leqslant k\leqslant n} (S_k \otimes 1)(1 \otimes S_{n-k}) \\
= & \left(\sum_{m\geqslant 0} t^m S_m \otimes 1\right)\left(1 \otimes \sum_{n\geqslant 0} S_n t^n\right) = (\sigma(t) \otimes 1)(1 \otimes \sigma(t)) \\
= & \ \sigma(t) \otimes \sigma(t). \quad \blacksquare
\end{aligned}$$

We saw in (1.2) that the logarithmic derivative of $\sigma(A;t)$ gave rise to two families of power sums, whose commutative images are equal. We also saw that Φ_n and Ψ_n are two families primitive with respect to the coproduct Δ. We can define the Lie bracket $[\cdot,\cdot]$: $\mathbf{Sym} \times \mathbf{Sym} \to \mathbf{Sym}$ by $[F, G] = FG - GF$, for any $F, G \in \mathbf{Sym}$. It can be shown [13] that the Lie algebras generated by $\{\Phi_n\}$ and $\{\Psi_n\}$ coincide. Denote by $L(\Psi)$ the Lie algebra generated by $\{\Psi_n\}$. Primitive elements of $\mathbf{Sym}_n$ can be characterized as follows (Corollary 5.17 of [13]).

THEOREM 1.2.5. *Let π_n be an homogeneous element of $\mathbf{Sym}_n$. The following assertions are equivalent:*

(1) *π_n belongs to the Lie algebra $L(\Psi)$.*

(2) *π_n is a primitive element for Δ.*

There is yet another product defined on **Sym**, the so-called internal product, denoted $*$, which endows **Sym** with an algebra structure.

The internal product $*$ has a simple combinatorial description [13], as follows.

PROPOSITION 1.2.6. *For any two compositions $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_q)$, one has*

$$S^I * S^J = \sum_{M \in \mathrm{Mat}(I,J)} S^M,$$

where $\mathrm{Mat}(I, J)$ denotes the set of matrices of non-negative integers $M = (m_{ij})$ of order $p \times q$ such that $\sum_s m_{rs} = i_r$ and $\sum_r m_{rs} = j_s$ for $r \in [p]$ and $s \in [q]$, and where

$$S^M = S_{m_{11}} S_{m_{12}} \cdots S_{m_{1p}} \cdots S_{m_{q1}} \cdots S_{m_{pq}}.$$

There is a compatibility condition between the concatenation product and the internal product $*$, of **Sym**, known as the Mackey formula, as follows.

PROPOSITION 1.2.7. *Let $F_1, \ldots, F_r, G \in \mathbf{Sym}$. Then*

$$F_1 \cdots F_r * G = \mu_r((F_1 \otimes \cdots \otimes F_r) * \Delta^r G),$$

where μ_r is the r-fold multiplication $F_1 \otimes \cdots \otimes F_r \to F_1 \cdots F_r$, and $*$ is the internal product induced to $\mathbf{Sym}^{\otimes r}$.

In [20], transformations of alphabets of **Sym** are considered. More precisely, let $X = \{x_1, x_2, \ldots\}$ be a commutative totally ordered alphabet and A a noncommutative alphabet. Identify X with the formal sum of elements of X. The noncommutative symmetric functions $S_n(XA)$ is defined by

$$\sigma(XA;t) = \sum_{n\geqslant 0} S_n(XA)t^n = \overleftarrow{\prod_{x\in X}} \sigma(A;xt) = \cdots \sigma(A;x_2t)\sigma(A;x_1t).$$

Other bases of **Sym** of alphabet XA are then defined via their generating functions, which are related to $\sigma(A;xt)$ by (1.2). A particular realization of $\sigma(XA;t)$ of much interest is the one with $X = \{1 < q < q^2 < \cdots\}$, which, under the identification $X \leftrightarrow \sum_{x\in X} x$, can be written as $\sigma(A/(1-q);t)$. It is not hard to show [13], [20], that

$$S_n(\frac{A}{1-q}) = \sum_{|I|=n} \frac{q^{\mathrm{maj}(I)}}{(1-q^{i_1})(1-q^{i_1+i_2})\cdots(1-q^{i_1+\cdots+i_k})} S^I(A), \tag{1.6}$$

where maj(I) is the major index of I, defined to be the sum of all element of $S(I)$. Other families of bases of **Sym** with alphabet $A/(1-q)$ can also be obtained by first expanding in terms of $S_n(A/(1-q))$ and then by expressing $S_n(A/(1-q))$ in terms of S^I using (1.6). We record none but one such expansion, namely

$$\varphi_n(q) = \frac{1}{n}\sum_{|I|=n} \frac{(-1)^{l(I)-1}}{\begin{bmatrix} n-1 \\ l(I)-1 \end{bmatrix}_q} q^{\mathrm{maj}(I)-\binom{l(I)}{2}} R_I = \frac{1-q^n}{n}\Psi_n(\frac{A}{1-q}). \tag{1.7}$$

A comparison of (1.7) with (1.4) reveals that the former is a q-analog of the latter. Moreover, it is shown in [20] that $\varphi_n(q)$ is a Lie idempotent interpolating between Ψ_n/n (when $q=0$), Φ_n/n (when $q=1$), and K_n/n (when $q=\zeta$ is a primitive nth root of unity). The latter is the image of the Klyachko idempotent, first discovered by Klyachko [19], in $\mathbf{Sym}_n$, which takes the form

$$\frac{1}{n}K_n = \frac{1}{n}\sum_{|I|=n} \zeta^{\mathrm{maj}(I)} R_I.$$

1.3 Duality

Gessel [16] utilized multipartite P-partitions to count permutations by descent sets. He remarked in closing that the internal coproduct on QSym_n is dual to the product in the descent algebra Σ_n, and it restricts to the commutative symmetric functions Sym_n to give a coalgebra which is dual to the algebra Sym_n with the internal product $*$. This duality was further generalized, by Malvenuto and Reutenauer [24], to that the Hopf algebra of quasi-symmetric functions is dual to the noncommutative concatenation Hopf algebra. This result is unified by Theorem 6.1 of [13], abstracted as follows.

THEOREM 1.3.1. *The pairing, defined by setting $\langle M_I, S^J\rangle = \delta_{IJ}$, for every compositions I, J, induces an isomorphism of Hopf algebras, given by $(S^J)^* \mapsto M_I$, between the dual $\mathbf{Sym}^*$ of $\mathbf{Sym}$ and the Hopf algebra QSym of quasi-symmetric functions. More precisely, one has for f, $g \in \mathrm{QSym}$ and P, $Q \in \mathbf{Sym}$*

$$\begin{aligned}
\langle f, PQ\rangle &= \langle \gamma f, P\otimes Q\rangle,\\
\langle fg, P\rangle &= \langle f\otimes g, \Delta P\rangle,\\
\langle \delta f, P\otimes Q\rangle &= \langle f, P * Q\rangle,
\end{aligned}$$

where $*$ is the internal product of $\mathbf{Sym}$.

1.4 Descent algebra of $\mathfrak{S}_n$

Solomon [31] established the existence of subalgebras of the group algebra of Coxeter groups which now bear his name. After Solomon's pioneering work little progress was made until Garsia and Reutenauer [11], which gave a concrete realization of the Solomon's descent algebra of the symmetric groups and a decomposition of its multiplicative structure.

Let I be a composition of n. Define the descent class A_I by

$$A_I = \sum_{\substack{\pi\in\mathfrak{S}_n \\ C(\pi)=I}} \pi.$$

The descent classes A_I span a subalgebra Σ_n of the group algebra $K[\mathfrak{S}_n]$, i.e.,

$$A_I A_J = \sum_K \gamma_{I,J}^K A_K,$$

for some constants $\gamma_{I,J}^K \in K$, where the product on the left is the linear extension of compositions of permutations. The descent algebra Σ_n is also spanned by elements of $K[\mathfrak{S}_n]$ defined by

$$B_I = \sum_{\substack{\pi \in \mathfrak{S}_n \\ C(\pi) \preccurlyeq I}} \pi = \sum_{J \preccurlyeq I} A_J.$$

For a given $h \times k$ matrix $M = (m_{ij})$ of non-negative integer entries, we let $r(M) = (r_1, \ldots, r_k)$, where $r_j = \sum_i m_{ij}$, $j = 1, 2, \ldots, k$, and $c(M) = (c_1, \ldots, c_h)$, where $c_i = \sum_j m_{ij}$, $i = 1, 2, \ldots, h$, and $w(M)$ the word obtained by reading the entries of M row by row starting from the top, with zero entries dropped. There is a simple combinatorial rule for the multiplication of B_I by B_J.

Proposition 1.4.1. *For any two compositions I, J of n, we have*

$$B_I B_J = \sum_{c(M)=I,\, r(M)=J} B_{w(M)}.$$

Now define a vector space isomorphism $\alpha\colon \Sigma_n \to \mathbf{Sym}_n$ by sending $B_I \mapsto S^I$, and by linear extension. A comparison of the multiplication rule for Σ_n (Proposition 1.4.1) and that for $(\mathbf{Sym}_n, *)$ (Proposition 1.2.6) shows that α is an algebra anti-isomorphism.

1.5 Descent algebra of B_n

After the paper of Garsia and Reutenauer, much interest was garnered in decomposing the descent algebras of other Coxeter families. The most notable progress made in this direction was Bergeron, Bergeron, Howlett, and Taylor [6], in which the descent algebra of the dihedral groups was studied, Bergeron and Bergeron [4], and

Bergeron [3], in which the descent algebra of the hyperoctahedral groups of type B was examined. The two papers by Bergeron and Bergeron are in the same spirit of Garsia and Reutenauer's.

Let $p = (p_1, p_2 \cdots, p_k)$ be a composition of $m \leqslant n$. The composition p is in bijective correspondence with subsets of $[0, n-1]$, that is,

$$p(p_1, p_2 \cdots, p_k) \leftrightarrow S(p) = \{p_0, p_0 + p_1, \ldots, p_0 + \cdots + p_{k-1}\},$$

where $p_0 = n - m$. A signed permutation $\pi \in B_n$ has a descent at position $i \in [0, n-1]$ if $\pi_i > \pi_{i+1}$, where $\pi_0 = 0$. Denote by $\mathrm{Des}(\pi)$ the descent set of π. Define the descent class in the group algebra $K[B_n]$ by

$$A_p = \sum_{\mathrm{Des}(\pi) = S(p)} \pi.$$

The collection of A_p, as p runs over all compositions of $m \leqslant n$, form a basis for the descent algebra of B_n, denoted ΣB_n. Another natural basis for ΣB_n is

$$B_p = \sum_{\mathrm{Des}(\pi) \subseteq S(p)} \pi = \sum_{q \preccurlyeq p} A_q,$$

where $q \preccurlyeq p$ iff $S(q) \subseteq S(p)$.

Let $k, l \geqslant 1$. Consider templates M of the following form

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0l} \\ & b_{11} & b_{12} & \cdots & b_{1l} \\ a_{10} & a_{11} & a_{12} & \cdots & a_{1l} \\ & \vdots & \vdots & \ddots & \vdots \\ & b_{k1} & b_{k2} & \cdots & b_{kl} \\ a_{k0} & a_{k1} & a_{k2} & \cdots & a_{kl} \end{pmatrix},$$

where a_{ij}, b_{ij} are non-negative integers. Define the reading word $w(M)$ to be

$$a_{01}\cdots a_{0l}b_{1l}\cdots b_{11}a_{10}a_{11}\cdots a_{1l}\cdots a_{kl},$$

that is, we read the rows of M from left to right on a lines and from right to left on b lines, with zeros ignored. The row sum word $q(M)$ to be $q(M) = q_1q_2\cdots q_k$, where

$$q_i = a_{i0} + \sum_{j=1}^{k}(a_{ij} + b_{ij}), \quad 1 \leqslant i \leqslant l.$$

The column sum word $p(M)$ to be $p(M) = p_0p_1\cdots p_l$, where

$$p_j = a_{0j} + \sum_{i=1}^{l}(a_{ji} + b_{ji}), \quad 1 \leqslant j \leqslant k.$$

The multiplication rule for B_p is given in Theorem 1 of [4], as follows.

THEOREM 1.5.1. *For two compositions $p \models m_1 \leqslant n$, and $q \models m_2 \leqslant n$, one has*

$$B_pB_q = \sum_{\substack{p(M)=p\\ q(M)=q}} B_{w(M)}.$$

Chapter 2

Quasi-Symmetric Functions of Type B

In this chapter we develop the theory of quasi-symmetric functions of type B. The theory developed here parallels much that of ordinary quasi-symmetric functions. The first section gives a generalization of ordinary P-partitions [32], namely P-partitions of type B, the generating functions of which are the quasi-symmetric functions of type B. The algebra structure of the quasi-symmetric functions is examined in section two. The coproducts of QSym extend to endow BQSym with a coalgebra and a comodule structure, whose discussion constitutes section three. Two specializations giving interesting identities are studied in section four.

2.1 P-partitions of type B

In this section, we introduce the P-partitions of type B. Our approach is similar to that of Gessel [16]. For the theory of P-partitions of type B from a Coxeter group perspective, see Reiner [27].

Let P be a poset with partial order $<_P$.

Definition 2.1.1. A B_n *poset* is a poset P labelled by $0, \pm 1, \pm 2, \ldots, \pm n$ such that if $i <_P j$ then $-j <_P -i$.

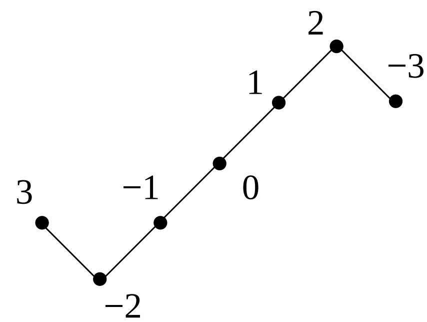

Figure 2-1: A B_3 poset.

An example of B_3 poset is given in Figure 2-1. From the definition of a B_n poset, it is enough to specify half of the poset including the element labelled 0. To recover the whole poset we join the specified half with its dual poset with negated labels at the point labelled 0. We will follow this way of specifying a B_n poset in the sequel. Denote by $\pm[n]$ the interval of integers $\{-n, -n+1, \ldots, -1, 0, 1, \ldots, n-1, n\}$.

DEFINITION 2.1.2. Let $(P, <_P)$ be a B_n poset. A B_n *P-partition* is a function $f\colon \pm[n] \to \mathbb{Z}$ that satisfies

(i) $f(i) \leqslant f(j)$ if $i <_P j$;

(ii) $f(i) < f(j)$ if $i <_P j$ and $i > j$;

(iii) $f(-i) = -f(i)$.

From the above definition we see that a B_n P-partition f is a P-partition with the added condition (iii). It is immediate from (iii) that $f(0) = 0$. We denote the set of B_n P-partitions by $A(P)$.

If π is a signed permutation of $[n]$, we may identify π with the total order

$$\pi(1) <_\pi \pi(2) <_\pi \cdots <_\pi \pi(n-1) <_\pi \pi(n)$$

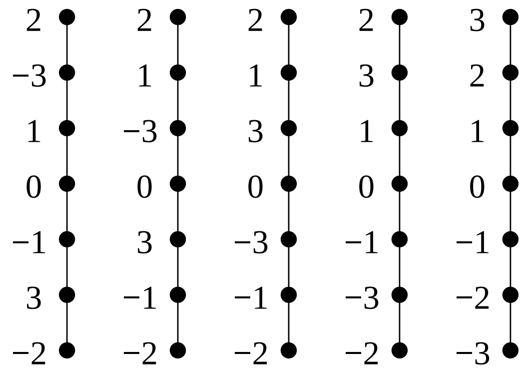

Figure 2-2: Linear extensions of the B_3 poset

in which $\pi(i) <_\pi \pi(j)$ if and only if $1 \leqslant i < j \leqslant n$. It is convenient to extend π to $\pm[n]$ by defining $\pi(-i) = -\pi(i)$, $0 \leqslant i \leqslant n$. Then $A(\pi)$ is the set of functions $f\colon \pm[n] \to X$ satisfying $f(-i) = -f(i)$ for all i, and $f(\pi(0)) \sim_0 f(\pi(1)) \sim_1 f(\pi(2)) \sim_2 \cdots \sim_{n-1} f(\pi(n))$, where $\sim_i$ is $\leqslant$ if $\pi(i) < \pi(i+1)$, and $\sim_i$ is $<$ if $\pi(i) > \pi(i+1)$, $0 \leqslant i \leqslant n-1$.

Define $L(P)$ to be the set of signed permutations of $[n]$ extending P to a total order, that is, $\pi \in L(P)$ if and only if $i <_P j$ implies $\pi^{-1}(i) < \pi^{-1}(j)$.

Example 2.1.3. The B_3 poset P in Figure 2-1 can be extended to one of the chains shown in Figure 2-2. Thus, $L(P) = \{1\bar{3}2, \bar{3}12, 312, 132, 123\}$.

The connection between $L(P)$ and $A(P)$ is given in the next theorem.

Theorem 2.1.4. $A(P) = \coprod_{\pi \in L(P)} A(\pi)$, *where* $\coprod$ *denotes disjoint union.*

Proof. Induction on the number of incomparable pairs of elements in P. If there is none then we are done. Otherwise, let i and j be incomparable in P. Let P_{ij} be the poset obtained by adding the relation $i <_P j$ and $-j <_P -i$, and let P_{ji} be similarly defined. Then $P_{ij} \cup P_{ji}$ has one less pair of incomparable elements,

$L(P) = L(P_{ij}) \cup L(P_{ji})$ and $A(P) = A(P_{ij}) \cup A(P_{ji})$. The theorem now follows by induction. ■

Let $X = \{x_0, x_1, x_2, \dots\}$ be an alphabet of commuting indeterminates, with $x_{-i} = x_i$ for all i, and $f\colon \pm[n] \to X$. Define the monomial x^f by

$$x^f = x_{f(1)} x_{f(2)} \cdots x_{f(n)}.$$

Let $\Gamma(P)$ be the generating function for P. By Theorem 2.1.4,

$$\Gamma(P) = \sum_{\pi \in L(P)} \Gamma(\pi),$$

where

$$\Gamma(\pi) = \sum_{f \in A(\pi)} x^f = \sum_{f \in A(\pi)} \prod_{i \in [n]} x_{f(i)}$$

So, it is crucial to study $\Gamma(\pi)$, which is the subject of the next section.

2.2 BQSym

Let $\pi \in B_n$. Then π can be represented as a signed permutation,

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix}$$

where $|\pi| = |\pi_1||\pi_2| \cdots |\pi_n|$ is a permutation in $\mathfrak{S}_n$. A signed permutation π is said to have a *descent* at position i, $i = 0, 1, 2, \dots, n-1$ if $\pi_i > \pi_{i+1}$, where $\pi_0 = 0$. Denote by $\mathrm{Des}(\pi) = \{i\colon \pi_i > \pi_{i+1}, 0 \leqslant i \leqslant n-1\}$ the *descent set* of π.

Define a pseudo-composition $p = (p_1, p_2, \dots, p_k)$ of n to be an ordered sequence of non-negative integers such that $p_1 + p_2 + \cdots + p_k = n$ with $p_1 \geqslant 0$, $p_i > 0$, $i = 2, 3, \dots, k$. We shall write $p \models n$ or $|p| = n$ to denote that p is a pseudo-composition of n. Since similar notations are also used in other work to denote compositions of n. We shall indicate explicitly when the latter is intended.

To each $p \models n$, we can associate a subset $S(p)$ of $[0, n-1] = \{0, 1, 2, \ldots, n-1\}$, namely,

$$S(p) = \{p_1, p_1 + p_2, \ldots, p_1 + p_2 + \cdots + p_{k-1}\}.$$

Conversely, for any subset $A = \{a_1, a_2, \ldots, a_k\}_<$ of $[0, n-1]$, we can associate a pseudo-composition $C(A)$ of n by

$$C(A) = (a_1, a_2 - a_1, \ldots, a_k - a_{k-1}, n - a_k).$$

The pseudo-compositions p of n can be partially ordered by reverse refinement, i.e., $p \preccurlyeq q$ if and only if $S(p) \subseteq S(q)$.

Let $X = \{x_0, x_1, x_2, \cdots\}$ be an infinite alphabet, and $p = (p_1, p_2, \ldots, p_k) \models n$.

Definition 2.2.1. The *fundamental quasi-symmetric function* $\tilde{F}_p(X)$ *of type* B, of alphabet X, also called *quasi-ribbon*, is defined by

$$\tilde{F}_p(X) = \sum_{\substack{0 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n \\ j \in S(p) \Rightarrow i_j < i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n},$$

where $i_0 = 0$.

Example 2.2.2. For $n = 2$,

$$\tilde{F}_{(2)} = \sum_{0 \leqslant i \leqslant j} x_i x_j, \quad \tilde{F}_{(1,1)} = \sum_{0 \leqslant i < j} x_i x_j, \quad \tilde{F}_{(0,2)} = \sum_{0 < i \leqslant j} x_i x_j, \quad \tilde{F}_{(0,1,1)} = \sum_{0 < i < j} x_i x_j. \quad \square$$

Definition 2.2.3. The *monomial quasi-symmetric function* $\tilde{M}_p(X)$ *of type* B, of alphabet X, is defined by

$$\tilde{M}_p(X) = \sum_{0 < i_2 < \cdots < i_k} x_0^{p_1} x_{i_2}^{p_2} \cdots x_{i_k}^{p_k}.$$

EXAMPLE 2.2.4. For $n = 2$,

$$\tilde{M}_{(2)} = x_0^2, \quad \tilde{M}_{(1,1)} = \sum_{0<i} x_0 x_i, \quad \tilde{M}_{(0,2)} = \sum_{0<i} x_i^2, \quad \tilde{M}_{(0,1,1)} = \sum_{0<i<j} x_i x_j. \quad \square$$

It is clear from these definitions that B_n quasi-symmetric functions involve the indeterminate x_0, which is absent in the ordinary quasi-symmetric functions. In the sequel, we shall drop the specification of underlying alphabet in the quasi-symmetric functions unless circumstances demand the contrary.

LEMMA 2.2.5. $\tilde{F}_p = \sum_{p \preccurlyeq q} \tilde{M}_q$, *and* $\tilde{M}_p = \sum_{p \preccurlyeq q} (-1)^{l(p)-l(q)} \tilde{F}_q$.

Proof. The first assertion follows by regrouping terms, and the second from the first by inclusion-exclusion. ■

Let $\pi = \pi_1 \pi_2 \cdots \pi_n \in B_n$ and let $\mathrm{Des}(\pi)$ be the descent set of π. We denote by $C(\pi) = C(\mathrm{Des}(\pi))$ the descent composition of π.

Let u, v be words with no letter in common. A word w is said to be a shuffle of u and v if the letters of u and v appear in w in the same order as in u and v, and when the letters of u in w are deleted, the word v remains. Denote by $u \sqcup\!\sqcup v$ the shuffle product of u and v, which is defined as the formal sum of all shuffles of u and v.

Let S be a set of positive integers. Denote by B_S the set of signed permutations of S. Like the ordinary quasi-symmetric functions the products of B_n quasi-symmetric functions are determined by shuffles, as the next proposition shows.

PROPOSITION 2.2.6. *Let* $\sigma = \sigma_1 \cdots \sigma_m \in B_m$ *and* $\tau = \tau_1 \cdots \tau_n \in B_{[m+1,m+n]}$. *Then*

$$\tilde{F}_{C(\sigma)} \tilde{F}_{C(\tau)} = \sum \tilde{F}_{C(\pi)}$$

where the sum is over all shuffles of the two words σ *and* τ.

Proof. Let P be the disjoint union of the two chains $\sigma_1 <_\sigma \cdots <_\sigma \sigma_m$ and $\tau_1 <_\tau \cdots <_\tau \tau_n$. Then $\Gamma(P) = \tilde{F}_{C(\sigma)}\tilde{F}_{C(\tau)}$. On the other hand, $L(P)$ is exactly the collection of shuffles of σ and τ. The proposition follows. ■

EXAMPLE 2.2.7. Let $\sigma = 1\bar{2} \in B_2$, $\tau = \bar{5}3\bar{4} \in B_{[3,5]}$ so that $C(\sigma) = (1,2)$ and $C(\tau) = (0,2,1)$.

$$\begin{aligned} \sigma \sqcup\!\sqcup \tau &= 1\bar{2} \sqcup\!\sqcup \bar{5}3\bar{4} \\ &= 1\bar{2}\bar{5}3\bar{4} + \bar{5}1\bar{2}3\bar{4} + \bar{5}31\bar{2}\bar{4} + \bar{5}3\bar{4}1\bar{2} + 1\bar{5}\bar{2}3\bar{4} + \bar{5}13\bar{2}\bar{4} + \bar{5}31\bar{4}\bar{2} + 1\bar{5}3\bar{2}\bar{4} \\ &\quad + \bar{5}13\bar{4}\bar{2} + 1\bar{5}3\bar{4}\bar{2} \end{aligned}$$

so that

$$\begin{aligned} \tilde{F}_{(1,2)}\tilde{F}_{(0,2,1)} &= \tilde{F}_{(1,1,3)} + 2\tilde{F}_{(0,2,2,1)} + \tilde{F}_{(0,2,1,1,1)} + \tilde{F}_{(1,3,1)} + \tilde{F}_{(0,3,1,1)} + \tilde{F}_{(0,2,1,2)} \\ &\quad + \tilde{F}_{(1,2,1,1)} + \tilde{F}_{(0,3,2)} + \tilde{F}_{(1,2,2)}. \quad \square \end{aligned}$$

DEFINITION 2.2.8. Let $\mathrm{BQSym}_n = K\text{-span}\{\tilde{F}_p \colon p \models n\}$ be the homogeneous component of quasi-symmetric functions of type B of degree n, and $\mathrm{BQSym} = \bigoplus_{n \geqslant 0} \mathrm{BQSym}_n$.

Proposition 2.2.6 says that BQSym is an algebra naturally graded by the weight function $\mathrm{wt}(\mathrm{BQSym}_n) = n$.

Since $\tilde{F}_p$ (resp., $\tilde{M}_p$), as p runs over all pseudo-compositions of n, form a basis for BQSym_n. There are 2^n pseudo-compositions of n. So, we have

LEMMA 2.2.9. $\dim \mathrm{BQSym}_n = 2^n$.

2.3 Coproduct of BQSym

There are two naturally defined coproducts on QSym [24]. As BQSym is obtained by adjoining x_0 to the alphabet of QSym, we can obtain coproduct(s) on BQSym by suitably extending those for QSym.

Let $X = \{x_0, x_1, x_2, \dots\}$, $Y = \{y_1, y_2, \dots\}$ be two infinite alphabets of commuting indeterminates, totally ordered by $x_i < x_j$ iff $i < j$, and similarly for Y. We shall denote by $\tilde{F}_p(X)$, $\tilde{M}_p(Y)$ the corresponding B_n quasi-symmetric functions with alphabet X and Y, respectively. Denote by $X + Y$ the disjoint sum of X and Y in which $x_i < y_j$ whenever $x_i \in X$, $y_j \in Y$, and x_i, y_j respect their total order in X and Y, respectively.

In QSym, the map $\gamma\colon f(X) \mapsto f(X+Y)$, $f \in \mathrm{QSym}$, defines a coproduct, called the external coproduct, where if $f(X+Y) = \sum g(X)h(Y)$ then $\gamma(f) = \sum g \otimes h$.

In BQSym, the same map $\gamma\colon f(X) \mapsto f(X+Y)$ is not a coproduct. It is only a map $\gamma_B\colon \mathrm{BQSym} \to \mathrm{BQSym} \otimes \mathrm{QSym}$. For explanation of this choice of extension of γ, see Remark 3.2.4.

Example 2.3.1. Consider the monomial quasi-symmetric function $\tilde{M}_{(2,1)}(X) = \sum_{0<i_2} x_0^2 x_{i_2}$.

$$\tilde{M}_{(2,1)}(X+Y) = \sum_{0<i_2} (x_0^2 x_{i_2} + x_0^2 y_{i_2}) = \tilde{M}_{(2,1)}(X) + \tilde{M}_{(2)}(X) M_{(1)}(Y).$$

so that

$$\gamma_B(\tilde{M}_{(2,1)}) = \tilde{M}_{(2,1)} \otimes 1 + \tilde{M}_{(2)} \otimes M_{(1)}. \quad \square$$

The external coproduct γ for QSym has a counit ε defined by $\varepsilon(F)$ = constant term of F. This counit readily extends to BQSym with the same definition.

Proposition 2.3.2. *We have* $(I \otimes \gamma) \circ \gamma_B = (\gamma_B \otimes I) \circ \gamma_B$, *and* $(I \otimes \varepsilon) \circ \gamma_B = i$, *where* $i\colon \mathrm{BQSym} \to \mathrm{BQSym} \otimes K$ *is the map sending* $f \in \mathrm{BQSym}$ *to* $f \otimes 1 \in \mathrm{BQSym} \otimes K$.

Proof. Let $n > 0$ and I a pseudo-composition of n. Then

$$(I \otimes \gamma) \circ \gamma_B(\tilde{M}_I) = \sum_{J \cdot K = I} \tilde{M}_J \otimes \gamma(M_K) = \sum_{J \cdot K \cdot L = I} \tilde{M}_J \otimes M_K \otimes M_L,$$
$$(\gamma_B \otimes I) \circ \gamma_B(\tilde{M}_I) = \sum_{J \cdot L = I} \gamma_B(\tilde{M}_J) \otimes M_L = \sum_{J \cdot K \cdot L = I} \tilde{M}_J \otimes M_K \otimes M_L,$$

and the first assertion follows. The second assertion holds because $\gamma_B(1) = 1 \otimes 1$. ■

Let $C = (C, \Delta_C, \varepsilon_C)$ be a coalgebra over a field k. A right C-comodule [25] is a pair (M, ψ), where M is a vector space and $\psi\colon M \to M \otimes C$ such that $(I \otimes \Delta_C) \circ \psi = (\psi \otimes I) \circ \psi$ and $(I \otimes \varepsilon_C) \circ \psi = i$, where $i\colon M \to M \otimes k$ is the map sending $m \in M$ to $m \otimes 1 \in M \otimes k$. So, for $C = (\mathrm{QSym}, \gamma, \varepsilon)$, and $(M, \psi) = (\mathrm{BQSym}, \gamma_B)$, Proposition 2.3.2 says precisely that BQSym is a right QSym-comodule.

Now let $X = \{x_0, x_1, x_2, \dots\}$ and $Y = \{y_0, y_1, y_2, \dots\}$ be two infinite alphabets of commuting indeterminates with the identification that $x_{-i} = x_i$, for $0 \leqslant i$, and similarly for Y. Denote by $XY = \{x_i y_j \colon x_i \in X, y_j \in Y\}$ the alphabet obtained from X and Y, where the pair of indices (i,j) are in lexicographic order, i.e., $(i,j) < (k,l)$ if and only if $i < k$ or $i = k$ and $j < l$.

In QSym, the map $\delta\colon f(X) \mapsto f(XY)$, $f \in \mathrm{QSym}$, defines a coproduct, called the internal coproduct, where if $f(XY) = \sum g(X)h(Y)$ then $\delta(f) = \sum g \otimes h$.

In BQSym, the same map $\delta\colon f(X) \mapsto f(XY)$ generalizes to a coproduct. A simple example follows.

Example 2.3.3.

$$\begin{aligned}
\tilde{M}_{(2,1)}(XY) &= \sum_{(0,0)<(i_2,j_2)} (x_0y_0)^2(x_{i_2}y_{j_2}) \\
&= \sum_{0<i_2} x_0^2 x_{i_2} \sum_{j_2} y_0^2 y_{j_2} + \sum_{0=i_2} x_0^2 x_{i_2} \sum_{0<j_2} y_0^2 y_{j_2} \\
&= \sum_{0<i_2} x_0^2 x_{i_2} \left(y_0^3 + 2\sum_{0<j_2} y_0^2 y_{j_2} \right) + x_0^3 \sum_{0<j_2} y_0^2 y_{j_2} \\
&= \tilde{M}_{(2,1)}(X)(\tilde{M}_{(3)}(Y) + 2\tilde{M}_{(2,1)}(Y)) + \tilde{M}_{(3)}(X)\tilde{M}_{(2,1)}(Y),
\end{aligned}$$

so that $\delta \tilde{M}_{(2,1)} = \tilde{M}_{(2,1)} \otimes (\tilde{M}_{(3)} + 2\tilde{M}_{(2,1)}) + \tilde{M}_{(3)} \otimes \tilde{M}_{(2,1)}$. □

There is a description of the internal coproduct in terms of descent sets of permutations in B_n.

THEOREM 2.3.4. *Let $\pi \in B_n$. Then*

$$\delta \tilde{F}_{C(\pi)} = \sum_{\tau\sigma=\pi} \tilde{F}_{C(\sigma)} \otimes \tilde{F}_{C(\tau)}.$$

Proof. By definition,

$$\delta \tilde{F}_{C(\pi)} = \sum_{\substack{(i_0,j_0)\leqslant(i_1,j_1)\leqslant\cdots\leqslant(i_n,j_n) \\ s\in\mathrm{Des}(\pi)\Rightarrow(i_s,j_s)<(i_{s+1},j_{s+1})}} x_{i_1}\cdots x_{i_n} y_{j_1}\cdots y_{j_n}, \tag{2.1}$$

where the inequalities are in lexicographic order, and $i_0 = j_0 = 0$. Let $0 \leqslant s \leqslant n-1$. If $\pi_s < \pi_{s+1}$, then $(i_s, j_s) \leqslant (i_{s+1}, j_{s+1})$, which is equivalent to

(i) $i_s \leqslant i_{s+1}$ and $j_s \leqslant j_{s+1}$,

(ii) $i_s < i_{s+1}$ and $j_s > j_{s+1}$.

Note that (i) and (ii) are mutually exclusive. If $\pi_s > \pi_{s+1}$, then $(i_s, j_s) < (i_{s+1}, j_{s+1})$, which can be decomposed as

(iii) $i_s < i_{s+1}$ and $j_s \geqslant j_{s+1}$,

(iv) $i_s \leqslant i_{s+1}$ and $j_s < j_{s+1}$,

which are also mutually exclusive. Since there are n lexicographic inequalities, each of which can be decomposed into 2 mutually exclusive cases, there are altogether 2^n mutually exclusive sets of inequalities, of which the inequalities in i_s are non-decreasing. So, it is convenient to index these inequalities in i_s by pseudo-compositions of n, so that if $L = (L_1, L_2, \ldots, L_k)$ is a pseudo-composition of n, then $i_0 \leqslant i_1 \leqslant \cdots \leqslant i_{L_1} < i_{L_1+1} \leqslant \cdots \leqslant i_{L_1+L_2} < \cdots$. Consider now the B_n poset P_L of $n+1$ elements labelled consecutively by $0 = \pi_0, \pi_1, \ldots, \pi_n$, and with k increasing runs of elements of respective lengths $L_1, \ldots, L_k$, i.e.,

$$\pi_0 <_{P_L} \pi_1 <_{P_L} \cdots <_{P_L} \pi_{L_1} >_{P_L} \pi_{L_1+1} <_{P_L} \cdots <_{P_L} \pi_{L_1+L_2} >_{P_L} \pi_{L_1+L_2+1} <_{P_L} \cdots.$$

So, $\pi_s <_{P_L} \pi_{s+1}$ (resp., $\pi_s >_{P_L} \pi_{s+1}$) is the same as saying that $i_s \leqslant i_{s+1}$ (resp., $i_s < i_{s+1}$). Let $f\colon \pm[n] \to \mathbb{Z}$ be a function defined by $f(\pi_s) = j_s$, for $0 \leqslant s \leqslant n$, and $f(-s) = -f(s)$. If $\pi_s <_{P_L} \pi_{s+1}$ and $\pi_s < \pi_{s+1}$ (resp., $\pi_s > \pi_{s+1}$), then (i) (resp., (iv)) implies that $f(\pi_s) = j_s \leqslant j_{s+1} = f(\pi_{s+1})$ (resp., $f(\pi_s) = j_s < j_{s+1} = f(\pi_{s+1})$). If $\pi_s >_{P_L} \pi_{s+1}$ and $\pi_s < \pi_{s+1}$ (resp., $\pi_s > \pi_{s+1}$), then (ii) (resp., (iii)) implies that $f(\pi_s) = j_s > j_{s+1} = f(\pi_{s+1})$ (resp., $f(\pi_s) = j_s \geqslant j_{s+1} = f(\pi_{s+1})$). But these conditions say exactly that f is a P-partition of P_L. So, the right hand side of (2.1) can be written as

$$\sum_{|L|=n} \tilde{F}_L(X) \cdot \Gamma(P_L)(Y), \tag{2.2}$$

where $\Gamma(P_L) = \sum_{\tau \in L(P_L)} \tilde{F}_{C(\tau)}$. A signed permutation $\tau \in L(P_L)$ means $\tau^{-1}(\pi_1) < \cdots < \tau^{-1}(\pi_{L_1}) > \tau^{-1}(\pi_{L_1+1}) < \cdots$, which is the same as saying that $C(\tau^{-1}\pi) = L$. Let now $\sigma = \tau^{-1}\pi$, so that $\tau\sigma = \pi$ and $C(\sigma) = L$. Then (2.2) becomes

$$\sum_{|L|=n} \sum_{C(\sigma)=L} \tilde{F}_{C(\sigma)}(X) \sum_{\tau\sigma=\pi} \tilde{F}_{C(\tau)}(Y) = \sum_{\tau\sigma=\pi} \tilde{F}_{C(\sigma)}(X)\tilde{F}_{C(\tau)}(Y),$$

proving the theorem. ■

Example 2.3.5. In B_2, $\bar{2}1 = (12)(\bar{2}1) = (\bar{1}2)(\bar{2}\bar{1}) = (1\bar{2})(21) = (\bar{1}\bar{2})(2\bar{1}) = (21)(\bar{1}2) = (\bar{2}1)(12) = (2\bar{1})(\bar{1}\bar{2}) = (\bar{2}\bar{1})(1\bar{2})$. Thus,

$$\begin{aligned} \delta\tilde{F}_{(0,2)} &= \tilde{F}_{(0,2)} \otimes \tilde{F}_{(2)} + \tilde{F}_{(2)} \otimes \tilde{F}_{(0,2)} + \tilde{F}_{(0,2)} \otimes \tilde{F}_{(0,2)} + \tilde{F}_{(1,1)} \otimes \tilde{F}_{(1,1)} \\ &\quad + \tilde{F}_{(1,1)} \otimes \tilde{F}_{(0,1,1)} + \tilde{F}_{(0,1,1)} \otimes \tilde{F}_{(1,1)} + \tilde{F}_{(0,2)} \otimes \tilde{F}_{(1,1)} + \tilde{F}_{(1,1)} \otimes \tilde{F}_{(0,2)}. \quad \square \end{aligned}$$

We may as well consider the internal coproduct $\delta\tilde{M}_I$ of monomial quasi-symmetric functions which has a simple combinatorial rule.

Let $k, l \geqslant 0$. Consider templates M of the following form

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0l} \\ & b_{11} & b_{12} & \cdots & b_{1l} \\ a_{10} & a_{11} & a_{12} & \cdots & a_{1l} \\ & \vdots & \vdots & \ddots & \vdots \\ & b_{k1} & b_{k2} & \cdots & b_{kl} \\ a_{k0} & a_{k1} & a_{k2} & \cdots & a_{kl} \end{pmatrix},$$

where a_{ij}, b_{ij} are non-negative integers. Define the row sum word $p(M)$ to be $p(M) = p_0 p_1 \cdots p_k$, where

$$p_0 = a_{00} + \sum_{j=1}^{l} a_{0j},$$
$$p_i = a_{i0} + \sum_{j=1}^{l} (a_{ij} + b_{ij}), \quad 1 \leqslant i \leqslant k.$$

Define the column sum word $q(M)$ to be $q(M) = q_0 q_1 \cdots q_l$, where

$$q_0 = a_{00} + \sum_{j=1}^{k} a_{j0},$$
$$q_i = a_{0i} + \sum_{j=1}^{k} (a_{ji} + b_{ji}), \quad 1 \leqslant i \leqslant l.$$

Define the reading word $w(M)$ to be

$$a_{00} a_{01} \cdots a_{0l} b_{1l} \cdots b_{11} a_{10} a_{11} \cdots a_{1l} \cdots a_{kl},$$

that is, we read the rows of M from left to right on a lines and from right to left on b lines, with non-leading zeros ignored.

EXAMPLE 2.3.6. The template $M = \begin{pmatrix} 0 & 1 & 0 \\ & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}$ has row sum word $p(M) = (1, 3)$, column sum word $q(M) = (2, 2)$, and reading word $w(M) = 0112$. Note that

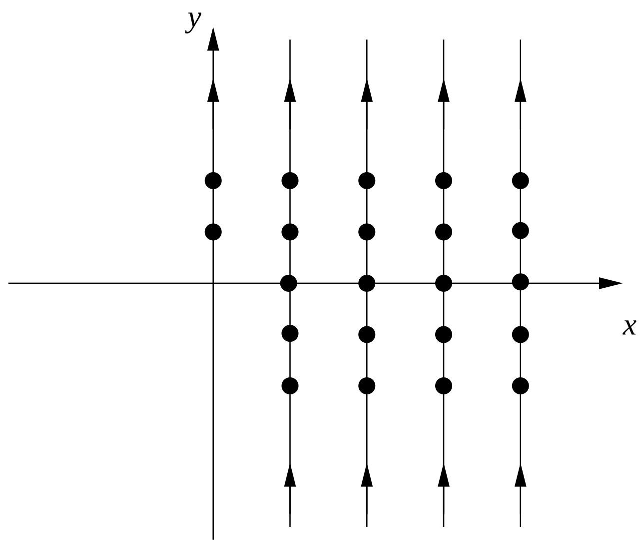

Figure 2-3: Lattice points $(0,0) < (i,j)$ in lexicographic order

the (leading) 0 at the upper left corner is kept while other (non-leading) 0's are dropped. $\square$

REMARK 2.3.7. The templates M just defined are closely related to, but differ slightly from, those defined by Bergeron and Bergeron [4] for computing products of descent classes in the descent algebra of B_n. The difference lies in the ways in which B_n quasi-symmetric functions and the descent classes of B_n are indexed as well as the order of operands. More precisely, Bergeron and Bergeron index descent classes by compositions $p \models m \leqslant n$ while we index B_n quasi-symmetric functions by pseudo-compositions of n. However, we can convert Bergeron and Bergeron's indexing composition $p \models m \leqslant n$ into a pseudo-composition of n by simply prepending $n-m$ to p.

With the row sum word, column sum word, and reading word defined, we can describe the combinatorial rule for computing $\delta\tilde{M}_I$.

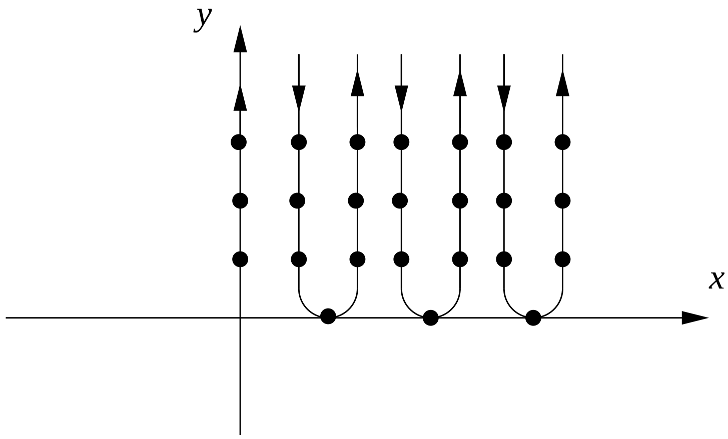

Figure 2-4: Lattice points $(0,0) < (i,j)$ in lexicographic order reflected with respect to the x-axis

THEOREM 2.3.8. *Let $r = (r_0, r_1, \ldots, r_t) \models n$. Then*

$$\delta \tilde{M}_r = \sum_{w(M)=r} \tilde{M}_{p(M)} \otimes \tilde{M}_{q(M)},$$

where the sum ranges over all templates M with reading word $w(M) = r$ and row sum word $p(M) \preccurlyeq r$.

Proof. By definition,

$$\tilde{M}_r(XY) = \sum_{(i_0,j_0)<(i_1,j_1)<\cdots<(i_t,j_t)} x_0^{r_0} x_{i_1}^{r_1} \cdots x_{i_t}^{r_t} y_0^{r_0} y_{j_1}^{r_1} \cdots y_{j_t}^{r_t},$$

where $i_0 = j_0 = 0$. The inequality $(i_s, j_s) < (i_{s+1}, j_{s+1})$ is equivalent to $i_s < i_{s+1}$ or, $i_s = i_{s+1}$ and $j_s < j_{s+1}$, $s = 0, 1, \ldots, t-1$. So, the chain of lexicographic inequalities for indices $(i_0, j_0) < (i_1, j_1) < \cdots < (i_t, j_t)$ can be represented by points in $\mathbb{Z}^2$ with ascending order as depicted in Figure 2-3. Since $y_{-i} = y_i$, those points in the lower-half plane can be reflected with respect to the x-axis, giving a folded path, as shown in Figure 2-4. Let k, l be respectively the number of distinct $i_s > 0$ and $|j_s| > 0$, and $u_0 = 0 < u_1 < \cdots < u_k$ and $v_0 = 0 < v_1 < \cdots < v_l$ be their values. We can represent the summation configuration by a template M with a_{ij} (resp., b_{ij}) equal to the exponent of y_{j_s} with $i_s = u_i$ and $j_s = v_j$ (resp., $-j_s = v_j$), for $0 \leqslant i \leqslant k$, $0 \leqslant j \leqslant l$, and 0 otherwise. The reading word $w(M)$ of M is clearly r, and $p(M) \preccurlyeq r$. Conversely, a template M with reading word r and column sum word $p(M) \preccurlyeq r$ clearly corresponds to a summand of $\tilde{M}_r(XY)$. ■

EXAMPLE 2.3.9. Consider $\delta\tilde{M}_{(1,2)}$. Those templates with reading word $(1,2)$ and row sum word $\preccurlyeq (1,2)$ are

$$\begin{pmatrix} 1 \\ \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ & 0 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ & 2 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \end{pmatrix}.$$

Thus,

$$\delta\tilde{M}_{(1,2)} = \tilde{M}_{(1,2)} \otimes \tilde{M}_{(3)} + 2\tilde{M}_{(1,2)} \otimes \tilde{M}_{(1,2)} + \tilde{M}_{(3)} \otimes \tilde{M}_{(1,2)}. \quad \square$$

The internal coproduct δ for QSym has a counit ε' defined by $\varepsilon'(F_I) = F_I$ if $l(I) = 1$ and 0 otherwise. This counit extend readily to BQSym with the same definition.

2.4 Two specializations

Let $X = \{x_0, x_1, \dots\}$ be an infinite alphabet of commuting indeterminates. Denote by BQSym(X) the algebra of quasi-symmetric functions of type B of alphabet X. Define a homomorphism $\Lambda_m\colon$ BQSym$_r(X) \to K[m]$ by setting, for each $f \in$ BQSym$_r(X)$, $x_k = 1$ for $|k| \leqslant m$, and $x_k = 0$ for $k > m$.

In particular, if $|I| = r$, then

$$\Lambda_m(\tilde{F}_I) = \sum_{\substack{0 \leqslant i_1 \leqslant \cdots \leqslant i_r \leqslant m \\ j \in S(I) \Rightarrow i_j < i_{j+1}}} 1. \tag{2.3}$$

Recall the “double-choose” notation [32],

$$\left(\!\!\binom{m}{n}\!\!\right) = \binom{m+n-1}{n},$$

which is equal to the number of $1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n \leqslant m$. Thus, the sum in (2.3) is the number of $0 \leqslant i_1 \leqslant \cdots \leqslant i_r \leqslant m$ with $l(I)-1$ strict inequalities. But this is exactly the order polynomial of type B, $\Omega_B(m)$, defined as

$$\begin{aligned}\Omega_B(m) = &\ \text{number of } B_r\ P\text{-partitions } f\colon \pm[r] \to \pm[m] \\ = &\ \left(\!\!\binom{m+2-l(I)}{r}\!\!\right) = \binom{m+1+r-l(I)}{r}.\end{aligned}$$

Let $\pi \in B_r$ with $C(\pi) = I$. Denote by $\gamma^I_{J,K}$ the number of pairs $(\tau, \sigma) \in B_n \times B_n$ such that $\tau\sigma = \pi$, $C(\tau) = K$, and $C(\sigma) = J$. Theorem 2.3.4 can be restated as

$$\tilde{F}_I(XY) = \sum_{J,K} \gamma^I_{J,K} \tilde{F}_J(X)\tilde{F}_K(Y),$$

with the sum ranges over all pseudo-compositions J, K of r.

Proposition 2.4.1. *Let $\pi \in B_r$. Then*

$$\binom{2mn+m+n+r-d(\pi)}{r} = \sum_{\tau\sigma=\pi} \binom{m+r-d(\sigma)}{r}\binom{n+r-d(\tau)}{r},$$

where $d(\pi)$ is the descent number of π.

Proof. Let $k = l(C(\pi))$. Apply $\Lambda_m \otimes \Lambda_n$ to $\delta\tilde{F}_{C(\pi)}$. It is easy to see that $(\Lambda_m \otimes \Lambda_n)(\delta\tilde{F}_{C(\pi)})$ is equal to the number of $(0,0) \leqslant (i_1, j_1) \leqslant \cdots \leqslant (i_k, j_k) \leqslant (m, n)$ in lexicographic order with $k-1$ inequalities strict, with $|i_s| \leqslant m$, $|j_s| \leqslant n$, for $s = 1, 2, \ldots, k$. This number is equal to

$$\left(\!\!\binom{2mn+m+n+1-d(\pi)}{r}\!\!\right) = \binom{2mn+m+n+r-d(\pi)}{r},$$

whence the proposition. ■

By replacing m by $(m-1)/2$, and n by $(n-1)/2$, Proposition 2.4.1 becomes

$$\binom{\frac{mn-1}{2}+r-d(\pi)}{r} = \sum_{\tau\sigma=\pi} \binom{\frac{m-1}{2}+r-d(\sigma)}{r}\binom{\frac{n-1}{2}+r-d(\tau)}{r}.$$

Now define an element of the group algebra $K[B_r]$ by

$$\phi(m) = \sum_{\pi\in B_r} \binom{\frac{m-1}{2}+r-d(\pi)}{r}\pi. \tag{2.4}$$

PROPOSITION 2.4.2. *The element $\phi(m)$ satisfies $\phi(mn)=\phi(m)\phi(n)$.*

Proof. Just compute

$$\begin{aligned}
\phi(m)\phi(n) &= \sum_{\sigma,\tau\in B_r} \binom{\frac{m-1}{2}+r-d(\sigma)}{r}\binom{\frac{n-1}{2}+r-d(\tau)}{r}\tau\sigma \\
&= \sum_{\pi\in B_r}\pi \sum_{\tau\sigma=\pi} \binom{\frac{m-1}{2}+r-d(\sigma)}{r}\binom{\frac{n-1}{2}+r-d(\tau)}{r} \\
&= \sum_{\pi\in B_r} \binom{\frac{mn-1}{2}+r-d(\pi)}{r}\pi \\
&= \phi(mn). \quad \blacksquare
\end{aligned}$$

Now define elements e_i of the group algebra $K[B_r]$ by $\phi(m) = \sum_{i=0}^{r} m^i e_i$.

THEOREM 2.4.3. *The elements e_i are orthogonal idempotents.*

Proof. Equating the coefficient of $m^i n^j$ on both sides of

$$\sum_{i=0}^{r} m^i e_i \sum_{j=0}^{r} n^j e_j = \phi(m)\phi(n) = \phi(mn) = \sum_{i=0}^{r}(mn)^i e_i,$$

we get that $e_i e_j = \delta_{i,j} e_i$, as desired. $\blacksquare$

In chapter 5 we will study a subalgebra of the group algebra of B_r containing $\phi(m)$.

Let $I = (i_1, i_2, \dots, i_k)$ is a pseudo-composition of n. Define the major index, $\text{maj}(I)$, of I to be the sum of all elements of the subset $S(I)$ of $[0, n-1]$ associated to I. More precisely, the subset $S(I)$ associated to I is $\{i_1, i_1 + i_2, \dots, i_1 + \dots + i_{k-1}\}$, so that

$$\text{maj}(I) = i_1 + (i_1 + i_2) + \cdots + (i_1 + \cdots + i_{k-1}) = \sum_{j=1}^{k-1} (k-j) i_j.$$

Define now a homomorphism $\Lambda_m^q\colon\ \text{BQSym} \to K(q)$ by $\Lambda_m^q(x_i) = q^{|i|}$ if $0 \leqslant |i| \leqslant m$ and 0 otherwise, and extend algebraically to all of BQSym. We can describe $\Lambda_m^q(\tilde{F}_C)$ by generating function, as follows. The proof is in the same spirit as the corresponding result for QSym, considered in [17].

Proposition 2.4.4. *Let $C = (c_1, \dots, c_k)$ be a pseudo-composition of r. Then*

$$\sum_{m \geqslant 0} t^m \Lambda_m^q(\tilde{F}_C) = \frac{t^{l(C)-1} q^{\text{maj}(C)}}{(t;q)_{r+1}}.$$

Proof. From the definitions of Λ_m^q and $\tilde{F}_C$,

$$\sum_{m \geqslant 0} t^m \Lambda_m^q(\tilde{F}_C) = \sum_{m \geqslant 0} \sum_{\substack{0 \leqslant i_1 \leqslant \cdots \leqslant i_r \leqslant m \\ s \in S(C) \Rightarrow i_s < i_{s+1}}} t^m q^{i_1 + \cdots + i_r}.$$

Now set $j_s = i_{s+1} - i_s - \epsilon_s$, $i_0 = 0$, $\epsilon_s = \chi(s \in S(C))$, $0 \leqslant s \leqslant r-1$, and $j_r = m - i_r$. The conditions $0 \leqslant i_1 \leqslant \cdots \leqslant i_r \leqslant m$ and $i_s < i_{s+1}$ if $s \in S(C)$ are equivalent to $j_s \geqslant 0$, $0 \leqslant s \leqslant r$. Solving for i_s, we have $i_s = \sum_{l=0}^{s-1} (j_l + \epsilon_l)$, $1 \leqslant s \leqslant r$, and

$m = j_r + i_r = j_0 + \cdots + j_r + \epsilon_0 + \cdots + \epsilon_{r-1}$. Thus,

$$\begin{aligned}
&\sum_{m \geqslant 0} \sum_{\substack{0 \leqslant i_1 \leqslant \cdots \leqslant i_r \leqslant m \\ s \in S(C) \Rightarrow i_s < i_{s+1}}} t^m q^{i_1 + \cdots + i_r} \\
&\quad = \sum_{j_0, \ldots, j_r \geqslant 0} t^{j_0 + \cdots + j_r + \epsilon_0 + \cdots + \epsilon_{r-1}} q^{r(j_0 + \epsilon_0) + (r-1)(j_1 + \epsilon_1) + \cdots + j_{r-1} + \epsilon_{r-1}} \\
&\quad = \sum_{j_0 \geqslant 0} (tq^r)^{j_0} \cdots \sum_{j_{r-1} \geqslant 0} (tq)^{j_{r-1}} \sum_{j_r \geqslant 0} t^{j_r} \cdot t^{\epsilon_0 + \cdots + \epsilon_{r-1}} q^{r\epsilon_0 + (r-1)\epsilon_1 + \cdots + \epsilon_{r-1}} \\
&\quad = \frac{t^{l(C)-1} q^{\mathrm{maj}(C)}}{(t;q)_{r+1}}. \quad \blacksquare
\end{aligned}$$

We have the following well-known result of Euler [18]:

THEOREM 2.4.5 (Euler).

$$\sum_{m \geqslant 0} \begin{bmatrix} m+p \\ p \end{bmatrix}_q t^m = \frac{1}{(1-t)(1-tq)(1-tq^2)\cdots(1-tq^p)}.$$

A comparison of Proposition 2.4.4 and Theorem 2.4.5 then yields that

COROLLARY 2.4.6. $\Lambda_m^q(\tilde{F}_C) = q^{\mathrm{maj}(C)} \begin{bmatrix} m+1+r-l(C) \\ r \end{bmatrix}_q.$

Chapter 3

Noncommutative Symmetric Functions of Type B

In this chapter we introduce the central object of our study, namely the noncommutative symmetric functions of type B, denoted **BSym**. In the first section, the module structure of **BSym** is demonstrated by giving the multiplication rules for two bases of **BSym**. A coalgebra structure of **BSym**, which is dual to the algebra structure of BQSym, is discussed in the second section. In the third section, the internal product is defined to endow **BSym** an algebra structure. In the fourth section, a rule which facilitates the computation of internal products is given. This chapter is closed by a discussion of the connection between **BSym** and the descent algebra of type B, studied by Bergeron and Bergeron [4], [3].

3.1 BSym

In this section we define the noncommutative symmetric functions of type B.

Let S_i be, as in the case of **Sym**, noncommuting indeterminates homogeneous of degree $i \geqslant 0$ with $S_0 = 1$. Let $\tilde{S}_i$ be another set of noncommuting indeterminates with $\tilde{S}_0 = 1$. Here S_i's play the role of complete homogeneous symmetric functions. Define the noncommutative symmetric functions **BSym** to be the free right **Sym**-module with a set of basis elements of the form

$$\tilde{S}^J = \tilde{S}_{j_1} S_{j_2} \cdots S_{j_k},$$

where $J = (j_1, j_2, \ldots, j_k)$ ranges over all pseudo-compositions. Note that if $j_1 = 0$ then $\tilde{S}^J = S_{j_2} \cdots S_{j_k} \in \mathbf{Sym}$. Thus, **Sym** sits right inside **BSym** as a submodule. The product in **BSym** is defined on basis elements by

$$\tilde{S}^I S^J = \tilde{S}^{I \cdot J}, \quad \tilde{S}^I \in \mathbf{BSym}, \quad S^J \in \mathbf{Sym},$$

and by linear extension to all of **BSym**, where $I \cdot J$ denotes the concatenation of pseudo-composition I and composition J.

Moreover, **BSym** is naturally graded,

$$\mathbf{BSym} = \bigoplus_{n \geqslant 0} \mathbf{BSym}_n,$$

where $\mathbf{BSym}_n$ is the right **Sym**-submodule spanned by $\tilde{S}^J$ with $J \models n$.

Also define the ribbon Schur function $\tilde{R}_I$ by

$$\tilde{S}^I = \sum_{J \preccurlyeq I} \tilde{R}_J.$$

It follows from inclusion-exclusion that

$$\tilde{R}_I = \sum_{J \preccurlyeq I} (-1)^{l(I)-l(J)} \tilde{S}^J.$$

We can describe the above relations between $\tilde{S}^I$ and $\tilde{R}_I$ in terms of transition matrices. We first note that the poset of pseudo-compositions of n is in bijective correspondence to the poset of compositions of $n+1$. The poset of pseudo-compositions of n, ordered by reverse refinement, can be obtained from that of compositions of $n+1$ by subtracting 1 from the first part of each element of the latter. This operation is clearly a bijection, the inverse being that which adds 1 to the first part of each pseudo-composition of n.

We shall order the pseudo-compositions in reverse lexicographic order. For instance,

$$(3) < (2,1) < (1,2) < (1,2) < (0,3) < (0,2,1) < (0,1,2) < (0,1,1,1).$$

When $n = 1$, $M_1(\tilde{R}, \tilde{S})\tilde{R} = \tilde{S}$, where $\tilde{S} = \begin{pmatrix} \tilde{S}^{(0,1)} \\ \tilde{S}^{(1)} \end{pmatrix}$, $\tilde{R} = \begin{pmatrix} \tilde{R}_{(0,1)} \\ \tilde{R}_{(1)} \end{pmatrix}$, and $M_1(\tilde{R}, \tilde{S}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The inverse of $M_1(\tilde{R}, \tilde{S})$ is $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, which is precisely the transition matrix $M_1(\tilde{S}, \tilde{R})$. For $n > 1$, the transition matrix $M_n(\tilde{R}, \tilde{S})$ is simply the Kronecker product of $M_1(\tilde{R}, \tilde{S})$. For instance,

$$M_2(\tilde{R}, \tilde{S}) = M_1(\tilde{R}, \tilde{S})^{\otimes 2} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is clear that the transition matrix is integral upper triangular with 1's on the main diagonal and is thus invertible. Therefore, $\tilde{R}_I$'s also form a basis of **BSym**.

Let $I = (i_1, i_2, \ldots, i_r)$, and $J = (j_1, j_2, \ldots, j_s)$. Let us define

Definition 3.1.1. $I \lhd J = (i_1, i_2, \ldots, i_r + j_1, j_2, \ldots, j_s)$.

In words, $I \lhd J$ is the pseudo-composition obtained by adding the head of J to the tail of I. As we saw from above that the product $\tilde{S}^I S^J$ is described by the concatenation of I and J. With $I \lhd J$ defined, we can describe the product of $\tilde{R}_I R_J$, as follows.

LEMMA 3.1.2. *Let I be a pseudo-composition and J a composition. Then $\tilde{R}_I R_J = \tilde{R}_{I \lhd J} + \tilde{R}_{I \cdot J}$.*

Proof. Write $I = (i_1, \ldots, i_r)$ and $J = (j_1, \ldots, j_s)$. Then

$$\tilde{R}_{I \cdot J} + \tilde{R}_{I \lhd J} = \sum_{K \preccurlyeq I \cdot J} (-1)^{l(I \cdot J) - l(K)} \tilde{S}^K + \sum_{K \preccurlyeq I \lhd J} (-1)^{l(I \lhd J) - l(K)} \tilde{S}^K.$$

Note that $S(I \lhd J) \subsetneqq S(I \cdot J)$ since $i_1 + \cdots + i_r \in S(I \cdot J) \setminus S(I \lhd J)$. So, $K \preccurlyeq I \cdot J$ and $K \not\preccurlyeq I \lhd J$ iff $i_1 + \cdots + i_r \in S(K)$. Since $l(I \cdot J) = l(I) + l(J) = l(I \lhd J) + 1$, the part of the first sum with $K \preccurlyeq I \lhd J$ exactly cancels the second sum, leaving

$$\tilde{R}_{I \cdot J} + \tilde{R}_{I \lhd J} = \sum_{\substack{K \preccurlyeq I \cdot J \\ K \not\preccurlyeq I \lhd J}} (-1)^{l(I) + l(J) - l(K)} \tilde{S}^K.$$

Now, for $K = (k_1, \ldots, k_t) \preccurlyeq I \cdot J$ and $K \not\preccurlyeq I \lhd J$, let q be such that $k_1 + \cdots + k_q = i_1 + \cdots + i_r$. Then $K = P \cdot Q$, where $P = (k_1, \ldots, k_q) \preccurlyeq I$ and $Q = (k_{q+1}, \ldots, k_t) \preccurlyeq J$. Thus,

$$\tilde{R}_{I \cdot J} + \tilde{R}_{I \lhd J} = \sum_{P \preccurlyeq I} (-1)^{l(I) - l(P)} \tilde{S}^P \sum_{Q \preccurlyeq J} (-1)^{l(J) - l(Q)} \tilde{S}^Q = \tilde{R}_I R_J. \quad \blacksquare$$

EXAMPLE 3.1.3. We have $\tilde{R}_{(3,2)} R_{(2,1)} = \tilde{R}_{(3,2,2,1)} + \tilde{R}_{(3,4,1)}$. $\square$

Let I be a composition. We shall write $(0, I)$ for the pseudo-composition $(0) \cdot I$.

LEMMA 3.1.4. *Let I be a composition. Then $R_I = \tilde{R}_{(0,I)} + \tilde{R}_I$.*

Proof. By inclusion-exclusion,

$$\tilde{R}_{(0,I)} + \tilde{R}_I = \sum_{J \preccurlyeq (0,I)} (-1)^{l(I)+1-l(J)} \tilde{S}^J + \sum_{J \preccurlyeq I} (-1)^{l(I)+1-l(J)} \tilde{S}^J.$$

For $J \preccurlyeq (0,I)$, J is of the form either $(0,J')$ or J', where $J' \preccurlyeq I$. So, the part of the first sum with $J \preccurlyeq I$ exactly cancels the second sum. Thus,

$$\tilde{R}_{(0,I)} + \tilde{R}_I = \sum_{\substack{(0,J)\preccurlyeq(0,I) \\ J\preccurlyeq I}} (-1)^{l(I)+1-l((0,J))}\tilde{S}^{(0,J)} = \sum_{J\preccurlyeq I}(-1)^{l(I)-l(J)}S^J = R_I. \quad \blacksquare$$

In the sequel, we write R_I as a short hand for $\tilde{R}_{(0,I)} + \tilde{R}_I$, with I a composition.

3.2 Duality

There are certain dualities between the products in **Sym** and the coproducts in QSym. Similar dualities exist between **BSym** and BQSym and are made precise in this section.

Define a pairing $\langle\cdot,\cdot\rangle\colon$ BQSym $\times$ **BSym** $\to K$ by requiring

$$\langle \tilde{M}_I, \tilde{S}^J\rangle = \delta_{IJ},$$

i.e., $\tilde{M}_I$ is dual to $\tilde{S}^I$. As $\tilde{F}_I$ and $\tilde{M}_I$ are also related by inclusion-exclusion, it is not surprising that

Lemma 3.2.1. $\langle \tilde{F}_I, \tilde{R}_J\rangle = \delta_{IJ}$.

Proof. By inclusion-exclusion,

$$\begin{aligned}\langle \tilde{F}_I, \tilde{R}_J\rangle &= \left\langle \sum_{I\preccurlyeq K}\tilde{M}_K, \sum_{L\preccurlyeq J}(-1)^{l(J)-l(L)}\tilde{S}^L\right\rangle = \sum_{I\preccurlyeq K}\sum_{L\preccurlyeq J}(-1)^{l(J)-l(L)}\delta_{KL} \\ &= \sum_{I\preccurlyeq L\preccurlyeq J}(-1)^{l(J)-l(L)} = \sum_{S(I)\subseteq T\subseteq S(J)}(-1)^{\#S(J)-\#T}.\end{aligned}$$

Let $i = \#S(I)$, and $j = \#S(J)$. There exist $\binom{j-i}{k}$ subsets T of $S(J)$ such that

$S(I) \subseteq T$ and $\#(T \setminus S(I)) = k$. The above sum is then equal to

$$\sum_{k \geqslant 0} (-1)^{j-(k+i)} \binom{j-i}{k} = (-1)^{j-i}(1-1)^{j-i} = \begin{cases} 0 & \text{if } j > i \\ 1 & \text{if } j = i \end{cases},$$

which is equal to δ_{IJ}. ■

Recall the coproduct $\Delta\colon \mathbf{Sym} \to \mathbf{Sym} \otimes \mathbf{Sym}$ defined by

$$\Delta S_n = \sum_{i=0}^{n} S_i \otimes S_{n-i},$$

and extended algebraically to all of $\mathbf{Sym}$. This coproduct is readily extended to a coproduct $\Delta\colon \mathbf{BSym} \to \mathbf{BSym} \otimes \mathbf{BSym}$ by defining

$$\Delta \tilde{S}_n = \sum_{i=0}^{n} \tilde{S}_i \otimes \tilde{S}_{n-i},$$
$$\Delta(\tilde{S}_{j_1} S_{j_2} \cdots S_{j_k}) = (\Delta \tilde{S}_{j_1})(\Delta S_{j_2}) \cdots (\Delta S_{j_k}),$$

and extended linearly to all of $\mathbf{BSym}$.

Recall also the map $\gamma_B\colon \mathrm{BQSym} \to \mathrm{BQSym} \otimes \mathrm{QSym}$ defined by $\gamma_B(f) = f(X + Y)$, for $f \in \mathrm{BQSym}$.

Proposition 3.2.2. *We have*

(i) $\langle f \otimes g, \Delta H \rangle = \langle fg, H \rangle$, $f, g \in \mathrm{BQSym}$, $H \in \mathbf{BSym}$*;*

(ii) $\langle \gamma_B(f), G \otimes H \rangle = \langle f, GH \rangle$, $f \in \mathrm{BQSym}$, $G \in \mathbf{BSym}$, $H \in \mathbf{Sym}$.

Proof. It suffices to show that (i) and (ii) hold for $f = \tilde{M}_I$, $G = \tilde{S}^J$. The general case follows from linearity.

(i) Let $f = \tilde{M}_I$, $g = \tilde{M}_J$, and $H = \tilde{S}^K$. Write $\tilde{M}_I \tilde{M}_J = \sum_K \beta_{IJ}^L \tilde{M}_L$, where β_{IJ}^K can be combinatorially interpreted as the number of ways of obtaining K by adding

J to I, with first parts left justified and with insertion of zeros after the first part allowed. See Example 3.2.3. The right hand side of (i) then becomes

$$\langle \tilde{M}_I \tilde{M}_J, \tilde{S}^K \rangle = \sum_L \beta_{IJ}^L \langle \tilde{M}_L, \tilde{S}^K \rangle = \sum_L \beta_{IJ}^L \delta_{LK} = \beta_{IJ}^K.$$

Write $K = (k_1, \dots, k_r)$.

$$\begin{aligned} \Delta S^K = \; & (\Delta \tilde{S}_{k_1})(\Delta S_{k_2}) \cdots (\Delta S_{k_r}) = \sum_{p_i+q_i=k_i} \tilde{S}_{p_1} S_{p_2} \cdots S_{p_r} \otimes \tilde{S}_{q_1} S_{q_2} \cdots S_{q_r} \\ = \; & \sum_{P+Q=K} \tilde{S}^P \otimes \tilde{S}^Q, \end{aligned}$$

where the sum is over all pseudo-compositions $P = (p_1, \dots, p_r)$, $Q = (q_1, \dots, q_r)$, with $p_i, q_i \geqslant 0$, and $P + Q = K$ partwise. Now,

$$\langle \tilde{M}_I \otimes \tilde{M}_J, \Delta S^K \rangle = \sum_{P+Q=K} \langle \tilde{M}_I, \tilde{S}^P \rangle \langle \tilde{M}_J, \tilde{S}^Q \rangle,$$

where $\langle \tilde{M}_I, \tilde{S}^P \rangle = 1$ if I can be obtained from P by dropping the zeros after the first part of P, and 0 otherwise, and similarly for $\langle \tilde{M}_J, \tilde{S}^Q \rangle$. But these conditions together with $P + Q = K$ say exactly that the sum above is equal to β_{IJ}^K, proving (i).

(ii) Let $f = \tilde{M}_I$, $G = \tilde{S}^J$, $H = S^K$. Write $\gamma_B(\tilde{M}_I) = \sum_{P \cdot Q = I, l(P) \geqslant 1} \tilde{M}_P \otimes M_Q$. Then

$$\langle \gamma_B(\tilde{M}_I), \tilde{S}^J \otimes S^K \rangle = \sum_{P \cdot Q = I, l(P) \geqslant 1} \langle \tilde{M}_P, \tilde{S}^J \rangle \langle M_Q, S^K \rangle,$$

which is 1 if $P = J$, $Q = K$ and $P \cdot Q = I$, and 0 otherwise. On the other hand,

$$\langle \tilde{M}_I, \tilde{S}^J S^K \rangle = \langle \tilde{M}_I, \tilde{S}^{J \cdot K} \rangle = \delta_{I, J \cdot K},$$

where $J \cdot K$ is the pseudo-composition obtained by concatenating J and K. (ii) follows. ■

Example 3.2.3. The right hand side of

$$\tilde{M}_{(1,2)}\tilde{M}_{(0,1,1)} = \sum_{0<i} x_0 x_i^2 \sum_{0<j<k} x_j x_k$$

can be expanded as a linear combination of monomial quasi-symmetric functions by extending the disjoint union of the two chains $0 < i$ and $0 < j < k$ to a total order. It is easy to see that the resulting chains of inequalities are

$$0 < j < k < i, \quad 0 < j < k = i, \quad 0 < j < i < k, \quad 0 < j = i < k, \quad 0 < i < j < k,$$

for which the corresponding monomial quasi-symmetric functions are

$$\tilde{M}_{(1,1,1,2)}, \quad \tilde{M}_{(1,1,3)}, \quad \tilde{M}_{(1,1,2,1)}, \quad \tilde{M}_{(1,3,1)}, \quad \tilde{M}_{(1,2,1,1)},$$

which can in turn be represented as the results of left justified addition of $(1,2)$ and $(0,1,1)$, listed below.

$$\begin{array}{ccccc} & 1 & & & 2 \\ + & 0 & 1 & 1 & \\ \hline & 1 & 1 & 1 & 2 \end{array}, \quad \begin{array}{cccc} & 1 & & 2 \\ + & 0 & 1 & 1 \\ \hline & 1 & 1 & 3 \end{array}, \quad \begin{array}{ccccc} & 1 & & 2 & \\ + & 0 & 1 & & 1 \\ \hline & 1 & 1 & 2 & 1 \end{array},$$

$$\begin{array}{cccc} & 1 & 2 & \\ + & 0 & 1 & 1 \\ \hline & 1 & 3 & 1 \end{array}, \quad \begin{array}{ccccc} & 1 & 2 & & \\ + & 0 & & 1 & 1 \\ \hline & 1 & 2 & 1 & 1 \end{array} \quad \square$$

Remark 3.2.4. The external coproduct $\gamma\colon$ QSym $\to$ QSym $\otimes$ QSym was extended to just a map $\gamma_B\colon$ BQSym $\to$ BQSym $\otimes$ QSym. This choice of extension allows the duality as in Proposition 3.2.2 (ii). If γ were extended to a coproduct by allowing the second tensor component of $\gamma_B(f)$ to be in BQSym, then the module structure of **BSym** would be more complicated, as we would have to keep track of the occurrences of $\tilde{S}_i$ in $\tilde{S}^I$.

The coproduct Δ has a counit ε, defined by $\varepsilon(G)$ = constant term of G, where $G \in \mathbf{BSym}$. Now, we have the right to say

PROPOSITION 3.2.5. *The triple* $(\mathbf{BSym}, \Delta, \varepsilon)$ *is a coalgebra.*

3.3 The internal product

We shall define in this section an important operation, namely the internal product $*$, in $\mathbf{BSym}$. Recall the internal coproduct δ defined in BQSym by $\delta(f) = f(XY)$, $f \in \mathrm{BQSym}$. We now define the internal product $*\colon\ \mathbf{BSym}_n \times \mathbf{BSym}_n \to \mathbf{BSym}_n$ to be dual to the coproduct δ on BQSym_n, that is,

$$\langle \delta(f), G \otimes H \rangle = \langle f, G * H \rangle, \quad f \in \mathrm{BQSym}_n, \quad G, H \in \mathbf{BSym}_n\,.$$

We can extend $*$ to all of $\mathbf{BSym}$ by defining $G * H = 0$ if $G \in \mathbf{BSym}_m$, $H \in \mathbf{BSym}_n$, $m \neq n$. We shall write $*_B$ in the sequel to indicate explicitly the internal product for $\mathbf{BSym}$, in contrast to $*_A$ defined in $\mathbf{Sym}$.

Since $*_B$ is defined to be dual to δ, the multiplication rule for $*_B$ follows from that for δ in a dual manner. See Chapter 2 for the definitions of templates M, reading words $w(M)$, row sum words $p(M)$, and column sum words $q(M)$.

THEOREM 3.3.1. *Let* $p = p_0 p_1 \cdots p_k$, $q = q_0 q_1 \cdots q_l$ *be compositions of* n.

$$\tilde{S}^p *_B \tilde{S}^q = \sum_{p(M)=p,\, q(M)=q} \tilde{S}^{w(M)},$$

where the sum ranges over all templates M *with* $p(M) = p$, $q(M) = q$.

EXAMPLE 3.3.2. Let $j_0 + \cdots + j_k = n$. Then $\tilde{S}_n *_B \tilde{S}_{j_0} S_{j_1} \cdots S_{j_k} = \tilde{S}_{j_0} S_{j_1} \cdots S_{j_k}$ because there is exactly one template M whose column sum word and row sum word are n and $j_0 j_1 \dots j_k$, respectively. The reading word $w(M)$ of M is precisely $j_0 j_1 \dots j_k$. $\square$

EXAMPLE 3.3.3. Let $j_0 + \cdots + j_k = n$. Then

$$S_n *_B \tilde{S}_{j_0} S_{j_1} \cdots S_{j_k} = \sum S_{p_k} \cdots S_{p_1} S_{j_0} S_{q_1} \cdots S_{q_k},$$

where the sum ranges over all integers p_i, $q_i \geqslant 0$ such that $p_i + q_i = j_i$, $1 \leqslant i \leqslant k$. $\square$

Example 3.3.2 and Example 3.3.3 are noteworthy. In **Sym**, the S_n are identity elements with respect to $*_A$. However, is it not the case in **BSym**; the $\tilde{S}_n$ serve as identity elements instead.

3.4 Mackey formula

We obtained the combinatorial rule for computing the internal product $*_B$ of **BSym** in the previous section. In the present section we give a rule, called the Mackey formula, which facilitates the computation of internal product of **BSym**.

Recall the generating function for S_i, $\sigma(t)$, defined in Section 1.2. We define similarly the generating function for $\tilde{S}_i$, denoted by $\tilde{\sigma}(t)$. We showed in Lemma 1.2.4 that $\sigma(t)$ is group-like, and we can show similarly that the same is true for $\tilde{\sigma}(t)$.

Recall that the reading word of a template M for $*_B$ is obtained by reading the top line from left to right and subsequent lines from right to left and then left to right, with non-leading zeros dropped. In contrast, the reading word of a template for $*_A$ is obtained by reading each line left to right, with zeros omitted. This difference of the definition of reading words suggests defining the "straightening" map $\Theta\colon$ **BSym** $\to$ **Sym** by

$$\Theta(\tilde{S}_n) = S_n,$$
$$\Theta(S_n) = \sum_{i=0}^{n} S_i S_{n-i}.$$

and if $F \in \mathbf{BSym}$ and $G \in \mathbf{Sym}$, then $\Theta(FG) = \mu'(\Theta(F) \otimes \Delta G)$ where $\mu'(A \otimes B \otimes C) = \bar{B}AC$, and $F \to \bar{F}$ is reversal, i.e., the anti-automorphism sending $S_n \to S_n$.

EXAMPLE 3.4.1. $\Delta S_3 = S_3 \otimes 1 + S_2 \otimes S_1 + S_1 \otimes S_2 + 1 \otimes S_3$ and $\Theta(\tilde{S}_2) = S_2$ imply that $\Theta(\tilde{S}_2 S_3) = \bar{S}_3 S_2 + \bar{S}_2 S_2 S_1 + \bar{S}_1 S_2 S_2 + S_2 S_3 = S_3 S_2 + S_2 S_2 S_1 + S_1 S_2 S_2 + S_2 S_3$. □

EXAMPLE 3.4.2. (cf. Example 1.2.3) $\Delta(\Phi_n) = \Phi_n \otimes 1 + 1 \otimes \Phi_n$ and Φ_n invariant under reversal imply that $\Theta(\Phi_n) = 2\Phi_n$. $\Delta(\Psi_n) = \Psi_n \otimes 1 + 1 \otimes \Psi_n$ implies that $\Theta(\Psi_n) = \Psi_n + \bar{\Psi}_n$, because Ψ_n is not invariant under reversal. □

Let $J = (j_0, j_1, \dots, j_r)$ be a pseudo-composition, with all $j_i \geqslant 0$. Multiplying the monomial $\tilde{S}^J = \tilde{S}_{j_0} S_{j_1} \cdots S_{j_r}$ by $t_0^{j_0} t_1^{j_1} \cdots t_r^{j_r}$ and and summing over all J, we obtain the generating function for monomials, namely

$$\tilde{\sigma}(t_0)\sigma(t_1)\sigma(t_2)\cdots = \sum_{j_i \geqslant 0} t_0^{j_0} t_1^{j_1} \cdots t_r^{j_r} \tilde{S}_{j_0} S_{j_1} \cdots S_{j_r}.$$

Since $\tilde{\sigma}(t_0)$ and $\sigma(t_i)$ are group-like and Δ is an algebra morphism, it follows that $\tilde{\sigma}(t_0)\sigma(t_1)\sigma(t_2)\cdots$ is group-like. This property of $\tilde{\sigma}(t_0)\sigma(t_1)\sigma(t_2)\cdots$ will be exploited in the proofs that follow.

The map Θ and the iterated coproduct Δ^n satisfy the following "commutation relation."

LEMMA 3.4.3. $\Delta^n \circ \Theta = \Theta^{\otimes n} \circ \Delta^n$.

Proof. The generating function for all monomials $\tilde{S}_{j_1} S_{j_2} \cdots S_{j_k}$ is $\tilde{\sigma}(t_0)\sigma(t_1)\sigma(t_2)\cdots$. By Lemma 1.2.4, $\Delta\sigma(t_1)\sigma(t_2)\cdots = \sigma(t_1)\sigma(t_2)\cdots \otimes \sigma(t_1)\sigma(t_2)\cdots$,

$$\Theta(\tilde{\sigma}(t_0)\sigma(t_1)\cdots) = \overline{\cdots\sigma(t_1)}\Theta(\sigma(t_0))\sigma(t_1)\cdots = \cdots\sigma(t_1)\sigma(t_0)\sigma(t_1)\cdots$$
$$\Delta^n \circ \Theta(\tilde{\sigma}(t_0)\sigma(t_1)\sigma(t_2)\cdots) = \underbrace{\cdots\sigma(t_1)\sigma(t_0)\sigma(t_1)\cdots \otimes \cdots \otimes \cdots\sigma(t_1)\sigma(t_0)\sigma(t_1)\cdots}_{n \text{ times}}$$

On the other hand,

$$\begin{aligned}\Theta^{\otimes n} \circ \Delta^n(\tilde{\sigma}(t_0)\sigma(t_1)\cdots) &= \Theta^{\otimes n}(\tilde{\sigma}(t_0)\sigma(t_1)\cdots \otimes \cdots \otimes \tilde{\sigma}(t_0)\sigma(t_1)\cdots) \\ &= \underbrace{\cdots\sigma(t_1)\sigma(t_0)\sigma(t_1)\cdots \otimes \cdots \otimes \cdots\sigma(t_1)\sigma(t_0)\sigma(t_1)\cdots}_{n \text{ times}}\end{aligned}$$

and the lemma is proved. ■

Let $C = (C, \Delta_C, \varepsilon_C)$, and $D = (D, \Delta_D, \varepsilon_D)$ be coalgebras. A coalgebra map [25] $f\colon C \to D$ is a linear map such that $(f \otimes f) \circ \Delta_C = \Delta_D \circ f$, and $\varepsilon_D \circ f = \varepsilon_C$, where Δ_C, and ε_C are coproduct and counit of C, and similarly for D. So, for $C = \mathbf{BSym}$, $D = \mathbf{Sym}$, and $f = \Theta$, Lemma 3.4.3 expresses that Θ is a coalgebra map (the second condition is trivial because $\Theta(1) = 1$).

Lemma 3.4.4. *For $G \in \mathbf{BSym}_n$, $S_n *_B G = \Theta(G)$.*

Proof. It suffices to consider $S_n *_B \tilde{S}_{j_1} S_{j_2} \cdots S_{j_k}$ with $j_1 + \cdots + j_k = n$. By the multiplication rule

$$S_n *_B \tilde{S}_{j_1} S_{j_2} \cdots S_{j_k} = \sum S_{p_k} \cdots S_{p_2} S_{j_1} S_{q_2} \cdots S_{q_k},$$

where the sum ranges over all $p_i, q_i \geqslant 0$ with sum j_i, $2 \leqslant i \leqslant k$. On the other hand,

$$\begin{aligned}\Delta(S_{j_2} \cdots S_{j_k}) &= \left(\sum_{p_2+q_2=j_2} S_{p_2} \otimes S_{q_2}\right) \cdots \left(\sum_{p_k+q_k=j_k} S_{p_k} \otimes S_{q_k}\right) \\ &= \sum S_{p_2} \cdots S_{p_k} \otimes S_{q_2} \cdots S_{q_k}\end{aligned}$$

where the sum ranges over all $p_i, q_i \geqslant 0$ with sum j_i, $2 \leqslant i \leqslant k$. Thus,

$$\begin{aligned}\Theta(\tilde{S}_{j_1} S_{j_2} \cdots S_{j_k}) &= \sum \overline{S_{p_2} \cdots S_{p_k}} \Theta(\tilde{S}_{j_1}) S_{q_2} \cdots S_{q_k} \\ &= \sum S_{p_k} \cdots S_{p_2} S_{j_1} S_{q_2} \cdots S_{q_k}\end{aligned}$$

and the lemma follows. ■

Recall that S_n serves as identity element for $*_A$. From the proof of Lemma 3.4.4, what Θ does is "straightening out" the template to give a template for computing $*_A$ in **Sym**. Complete reduction of the computation of $*_B$ to $*_A$ is also possible, that is,

PROPOSITION 3.4.5. *If $F \in \mathbf{Sym}$ then $F *_B G = F *_A \Theta(G)$, where $*_A$ is the internal product in* **Sym**.

Proof. It suffices to prove this when $F = S_{j_1} \cdots S_{j_k}$. The proof proceeds by induction on k. The case $k = 1$ is simply Lemma 3.4.4. Assume it is true for $k-1$. Let $\Delta G = \sum_{(G)} G_{(1)} \otimes G_{(2)}$ in Sweedler notation. Then by the induction hypothesis and Lemma 3.4.3,

$$\begin{aligned}(S_{j_1} \cdots S_{j_{k-1}})S_{j_k} *_B G = \; & \mu_2((S_{j_1} \cdots S_{j_{k-1}} \otimes S_{j_k}) * (I \otimes \Theta) \circ \Delta G) \\ = \; & \sum_{(G)} (S_{j_1} \cdots S_{j_{k-1}} *_B G_{(1)})(S_{j_k} *_A \Theta(G_{(2)})) \\ = \; & \sum_{(G)} (S_{j_1} \cdots S_{j_{k-1}} *_A \Theta(G_{(1)}))(S_{j_k} *_A \Theta(G_{(2)})) \\ = \; & \mu_2((S_{j_1} \cdots S_{j_{k-1}} \otimes S_{j_k}) * (\Theta \otimes \Theta) \circ \Delta G) \\ = \; & \mu_2((S_{j_1} \cdots S_{j_{k-1}} \otimes S_{j_k}) * \Delta \circ \Theta(G)) \\ = \; & (S_{j_1} \cdots S_{j_{k-1}})S_{j_k} *_A \Theta(G). \quad \blacksquare\end{aligned}$$

The main result of this section is

THEOREM 3.4.6 (Mackey Formula). *Let $F_1, G \in \mathbf{BSym}$, $F_2, \ldots, F_r \in \mathbf{Sym}$. Then*

$$(F_1 F_2 \cdots F_r) *_B G = \mu_r[(F_1 \otimes F_2 \otimes \cdots \otimes F_r) * (I \otimes \Theta \otimes \cdots \otimes \Theta) \circ \Delta^r G],$$

where μ_r is the r-fold multiplication $F_1 \otimes \cdots \otimes F_r \to F_1 \cdots F_r$, and $*$ is the internal product induced to $\mathbf{BSym} \otimes \mathbf{Sym}^{\otimes(r-1)}$.

Proof. We first consider the case $F_1 = \tilde{S}_{j_1}$, $F_k = S_{i_k}$, $2 \leqslant k \leqslant r$ and $G = \tilde{S}^J = \tilde{S}_{j_1} S_{j_2} \cdots S_{j_s}$. In this case (3.1) is equivalent to the multiplication formula for calculating $\tilde{S}^I *_B \tilde{S}^J$. Next, let $I^{(k)} = (i_1^{(k)}, i_2^{(k)}, \cdots, i_{n_k}^{(k)})$, $1 \leqslant k \leqslant r$. In Sweedler notation,

$\Delta^r G = \sum_{(G)} G_{(1)} \otimes G_{(2)} \otimes \cdots \otimes G_{(r)}$. Then

$$\begin{aligned}
&\mu_r((\tilde{S}^{I^{(1)}} \otimes S^{I^{(2)}} \otimes \cdots \otimes S^{I^{(r)}}) * (I \otimes \Theta^{\otimes(r-1)}) \circ \Delta^r G) \\
&\quad = \mu_r\left(\sum_{(G)} (\tilde{S}^{I^{(1)}} *_B G_{(1)}) \otimes (S^{I^{(2)}} *_A \Theta(G_{(2)})) \otimes \cdots \otimes (S^{I^{(r)}} *_A \Theta(G_{(r)}))\right) \\
&\quad = \mu_r\Bigg(\sum_{(G)} \mu_{n_1}((\tilde{S}_{i_1^{(1)}} \otimes S_{i_2^{(1)}} \otimes \cdots S_{i_{n_1}^{(1)}}) * (I \otimes \Theta^{\otimes(n_1-1)}) \circ \Delta^{n_1} G_{(1)}) \\
&\qquad\qquad \otimes \mu_{n_2}((S_{i_1^{(2)}} \otimes \cdots \otimes S_{i_{n_2}^{(2)}}) * \Delta^{n_2} \circ \Theta(G_{(2)})) \otimes \cdots \\
&\qquad\qquad \otimes \mu_{n_r}((S_{i_1^{(r)}} \otimes \cdots \otimes S_{i_{n_r}^{(r)}}) * \Delta^{n_r} \circ \Theta(G_{(r)}))\Bigg) \\
&\quad = \mu_r \circ (\mu_{n_1} \otimes \cdots \otimes \mu_{n_r})\Big((\tilde{S}_{i_1^{(1)}} \otimes \cdots \otimes S_{i_{n_1}^{(1)}} \otimes S_{i_1^{(2)}} \otimes \cdots \otimes S_{i_{n_r}^{(r)}}) * \\
&\qquad\qquad (I \otimes \Theta^{\otimes(n_1-1)} \otimes \Theta^{\otimes n_2} \otimes \cdots \otimes \Theta^{\otimes n_r}) \circ \Delta^{n_1+n_2+\cdots+n_r} G\Big) \\
&\quad = (\tilde{S}_{i_1^{(1)}} \cdots S_{i_{n_1}^{(1)}} S_{i_1^{(2)}} \cdots S_{i_{n_r}^{(r)}}) *_B G \\
&\quad = (\tilde{S}^{I^{(1)}} S^{I^{(2)}} \cdots S^{I^{(r)}}) *_B G
\end{aligned}$$

by the associativity, coassociativity and Lemma 3.4.3. ■

EXAMPLE 3.4.7. We compute $\tilde{S}^{(4,1)} *_B \tilde{S}^{(2,3)}$. By Theorem 3.4.6,

$$\tilde{S}^{(4,1)} *_B \tilde{S}^{(2,3)} = \mu_2[(\tilde{S}_4 \otimes S_1) * (I \otimes \Theta) \circ \Delta \tilde{S}^{(2,3)}]. \tag{3.1}$$

In $\Delta\tilde{S}^{(2,3)} = (\tilde{S}_2 \otimes 1 + \tilde{S}_1 \otimes \tilde{S}_1 + 1 \otimes \tilde{S}_2)(S_3 \otimes 1 + S_2 \otimes S_1 + S_1 \otimes S_2 + 1 \otimes S_3)$, only the two terms $\tilde{S}_2 S_2 \otimes S_1$ and $\tilde{S}_1 S_3 \otimes \tilde{S}_1$ give non-zero contributions to the internal product being computed. Noting that $\Theta(S_1) = 2S_1$ and $\Theta(\tilde{S}_1) = S_1$, the right hand side of (3.1) becomes

$$\begin{aligned}
(\tilde{S}_4 *_B \tilde{S}_2 S_2)(S_1 *_A 2S_1) + (\tilde{S}_4 *_B \tilde{S}_1 S_3)(S_1 *_A S_1) &= 2\tilde{S}_2 S_2 S_1 + \tilde{S}_1 S_3 S_1 \\
&= 2\tilde{S}^{(2,2,1)} + \tilde{S}^{(1,3,1)}. \quad \square
\end{aligned}$$

LEMMA 3.4.8. $\tilde{\sigma}(t_0) *_B \tilde{\sigma}(u_0)\sigma(u_1)\sigma(u_2)\cdots = \tilde{\sigma}(t_0 u_0)\sigma(t_0 u_1)\sigma(t_0 u_2)\cdots$.

Proof.

$$\begin{aligned}\tilde\sigma(t_0) *_B \tilde\sigma(u_0)\sigma(u_1)\sigma(u_2)\cdots &= \sum_{m\geqslant 0} t_0^m \tilde S_m *_B \tilde\sigma(u_0)\sigma(u_1)\sigma(u_2)\cdots \\ &= \sum_{m\geqslant 0} t_0^m \tilde\sigma(u_0)\sigma(u_1)\sigma(u_2)\cdots \\ &= \tilde\sigma(t_0u_0)\sigma(t_0u_1)\sigma(t_0u_2)\cdots . \quad \blacksquare\end{aligned}$$

Lemma 3.4.9.

$$\begin{aligned}&\sigma(t_1)\sigma(t_2)\cdots *_A (\cdots\sigma(u_1)\sigma(u_0)\sigma(u_1)\cdots) \\ &\qquad = (\cdots\sigma(t_1u_1)\sigma(t_1u_0)\sigma(t_1u_1)\cdots)(\cdots\sigma(t_2u_1)\sigma(t_2u_0)\sigma(t_2u_1)\cdots)\cdots .\end{aligned}$$

Proof. Since the $\sigma(t_i)$'s are group-like,

$$\Delta(\cdots\sigma(u_1)\sigma(u_0)\sigma(u_1)\cdots) = (\cdots\sigma(u_1)\sigma(u_0)\sigma(u_1)\cdots)\otimes(\cdots\sigma(u_1)\sigma(u_0)\sigma(u_1)\cdots).$$

By Theorem 1.2.7,

$$\begin{aligned}&\sigma(t_1)\sigma(t_2)\cdots *_A (\cdots\sigma(u_1)\sigma(u_0)\sigma(u_1)\cdots) \\ &\quad = (\sigma(t_1) *_A (\cdots\sigma(u_1)\sigma(u_0)\sigma(u_1)\cdots))(\sigma(t_2)\cdots *_A (\cdots\sigma(u_1)\sigma(u_0)\sigma(u_1)\cdots)) \\ &\quad = \sum_{m\geqslant 0} t_1^m(\cdots\sigma(u_1)\sigma(u_0)\sigma(u_1)\cdots)(\sigma(t_2)\cdots *_A (\cdots\sigma(u_1)\sigma(u_0)\sigma(u_1)\cdots)) \\ &\quad = (\cdots\sigma(t_1u_1)\sigma(t_1u_0)\sigma(t_1u_1)\cdots)(\sigma(t_2)\cdots *_A (\cdots\sigma(u_1)\sigma(u_0)\sigma(u_1)\cdots)).\end{aligned}$$

Since the second factor on the right hand side is the first on the left with indices shifted by one. Iterating the above argument, the result follows. ■

With the above two lemmas, we can prove the more general

Proposition 3.4.10.

$$\begin{aligned}&(\tilde\sigma(t_0)\sigma(t_1)\cdots) *_B (\tilde\sigma(u_0)\sigma(u_1)\cdots) \\ &\qquad = (\tilde\sigma(t_0u_0)\sigma(t_0u_1)\sigma(t_0u_2)\cdots)(\cdots\sigma(t_1u_1)\sigma(t_1u_0)\sigma(t_1u_1)\cdots) \\ &\qquad\quad \times(\cdots\sigma(t_2u_1)\sigma(t_2u_0)\sigma(t_2u_1)\cdots)\cdots \\ &\qquad = \tilde\sigma(t_0u_0)\prod_{i=1}^{\infty}\sigma(t_0u_i)\prod_{j=1}^{\infty}\prod_{k=-\infty}^{\infty}\sigma(t_ju_{|k|}).\end{aligned}$$

Proof. Let $F_1 = \tilde{\sigma}(t_0)$, $F_2 = \sigma(t_1)\sigma(t_2)\cdots$, and $G = \tilde{\sigma}(u_0)\sigma(u_1)\cdots$. Since G is a product of group-like elements, G is also group-like, i.e., $\Delta G = G \otimes G$. Also,

$$(I \otimes \Theta) \circ \Delta G = G \otimes \Theta(G) = G \otimes (\cdots \sigma(u_1)\sigma(u_0)\sigma(u_1)\cdots).$$

Theorem 3.4.6 with $r = 2$ yields

$$\begin{aligned} F_1 F_2 *_B G = &\ \mu_2((F_1 \otimes F_2) * (I \otimes \Theta) \circ \Delta G) \\ = &\ (F_1 *_B G)(F_2 *_A (\cdots \sigma(u_1)\sigma(u_0)\sigma(u_1)\cdots)). \end{aligned}$$

The proposition follows from Lemma 3.4.8 and Lemma 3.4.9. ■

As a causal application of Proposition 3.4.10,

EXAMPLE 3.4.11. We compute $\tilde{S}^{(4,1)} *_B \tilde{S}^{(2,3)}$, which is the coefficient of $t_0^4 t_1 u_0^2 u_1^3$ on the left side of Proposition 3.4.10. The monomial $t_0^4 t_1 u_0^2 u_1^3$ can be factored as either $(t_0 u_0)^2 (t_0 u_1)^2 (t_1 u_1)$ or $(t_0 u_0)(t_0 u_1)^3 (t_1 u_0)$, giving $2\tilde{S}^{(2,2,1)}$ and $\tilde{S}^{(1,3,1)}$, respectively. Thus, $\tilde{S}^{(4,1)} *_B \tilde{S}^{(2,3)} = 2\tilde{S}^{(2,2,1)} + \tilde{S}^{(1,3,1)}$. □

As a formal application of Proposition 3.4.10,

COROLLARY 3.4.12. *For any k, l,*

(i) $\tilde{\sigma}(1)\sigma(1)^k *_B \tilde{\sigma}(1)\sigma(1)^l = \tilde{\sigma}(1)\sigma(1)^{2kl+k+l}$,

(ii) $\tilde{\sigma}(1)\sigma(1)^k *_B \sigma(1)^l = \sigma(1)^{l(2k+1)}$,

(iii) $\sigma(1)^k *_B \tilde{\sigma}(1)\sigma(1)^l = \sigma(1)^{k(2l+1)}$,

(iv) $\sigma(1)^k *_B \sigma(1)^l = \sigma(1)^{2kl}$.

Proof. For k, l non-negative integers, setting

$$t_0 = t_1 = \cdots = t_k = u_0 = u_1 = \cdots = u_l = 1, t_{k+1} = t_{k+2} = \cdots = u_{l+1} = u_{l+2} = \cdots = 0,$$
$$t_0 = t_1 = \cdots = t_k = u_1 = \cdots = u_l = 1, u_0 = t_{k+1} = t_{k+2} = \cdots = u_{l+1} = u_{l+2} = \cdots = 0,$$
$$u_0 = t_1 = \cdots = t_k = u_1 = \cdots = u_l = 1, t_0 = t_{k+1} = t_{k+2} = \cdots = u_{l+1} = u_{l+2} = \cdots = 0,$$
$$t_1 = \cdots = t_k = u_1 = \cdots = u_l = 1, t_0 = t_{k+1} = t_{k+2} = \cdots = u_0 = u_{l+1} = u_{l+2} = \cdots = 0,$$

in Proposition 3.4.10 gives (i)–(iv), respectively. (i)–(iv) also hold for general k and l because the coefficients on both sides are polynomials in k and l. ∎

In **Sym**, the r-fold coproduct Δ^r is a homomorphism $\Delta^r\colon \mathbf{Sym} \to \mathbf{Sym}^{\otimes r}$, that is,

$$\Delta^r(F *_A G) = \Delta^r F * \Delta^r G,$$

where F, $G \in \mathbf{Sym}$, and the $*$ on the right is the induced internal product on $\mathbf{Sym}^{\otimes r}$. This result generalizes to **BSym** and is proved in two steps:

LEMMA 3.4.13. *For $F, G \in$ **BSym**, $\Delta(F *_B G) = \Delta F * \Delta G$, where $*$ on the right is the induced internal product on $\mathbf{BSym}^{\otimes 2}$.*

Proof. The generating function for all monomials $\tilde{S}_{j_1} S_{j_2} \cdots S_{j_k}$ is $\tilde{\sigma}(t_0)\sigma(t_1)\cdots$. By Lemma 1.2.4 and Lemma 3.4.10,

$$\begin{aligned}
&\Delta(\tilde{\sigma}(t_0)\sigma(t_1)\cdots *_B \tilde{\sigma}(u_0)\sigma(u_1)\cdots)\\
&\quad= \Delta\left(\tilde{\sigma}(t_0u_0)\prod_{i=1}^{\infty}\sigma(t_0u_i)\prod_{j=1}^{\infty}\prod_{k=-\infty}^{\infty}\sigma(t_ju_{|k|})\right)\\
&\quad= \tilde{\sigma}(t_0u_0)\prod_{i=1}^{\infty}\sigma(t_0u_i)\prod_{j=1}^{\infty}\prod_{k=-\infty}^{\infty}\sigma(t_ju_{|k|}) \otimes \tilde{\sigma}(t_0u_0)\prod_{i=1}^{\infty}\sigma(t_0u_i)\prod_{j=1}^{\infty}\prod_{k=-\infty}^{\infty}\sigma(t_ju_{|k|})\\
&\quad= (\tilde{\sigma}(t_0)\sigma(t_1)\cdots *_B \tilde{\sigma}(u_0)\sigma(u_1)\cdots) \otimes (\tilde{\sigma}(t_0)\sigma(t_1)\cdots *_B \tilde{\sigma}(u_0)\sigma(u_1)\cdots)\\
&\quad= (\tilde{\sigma}(t_0)\sigma(t_1)\cdots \otimes \tilde{\sigma}(t_0)\sigma(t_1)\cdots) * (\tilde{\sigma}(u_0)\sigma(u_1)\cdots \otimes \tilde{\sigma}(u_0)\sigma(u_1)\cdots)\\
&\quad= \Delta(\tilde{\sigma}(t_0)\sigma(t_1)\cdots) * \Delta(\tilde{\sigma}(u_0)\sigma(u_1)\cdots).
\end{aligned}$$

It follows that $\Delta(F *_B G) = \Delta F * \Delta G$ holds for F, G monomials. The general case now follows by linearity. ∎

With this special case asserted, we can prove

PROPOSITION 3.4.14. *$\Delta^r(F *_B G) = \Delta^r F * \Delta^r G$, where $*$ on the right is the induced internal product on $\mathbf{BSym}^{\otimes r}$.*

Proof. Induction on r. The case $r = 1$ is trivial because $\Delta^1 = I$. The case $r = 2$ is the content of Lemma 3.4.13. Assume that it is true for r. Write $\Delta^r F = \sum_{(F)} F_{(1)} \otimes \cdots \otimes F_{(r)}$ and similarly for $\Delta^r G$.

$$
\begin{aligned}
&\Delta^{r+1}(F *_B G) \\
&\qquad = (I^{\otimes(r-1)} \otimes \Delta) \circ \Delta^r(F *_B G) \\
&\qquad = (I^{\otimes(r-1)} \otimes \Delta)(\Delta^r F * \Delta^r G) \\
&\qquad = (I^{\otimes(r-1)} \otimes \Delta) \left(\sum_{(F),(G)} (F_{(1)} *_B G_{(1)}) \otimes \cdots \otimes (F_{(r)} *_B G_{(r)}) \right) \\
&\qquad = \sum_{(F),(G)} (F_{(1)} *_B G_{(1)}) \otimes \cdots \otimes (F_{(r-1)} *_B G_{(r-1)}) \otimes \Delta(F_{(r)} *_B G_{(r)}) \\
&\qquad = \sum_{(F),(G)} (F_{(1)} *_B G_{(1)}) \otimes \cdots \otimes (F_{(r-1)} *_B G_{(r-1)}) \otimes (\Delta(F_{(r)}) * \Delta(G_{(r)})) \\
&\qquad = \sum_{(F)} (F_{(1)} \otimes \cdots \otimes F_{(r-1)} \otimes \Delta F_{(r)}) * \sum_{(G)} (G_{(1)} \otimes \cdots \otimes G_{(r-1)} \otimes \Delta G_{(r)}) \\
&\qquad = \Delta^{r+1} F * \Delta^{r+1} G. \quad \blacksquare
\end{aligned}
$$

In **Sym**, the internal product $*_A$ preserves the primitive Lie algebra $L(\Psi)$. The same is true of $*_B$.

COROLLARY 3.4.15. *Let $F, G \in \mathbf{BSym}$ be primitive. Then $F *_B G$ is also primitive. In particular, the internal product $*_B$ preserves the primitive Lie algebra $L(\Psi)$.*

Proof. This follows from

$$
\begin{aligned}
\Delta(F *_B G) = {} & \Delta F * \Delta G \\
= {} & (F \otimes 1 + 1 \otimes F) * (G \otimes 1 + 1 \otimes G) \\
= {} & (F *_B G) \otimes 1 + 1 \otimes (F *_B G). \quad \blacksquare
\end{aligned}
$$

3.5 Connection with descent algebra of $K[B_n]$

Let I be a pseudo-composition of n. Define the descent class A_I to be the formal sum of signed permutations of B_n with descent compositions equal to I, i.e.,

$$A_I = \sum_{C(\pi)=I} \pi.$$

Also define B_I as the formal sum of signed permutations of B_n with descent compositions $\preccurlyeq I$, i.e.,

$$B_I = \sum_{C(\pi)\preccurlyeq I} \pi.$$

It is obvious that A_I and B_I are related by inclusion-exclusion:

$$B_I = \sum_{J\preccurlyeq I} A_J, \qquad A_I = \sum_{J\preccurlyeq I} (-1)^{l(I)-l(J)} B_J.$$

It is a celebrated result of Solomon [31] which asserts that A_I (or B_I), as I ranges over all pseudo-compositions of n, span a subalgebra of the group algebra $K[B_n]$ of B_n, known as the descent algebra of type B_n. (Solomon actually showed that this is true for general finite Coxeter groups.) The subalgebra ΣB_n spanned by A_I (or B_I) is called the descent algebra of type B.

Bergeron and Bergeron [4], Bergeron [3], and Bergeron and Bergeron [5] examine the descent algebra ΣB_n of $K[B_n]$. We shall establish the connection between the descent algebra of $K[B_n]$ and $\mathbf{BSym}_n$ in this section. Let us first make the

DEFINITION 3.5.1. Define the map $\alpha\colon \Sigma B_n \to (\mathbf{BSym}_n, *_B)$ by $\alpha(A_I) = \tilde{R}_I$, and extend linearly to all of ΣB_n.

LEMMA 3.5.2. *The map α sends B_I to $\tilde{S}^I$, for all pseudo-compositions I.*

Proof. Just compute

$$\alpha(B_I) = \sum_{J \preccurlyeq I} \alpha(A_J) = \sum_{J \preccurlyeq I} \tilde{R}_J = \tilde{S}^I. \quad \blacksquare$$

We can be more precise about α.

PROPOSITION 3.5.3. *The map α is an anti-isomorphism.*

Proof. The map α is a vector space isomorphism $\Sigma B_n \to \mathbf{BSym}_n$ since it sends a basis to a basis. An examination of the multiplication rules (Theorem 1.5.1 for ΣB_n and Theorem 3.3.1 for $(\mathbf{BSym}_n, *_B)$) reveals that α is anti-homomorphic with respect to the multiplications in Σ_n and $\mathbf{BSym}_n$. Hence, α is an anti-isomorphism. $\blacksquare$

Chapter 4

Idempotents

We established in Chapter 3 that $\mathbf{BSym}_n$ and ΣB_n are anti-isomorphic. The anti-isomorphism $\alpha\colon \mathbf{BSym}_n \to \Sigma B_n$ enables us to infer properties of one space from the other. For instance, an idempotent of $\mathbf{BSym}_n$ gives an idempotent of Σ_n, and vice versa. It is the purpose of the present chapter to study idempotents of either $\mathbf{BSym}_n$ or ΣB_n. In the first section, we investigate how quasi-idempotents of $(\mathbf{Sym}, *_A)$ behave with respect to $*_B$, demonstrate explicitly an idempotent of **BSym** which is not in **Sym**, and show how to construct new idempotents from them. In the second section, we study the Eulerian idempotents of type A, construct their type B counterparts, and show how they interact. In the third section, we examine the images of the idempotents of Bergeron-Bergeron in **BSym**. In the fourth section, we give a one-parameter family of Lie idempotents which interpolates interesting idempotents. In the fifth section, we study the action of a Lie idempotent on the free associative algebra $K\langle A\rangle$. In the final section, we show that a Lie-like representation of B_n is obtained by inducing a one-dimensional representation of a subgroup of B_n. A character formula will also be given.

4.1 Idempotents of BSym

Let K be a field, $\mathcal{A}$ a K-algebra, and $k \in K$. An element $x \in \mathcal{A}$ is a quasi-idempotent with constant k if $x^2 = kx$, and is an idempotent if $k = 1$.

Since **Sym** sits right inside **BSym** and has a large supply of idempotents with respect to $*_A$, e.g., Ψ_n, Φ_n, for each $n > 0$, it is of interest to see how these idempotents behave with respect to $*_B$. The first result in this vein is the following.

PROPOSITION 4.1.1. *Let* $F \in \mathbf{Sym}$ *be a primitive quasi-idempotent of* $(\mathbf{Sym}, *_A)$ *with constant* k. *Then* $F + \bar{F}$ *is a quasi-idempotent of* $(\mathbf{BSym}, *_B)$ *with constant* $4k$.

LEMMA 4.1.2. *Let* $F \in \mathbf{Sym}_n$ *be primitive and* $I = (i_1, \ldots, i_k) \models n$. *Then*

$$S^I *_A F = \begin{cases} 0 & \text{if } k > 1 \\ F & \text{if } k = 1. \end{cases}$$

Proof. By Theorem 1.2.7,

$$\begin{aligned} S^I *_A F = & \ \mu_k((S_{i_1} \otimes S_{i_2} \cdots S_{i_k}) * \Delta F) \\ = & \ \mu_k((S_{i_1} \otimes S_{i_2} \cdots S_{i_k}) * (F \otimes 1 + 1 \otimes F)) \\ = & \begin{cases} 0 & \text{if } k > 1 \\ F & \text{if } k = 1. \end{cases} \quad \blacksquare \end{aligned}$$

Lemma 4.1.2 implies that if $F, G \in \mathbf{Sym}_n$ with F primitive, then $G *_A F = ([S_n]G)F$, where $[S_n]G$ denotes the coefficient of S_n in G.

We are now ready to prove Proposition 4.1.1. Since F is primitive, $\Delta F = F \otimes 1 + 1 \otimes F$. This implies that $\Delta \bar{F} = \bar{F} \otimes 1 + 1 \otimes \bar{F}$, and hence $\Theta(F) = F + \bar{F} = \Theta(\bar{F})$. Then, by Proposition 3.4.5,

$$(F + \bar{F}) *_B (F + \bar{F}) = (F + \bar{F}) *_A \Theta(F + \bar{F}) = 2(F + \bar{F}) *_A (F + \bar{F}). \qquad (4.1)$$

To finish the proof, it is enough to compute the latter internal product. By the above remark, we have $F *_A F = ([S_n]F)F$. But $F *_A F = kF$, forcing $[S_n]F = k$. Note that $[S_n]\bar{F} = [S_n]F$, hence $\bar{F} *_A F = kF$. Replacing F by $\bar{F}$, we get $F *_A \bar{F} = \bar{F} *_A \bar{F} = k\bar{F}$.

Putting pieces together, we have that

$$(F+\bar{F}) *_B (F+\bar{F}) = 2(F *_A F + F *_A \bar{F} + \bar{F} *_A F + \bar{F} *_A \bar{F}) = 4k(F+\bar{F}),$$

as we wished to prove. ■

It is trivial to see that if x is a quasi-idempotent with constant $k \neq 0$, then $k^{-1}x$ is an idempotent. Thus we have

COROLLARY 4.1.3. *$(4n)^{-1}(\Psi_n + \bar{\Psi}_n)$ and $(2n)^{-1}\Phi_n$ are idempotents of* **BSym**.

Proof. Since in $\mathbf{Sym}_n$, $\Psi_n *_A \Psi_n = n\Psi_n$, and $\Phi_n *_A \Phi_n = n\Psi_n$. The first assertion is immediate. The second assertion holds because $\bar{\Phi}_n = \Phi_n$. ■

We just saw how quasi-idempotents of **Sym** gave rise to quasi-idempotents of **BSym**. Our next goal is to study quasi-idempotents of **BSym** which are not elements of **Sym**. Towards this end, we first make the

DEFINITION 4.1.4. $\eta(x) = \tilde{\sigma}(x)\sigma(x)^{-1/2} = \sum_{n \geqslant 0} \eta_n x^n.$

DEFINITION 4.1.5. Define

$$Z_i = (-1)^i \frac{1 \cdot 3 \cdot 5 \cdots (2i-1)}{2^i i!} = \binom{-\frac{1}{2}}{i}, \quad N_i = \frac{(-1)^{i-1}}{i}, \quad i \geqslant 1.$$

We also define $Z_0 = N_0 = 1$ for convenience.

The next lemma records several properties of $\eta(1)$, the second of which says exactly that $\eta(1)$ is an idempotent of **BSym**.

LEMMA 4.1.6. *Let $\eta(1)$ be defined as above. Then*

(i) $\Theta(\eta(1)) = 1$.

(ii) $\eta(1) *_B \eta(1) = \eta(1)$.

(iii) $\eta_m *_B \eta_n = \delta_{m,n}\eta_n$.

(iv) $\eta_n = \sum_{|I|=n} Z_{l(I)-1}\tilde{S}^I$, *where the sum ranges over all pseudo-compositions of* n.

Proof. Since $\sigma(1)^{-1/2}$ is group-like, $\Theta(\tilde{\sigma}(1)\sigma(1)^{-1/2}) = \sigma(1)^{-1/2}\sigma(1)\sigma(1)^{-1/2} = 1$, and (i) holds. (ii) is simply Corollary 3.4.12 with $k = l = -\frac{1}{2}$. From (ii), we have

$$\sum_{m\geqslant 0} \eta_m *_B \sum_{n\geqslant 0} \eta_n = \sum_{n\geqslant 0} \eta_n.$$

Taking components of weight n, we have (iii). We also have

$$\tilde{\sigma}(1)\sigma(1)^{-1/2} = \sum_{m\geqslant 0} \tilde{S}_m \sum_{i\geqslant 0} \binom{-\frac{1}{2}}{i} \sum_{l(I)=i} S^I = \sum_{n\geqslant 0} \sum_{|I|=n} Z_{l(I)-1}\tilde{S}^I.$$

(iv) is nothing but the components of weight n on both sides of this equation. ∎

The next lemma shows a way to construct quasi-idempotents of **BSym** from $\eta(1)$ and from those of **Sym**.

LEMMA 4.1.7. *If* β *is a primitive quasi-idempotent in* **Sym** *and* $\beta = \bar{\beta}$ *then* $\eta(1)\beta$ *is a quasi-idempotent in* **BSym**.

Proof. Suppose that $\beta *_A \beta = k\beta$, for some k. Since $\Delta\beta = \beta \otimes 1 + 1 \otimes \beta$. By Proposition 4.1.6, $\Theta(\eta(1)\beta) = \bar{\beta}\Theta(\eta(1)) + \Theta(\eta(1))\beta = 2\beta$. Then by the Mackey formula,

$$\begin{aligned}
\eta(1)\beta *_B \eta(1)\beta = \;& \mu_2((\eta(1)\otimes\beta) * (I \otimes \Theta)\circ\Delta(\eta(1)\beta)) \\
= \;& \mu_2((\eta(1)\otimes\beta) * (I\otimes\Theta)(\eta(1)\beta\otimes\eta(1) + \eta(1)\otimes\eta(1)\beta)) \\
= \;& \mu_2((\eta(1)\otimes\beta) * (\eta(1)\beta\otimes 1 + \eta(1)\otimes 2\beta)) \\
= \;& \mu_2((\eta(1) *_B \eta(1)\beta)\otimes(\beta *_A 1) + (\eta(1) *_B \eta(1))\otimes(\beta *_A 2\beta)) \\
= \;& 2k\eta(1)\beta. \quad \blacksquare
\end{aligned}$$

Let $C = (C, \Delta, \varepsilon)$ be a coalgebra, and g a group-like element of C. An element x of C is called g-primitive [25] if $\Delta x = x \otimes g + g \otimes x$. For $C = (\mathbf{BSym}, \Delta, \varepsilon)$, and $g = \eta(1)$, the calculation

$$\Delta\eta(1)\beta = (\eta(1) \otimes \eta(1))(\beta \otimes 1 + 1 \otimes \beta) = \eta(1)\beta \otimes \eta(1) + \eta(1) \otimes \eta(1)\beta$$

shows that $\eta(1)\beta$ is $\eta(1)$-primitive, but not primitive.

4.2 Eulerian idempotents

In **Sym**,

$$\sigma(1)^x *_A \sigma(1)^y = \sigma(1)^{xy}, \tag{4.2}$$

so that $\sigma(1)^x$ span a commutative subalgebra of $(\mathbf{Sym}, *_A)$, known as the Eulerian subalgebra, denoted by $\mathbf{E} = \bigoplus_{n \geqslant 0} \mathbf{E}_n$, where $\mathbf{E}_n$ is the homogeneous component of degree n. If we let $\sigma(1)^x = \sum_{n \geqslant 0} x^n E^{[n]}$, then by comparing the coefficients of $x^m y^n$ on both sides of (4.2), we obtain the orthonormality relation

$$E^{[m]} *_A E^{[n]} = \delta_{m,n} E^{[n]}, \tag{4.3}$$

where $\delta_{m,n}$ is the Kronecker delta. Equation (4.3) also implies that $E^{[n]}$ are idempotents, known as the Eulerian idempotents of type A (the designation in type is specifically introduced in order to distinguish the type B counterparts, to be defined shortly).

With respect to $*_B$, the Eulerian idempotents $E^{[n]}$ are still quasi-idempotents and satisfy an orthogonality relation. This is the content of the next proposition.

PROPOSITION 4.2.1. *We have*

(i) $\sigma(1)^x *_B \sigma(1)^y = \sigma(1)^{2xy}$,

(ii) $E^{[m]} *_B E^{[n]} = 2^n \delta_{m,n} E^{[n]}$.

Proof. (i) follows upon setting $k = x$, and $l = y$ in Corollary 3.4.12. (ii) follows from comparing coefficients of $x^m y^n$ on both sides of (i). ∎

PROPOSITION 4.2.2. $\tilde{\sigma}(1)\sigma(1)^{(x-1)/2} *_B \tilde{\sigma}(1)\sigma(1)^{(y-1)/2} = \tilde{\sigma}(1)\sigma(1)^{(xy-1)/2}$.

Proof. Set $k = (x-1)/2$ and $l = (y-1)/2$ in Corollary 3.4.12. ∎

Proposition 4.2.2 implies that $\tilde{\sigma}(1)\sigma(1)^{(x-1)/2}$ span a commutative subalgebra $\tilde{\mathbf{E}} = \bigoplus_{n\geqslant 0} \tilde{\mathbf{E}}_n$ of $(\mathbf{BSym}, *_B)$, called the Eulerian subalgebra of type B, where $\tilde{\mathbf{E}}_n$ is the homogeneous component of degree n. We now make the

DEFINITION 4.2.3. The Eulerian idempotents of type B, denoted $\tilde{E}^{[n]}$, are defined by $\tilde{\sigma}(1)\sigma(1)^{(x-1)/2} = \sum_{n\geqslant 0} x^n \tilde{E}^{[n]}$.

The definition made is perhaps unjustified as far as the idempotency of $\tilde{E}^{[n]}$ is concerned. However, this is remedied by the next proposition.

PROPOSITION 4.2.4. *The elements $\tilde{E}^{[n]}$ defined above are idempotents.*

Proof. Comparing the coefficients of $x^m y^n$ on both sides of Proposition 4.2.2, we obtain the orthogonal relation

$$\tilde{E}^{[m]} *_B \tilde{E}^{[n]} = \delta_{m,n} \tilde{E}^{[n]},$$

which says that $\tilde{E}^{[n]}$, besides being orthogonal, are idempotents. ∎

Not just elements of $\tilde{\mathbf{E}}_n$ commute, elements of $\tilde{\mathbf{E}}_n$ and of $\mathbf{E}_n$ also commute, as next proposition shows.

PROPOSITION 4.2.5. $\tilde{\sigma}(1)\sigma(1)^{(x-1)/2} *_B \sigma(1)^y = \sigma(1)^{xy} = \sigma(1)^y *_B \tilde{\sigma}(1)\sigma(1)^{(x-1)/2}$.

Proof. Setting $k = (x-1)/2$ and $l = y$ in Corollary 3.4.12 (ii) gives the first equality, whereas setting $k = y$ and $l = (x-1)/2$ in Corollary 3.4.12 (iii) yields the second. ∎

4.3 Idempotents of Bergeron-Bergeron

Let K be a field, and A_1 the totally ordered alphabet $\{\bar{n} < \cdots < \bar{2} < \bar{1} < 1 < 2 < \cdots < n\}$. As usual, A_1^* and A_1^n denote the collection of all A_1-words and all A_1-words of length n, respectively. The free associative algebra $K\langle A_1\rangle$ consists of polynomials

$$P = \sum_{w \in A_1^*} c_w w,$$

with a finite number of non-zero coefficient $c_w \in K$. The multiplication for $K\langle A_1\rangle$ is the linear extension of the concatenation product on A_1^*. Denote by $K\langle A_1\rangle_n$ the homogeneous component of degree n.

The free Lie algebra $\mathrm{Lie}(A_1)$ generated by A_1 is the smallest subalgebra containing A_1 of $K\langle A_1\rangle$ closed under the Lie bracket $[f,g] = fg - gf$. The elements of $\mathrm{Lie}(A_1)$ are called Lie polynomials. Denote by $\mathrm{Lie}_n(A_1)$ the homogeneous component of degree n.

The right action of $\pi = \pi_1\pi_2\cdots\pi_n \in B_n$ on an A_1-word $w = b_1b_2\cdots b_k$ is defined by

$$w\pi = \begin{cases} b_{\pi_1}b_{\pi_2}\cdots b_{\pi_n} & \text{if } k \neq n, \\ 0 & \text{if } k = n, \end{cases}$$

where we have identified $-i = \bar{i}$, $i = 1, 2, \ldots, n$. This defines by linear extension a right action of $K[B_n]$ on $K\langle A_1\rangle$. We can identify a element of B_n with the corresponding word in A_1^*. With this identification we can identify $K[B_n]$, considered as a right B_n module, with a submodule of $K\langle A_1\rangle$.

A Lie idempotent is an idempotent of the group algebra $K[B_n]$ which acts as a projector from the free associative algebra $K\langle A_1\rangle$ onto the free Lie algebra $\mathrm{Lie}(A_1)$ (for the right action of the hyperoctahedral group B_n on A_1-words).

In [4] Bergeron and Bergeron showed that

$$\frac{1}{2}I_{(n)} = \frac{1}{2}\sum_{p\models n}\frac{(-1)^{l(p)-1}}{l(p)}B_p = \frac{1}{2}\sum_{p\models n}N_{l(p)}B_p$$

is a Lie idempotent in the descent algebra ΣB_n. This gives an idempotent in **BSym** via the anti-isomorphism $\alpha\colon \Sigma B_n \to \mathbf{BSym}_n$.

PROPOSITION 4.3.1. *The image of* $\frac{1}{2}I_{(n)}$ *in* $\mathbf{BSym}_n$ *is* $\frac{1}{2n}\Phi_n$.

Proof. Here, in Bergeron-Bergeron's notation, their p corresponds to $(0,p)$ in our indexing scheme. See Remark 2.3.7. So,

$$\alpha(\frac{1}{2}I_{(n)}) = \frac{1}{2}\sum_{p\models n}\frac{(-1)^{l(p)-1}}{l(p)}\alpha(B_p) = \frac{1}{2}\sum_{p\models n}\frac{(-1)^{l(p)-1}}{l(p)}S^p = \frac{1}{2n}\Phi_n. \quad \blacksquare$$

In [3], Bergeron introduced yet another idempotent, namely

$$I_\varnothing = \sum_{p\models m\leqslant n} Z_{l(p)}B_p.$$

This gives rise to an idempotent in $\mathbf{BSym}_n$.

PROPOSITION 4.3.2. *The image of* $I_\varnothing$ *in* $\mathbf{BSym}_n$ *is* η_n*, as in Definition 4.1.4.*

Proof. Here, the sum in $I_\varnothing$, translated into our indexing scheme, is over all pseudo-compositions I of n. Also, $l(p) = l(I) - 1$. Therefore,

$$\alpha(I_\varnothing) = \sum_{p\models m\leqslant n} Z_{l(p)}\alpha(B_p) = \sum_I Z_{l(I)-1}\tilde{S}^I = \eta_n,$$

by Lemma 4.1.6 (iv). $\blacksquare$

Taking $\frac{1}{2}I_{(n)}$ and $I_\varnothing$ as fundamental building blocks, Bergeron [3] then constructed quasi-idempotents I_p, indexed by composition $p = (p_1, \ldots, p_k) \models m \leqslant n$, namely

$$I_p = \sum_{\substack{q_0 \models r \leqslant n-m \\ q_i \models p_i}} Z_{l(q_0)} N_{l(q_1)} \cdots N_{l(q_k)} B_{q_0 q_1 \cdots q_k},$$

where the subscript $q_0 q_1 \cdots q_k$ of B denotes concatenation of compositions.

PROPOSITION 4.3.3. *Let $p = (p_1, \ldots, p_k) \models m \leqslant n$. The image of I_p in $\mathbf{BSym}_n$ is $\eta_{n-m} \Phi^p / \pi(p)$, where $\pi(p) = p_1 \cdots p_k$.*

Proof. Just compute

$$\begin{aligned}
\alpha(I_p) &= \sum_{\substack{q_0 \models r \leqslant n-m \\ q_i \models p_i}} Z_{l(q_0)} N_{l(q_1)} \cdots N_{l(q_k)} \alpha(B_{q_0 q_1 \cdots q_k}) \\
&= \sum_{q_0 \models r \leqslant n-m} Z_{l(q_0)-1} \tilde{S}^{q_0} \sum_{q_1 \models p_1} N_{l(q_1)} S^{q_1} \cdots \sum_{q_k \models p_k} N_{l(q_k)} S^{q_k} \\
&= \eta_{n-m} \frac{\Phi_{p_1}}{p_1} \cdots \frac{\Phi_{p_k}}{p_k} \\
&= \eta_{n-m} \frac{\Phi^p}{\pi(p)}. \quad \blacksquare
\end{aligned}$$

Let $\mathcal{A}$ be an algebra with 1. Two idempotents e and f in $\mathcal{A}$ are orthogonal if $ef = 0$. A sequence of pairwise orthogonal idempotents $e_1, \ldots, e_n$ of $\mathcal{A}$ is complete if $e_1 + \cdots + e_n = 1$.

With $I_\varnothing$ and I_p at hand, Bergeron [3] then constructed a complete family of orthogonal idempotents $\{\tilde{E}_\lambda\}$, indexed by partitions $\lambda \vdash m \leqslant n$, of the descent algebra ΣB_n, More precisely, let $\lambda \vdash m \leqslant n$ with $l(\lambda) = k$. The idempotent $\tilde{E}_\lambda$ is then defined by

$$\tilde{E}_\lambda = \frac{1}{2^k k!} \sum_{\lambda(p) = \lambda} I_p,$$

with the sum ranges over all compositions p which are rearrangement of the partition λ.

In concluding the study of descent algebras of type B, Bergeron and Bergeron [5] consider a family of idempotents ρ_n^k defined by

$$\rho_n^k = \sum_{\substack{\lambda \vdash m \leqslant n \\ l(\lambda)=k}} \tilde{E}_\lambda, \quad 1 \leqslant k \leqslant n,$$

of which the generating function $\rho_n(x)$ is defined by

$$\rho_n(x) = \sum_{0 \leqslant k \leqslant n} \rho_n^k x^k,$$

The latter can also be expressed in terms of B_q, that is,

$$\rho_n(x) = \sum_{0 \leqslant r \leqslant n} \frac{(x-1)(x-3)\cdots(x-2r+1)}{2^r r!} \sum_{\substack{q \models s \leqslant n \\ l(q)=r}} B_q.$$

The image of $\rho_n(x)$ in $\mathbf{BSym}_n$ is easily seen to be

$$\alpha(\rho_n(x)) = \sum_{0 \leqslant r \leqslant n} \binom{\frac{x-1}{2}}{r} \sum_{\substack{q \models s \leqslant n \\ l(q)=r}} \alpha(B_q) = \sum_{r \geqslant 1} \binom{\frac{x-1}{2}}{r-1} \sum_{\substack{|q|=n \\ l(q)=r}} \tilde{S}^q.$$

Now we make the

DEFINITION 4.3.4. Define $\rho(x) = \sum_{n \geqslant 0} \rho_n(x)$.

The connection between $\rho(x)$ and $\tilde{\sigma}(1)\sigma(1)^{(x-1)/2}$ is clarified by

PROPOSITION 4.3.5. *The image of $\rho(x)$ in* **BSym** *is $\tilde{\sigma}(1)\sigma(1)^{(x-1)/2}$.*

Proof. This follows from

$$\alpha(\rho(x)) = \sum_{n \geqslant 0} \alpha(\rho_n(x)) = \sum_{n \geqslant 0} \sum_{r \geqslant 1} \binom{\frac{x-1}{2}}{r-1} \sum_{\substack{|q|=n \\ l(q)=r}} \tilde{S}^q = \tilde{\sigma}(1)\sigma(1)^{(x-1)/2}. \quad \blacksquare$$

We have seen in Proposition 4.2.2 that $\tilde{\sigma}(1)\sigma(1)^{(x-1)/2}$ span the Eulerian subalgebra of **BSym**, which is commutative. Since $(\mathbf{BSym}_n, *_B)$ is anti-isomorphic to ΣB_n, we conclude that

COROLLARY 4.3.6. *The elements $\rho(x)$ span a commutative subalgebra of the descent algebra of type B.*

We shall denote by $\mathcal{E}_n$ the commutative subalgebra of the descent algebra of type B spanned by $\rho(x)$, and call it the Eulerian subalgebra.

4.4 A one-parameter family of Lie idempotents

Let q be an indeterminate which commutes with all $f \in \mathbf{BSym}$.

DEFINITION 4.4.1. Let I be a composition of n. Denote by $\bar{I}$ the reversal of I. Define

$$\varphi_n^B(q) = \frac{1}{4n} \sum_{|I|=n} \frac{(-1)^{l(I)-1}}{\begin{bmatrix} n-1 \\ l(I)-1 \end{bmatrix}_q} q^{\mathrm{maj}(I) - \binom{l(I)}{2}} (R_I + R_{\bar{I}}),$$

where the sum ranges over all compositions of n.

Note that on the right side, R_I are the ribbon Schur functions as elements of **BSym**. We may replace each occurrence of R_I by $\tilde{R}_I + \tilde{R}_{(0,I)}$, thus expressing $\varphi_n^B(q)$ in terms of $\tilde{R}_I$'s.

EXAMPLE 4.4.2. For $n = 2, 3, 4$,

$$\begin{aligned}
\varphi_2^B(q) = {} & \tfrac{1}{4}(\tilde{R}_{(2)} + \tilde{R}_{(0,2)} - \tilde{R}_{(1,1)} - \tilde{R}_{(0,1,1)}), \\
\varphi_3^B(q) = {} & \tfrac{1}{12}(2\tilde{R}_{(3)} + 2\tilde{R}_{(0,3)} - \tilde{R}_{(2,1)} - \tilde{R}_{(0,2,1)} - \tilde{R}_{(1,2)} - \tilde{R}_{(0,1,2)} + 2\tilde{R}_{(1,1,1)} + 2\tilde{R}_{(0,1,1,1)}), \\
\varphi_4^B(q) = {} & \tfrac{1}{16}(2\tilde{R}_{(4)} + 2\tilde{R}_{(0,4)} - \tfrac{1+q^2}{1+q+q^2}(\tilde{R}_{(3,1)} + \tilde{R}_{(0,3,1)} + \tilde{R}_{(1,3)} + \tilde{R}_{(0,1,3)}) \\
& - \tfrac{2q}{1+q+q^2}(\tilde{R}_{(2,2)} + \tilde{R}_{(0,2,2)}) + \tfrac{1+q^2}{1+q+q^2}(\tilde{R}_{(2,1,1)} + \tilde{R}_{(0,2,1,1)} + \tilde{R}_{(1,1,2)} + \tilde{R}_{(0,1,1,2)}) \\
& + \tfrac{2q}{1+q+q^2}(\tilde{R}_{(1,2,1)} + \tilde{R}_{(0,1,2,1)}) - 2\tilde{R}_{(1,1,1,1)} - 2\tilde{R}_{(0,1,1,1,1)}). \quad \square
\end{aligned}$$

The expression for $\varphi_n^B(q)$ looks formidable. However, a comparison of $\varphi_n^B(q)$ with (1.7) reveals that

$$\varphi_n^B(q) = \frac{1-q^n}{4n}(\Psi_n(\frac{A}{1-q}) + \bar{\Psi}_n(\frac{A}{1-q})). \tag{4.4}$$

The latter expression for $\varphi_n(q)$ is preferred over Definition 4.4.1 because the power sum symmetric functions Ψ_n are easier to work with as far as the coproduct Δ (hence the internal product $*_B$) is concerned. We can say more about the properties of $\varphi_n^B(q)$, the most important one being the following.

THEOREM 4.4.3. *The element $\varphi_n^B(q)$ is an idempotent of $\mathbf{BSym}_n$.*

Proof. First note that by (1.7),

$$\Psi_n(\frac{A}{1-q}) = \frac{n}{1-q^n}\varphi_n(q),$$

where $\varphi_n(q)$ is a primitive idempotent in $\mathbf{Sym}_n$, so that $\Psi_n(A/(1-q))$ is a quasi-idempotent of $\mathbf{Sym}$ with constant $n/(1-q^n)$. Proposition 4.1.1 then implies that $\Psi_n(A/(1-q))+\bar{\Psi}_n(A/(1-q))$ is a quasi-idempotent of $\mathbf{BSym}$ with constant $4n/(1-q^n)$. Normalizing $\Psi_n(A/(1-q))+\bar{\Psi}_n(A/(1-q))$ by $4n/(1-q^n)$ and noting (4.4), the assertion follows. ■

The element $\varphi_n^B(q)$ of $\mathbf{BSym}_n$ interpolates between interesting idempotents, as summarized in the following proposition.

PROPOSITION 4.4.4. *We have $\varphi_n^B(1) = (2n)^{-1}\Phi_n$, $\varphi_n^B(0) = (4n)^{-1}(\Psi_n + \bar{\Psi}_n)$.*

Proof. Setting $q = 1$ and noting that $\varphi_n^B(1)$ is invariant under reversal, $\varphi_n^B(1) = (2n)^{-1}\Phi_n$ follows. Setting $q = 0$, the summand is non-zero if and only if $\mathrm{maj}(I) = \binom{l(I)}{2}$ if and only if $I = (1^k, n-k)$, $0 \leqslant k \leqslant n-1$. Thus, we conclude by (1.5) that

$$\varphi_n^B(0) = \frac{1}{4n}\sum_{k=0}^{n-1}(-1)^k(R_{(1^k,n-k)} + R_{(n-k,1^k)}) = \frac{1}{4n}(\Psi_n + \bar{\Psi}_n). \quad ■$$

We proved Theorem 4.4.3 by Proposition 4.1.1. Theorem 4.4.3 can also be seen as an immediate consequence of the more general

THEOREM 4.4.5. *Let q_1 and q_2 be indeterminates that commute with* **BSym**. *Then* $\varphi_n^B(q_1) *_B \varphi_n^B(q_2) = \varphi_n^B(q_2)$.

LEMMA 4.4.6. *We have* $S_n(\frac{A}{1-q}) *_A \Psi_n = \frac{\Psi_n}{1-q^n}$.

Proof. Let $I = (i_1, \dots, i_k)$. By (1.6) and Lemma 4.1.2,

$$\begin{aligned} S_n(\frac{A}{1-q}) *_A \Psi_n &= \sum_{|I|=n} \frac{q^{\mathrm{maj}(I)}}{(1-q^{i_1})(1-q^{i_1+i_2})\cdots(1-q^{i_1+\cdots+i_k})} S^I *_A \Psi_n \\ &= \frac{1}{1-q^n} S_n *_A \Psi_n = \frac{\Psi_n}{1-q^n}. \quad \blacksquare \end{aligned}$$

Proof (of Theorem 4.4.5). By Proposition 3.4.5, and Example 3.4.2,

$$\begin{aligned} &\varphi_n^B(q_1) *_B \varphi_n^B(q_2) \\ &= \frac{(1-q_1^n)(1-q_2^n)}{(4n)^2}(\Psi_n(\frac{A}{1-q_1}) + \bar{\Psi}_n(\frac{A}{1-q_1})) *_B (\Psi_n(\frac{A}{1-q_2}) + \bar{\Psi}_n(\frac{A}{1-q_2})) \\ &= \frac{(1-q_1^n)(1-q_2^n)}{(4n)^2}(\Psi_n(\frac{A}{1-q_1}) + \bar{\Psi}_n(\frac{A}{1-q_1})) *_A \Theta(\Psi_n(\frac{A}{1-q_2}) + \bar{\Psi}_n(\frac{A}{1-q_2})) \\ &= \frac{2(1-q_1^n)(1-q_2^n)}{(4n)^2}(\Psi_n(\frac{A}{1-q_1}) + \bar{\Psi}_n(\frac{A}{1-q_1})) *_A (\Psi_n(\frac{A}{1-q_2}) + \bar{\Psi}_n(\frac{A}{1-q_2})). \end{aligned}$$

We compute only $\Psi_n(\frac{A}{1-q_1}) *_A (\Psi_n(\frac{A}{1-q_2}) + \bar{\Psi}_n(\frac{A}{1-q_2}))$. The other is similar. We have

$$\begin{aligned}
&\Psi_n(\frac{A}{1-q_1}) *_A (\Psi_n(\frac{A}{1-q_2}) + \bar{\Psi}_n(\frac{A}{1-q_2})) \\
&= \sum_{i=0}^{n-1}(-1)^i(n-i)\Lambda_i(\frac{A}{1-q_1})S_{n-i}(\frac{A}{1-q_1}) *_A (\Psi_n(\frac{A}{1-q_2}) + \bar{\Psi}_n(\frac{A}{1-q_2})) \\
&= \sum_{i=0}^{n-1}(-1)^i(n-i)\mu_2((\Lambda_i(\frac{A}{1-q_1}) \otimes S_{n-i}(\frac{A}{1-q_1})) * \Delta(\Psi_n(\frac{A}{1-q_2}) + \bar{\Psi}_n(\frac{A}{1-q_2})) \\
&= \sum_{i=0}^{n-1}(-1)^i(n-i)\mu_2((\Lambda_i(\frac{A}{1-q_1}) \otimes S_{n-i}(\frac{A}{1-q_1})) \\
&\quad *((\Psi_n(\frac{A}{1-q_2}) + \bar{\Psi}_n(\frac{A}{1-q_2})) \otimes 1 + 1 \otimes ((\Psi_n(\frac{A}{1-q_2}) + \bar{\Psi}_n(\frac{A}{1-q_2})))) \\
&= nS_n(\frac{A}{1-q_1}) *_A (\Psi_n(\frac{A}{1-q_2}) + \bar{\Psi}_n(\frac{A}{1-q_2})) \\
&= \frac{n}{1-q_1^n}(\Psi_n(\frac{A}{1-q_2}) + \bar{\Psi}_n(\frac{A}{1-q_2})),
\end{aligned}$$

where the last equality follows from Lemma 4.4.6. Thus,

$$\begin{aligned}
\varphi_n^B(q_1) *_B \varphi_n^B(q_2) = &\quad \frac{2(1-q_1^n)(1-q_2^n)}{(4n)^2}\Big(\Psi_n(\frac{A}{1-q_1}) *_A (\Psi_n(\frac{A}{1-q_2}) + \bar{\Psi}_n(\frac{A}{1-q_2})) \\
&\quad + \bar{\Psi}_n(\frac{A}{1-q_1}) *_A (\Psi_n(\frac{A}{1-q_2}) + \bar{\Psi}_n(\frac{A}{1-q_2}))\Big) \\
= &\quad \frac{2(1-q_1^n)(1-q_2^n)}{(4n)^2} \cdot \frac{2n}{1-q_1^n}(\Psi_n(\frac{A}{1-q_2}) + \bar{\Psi}_n(\frac{A}{1-q_2})) \\
= &\quad \frac{1-q_2^n}{4n}(\Psi_n(\frac{A}{1-q_2}) + \bar{\Psi}_n(\frac{A}{1-q_2})) \\
= &\quad \varphi_n^B(q_2). \quad \blacksquare
\end{aligned}$$

Setting $q_1 = q_2 = q$, we recover Theorem 4.4.3.

Theorem 4.4.5, besides being a generalization of Theorem 4.4.3, also has an implication for the left ideals $K\langle A_1\rangle\alpha^{-1}(\varphi_n^B(q))$ of $K\langle A_1\rangle$. The commutation relation $\varphi_n^B(q_1) *_B \varphi_n^B(q_2) = \varphi_n^B(q_2)$ implies that in ΣB_n, $\alpha^{-1}(\varphi_n^B(q_2))\alpha^{-1}(\varphi_n^B(q_1)) = \alpha^{-1}(\varphi_n^B(q_2))$, which in turn implies that $K\langle A_1\rangle\alpha^{-1}(\varphi_n^B(q_2)) \subseteq K\langle A_1\rangle\alpha^{-1}(\varphi_n^B(q_1))$. Interchanging q_1 and q_2, we obtain the reverse inclusion. Therefore, the two left ideals $K\langle A_1\rangle\alpha^{-1}(\varphi_n^B(q_1))$ and $K\langle A_1\rangle\alpha^{-1}(\varphi_n^B(q_2))$ are identical.

Since $\varphi_n^B(1) = (2n)^{-1}\Phi_n$ has image $\frac{1}{2}I_{(n)}$ in ΣB_n, which projects the free associative algebra $K\langle A_1\rangle$ onto the space of negative Lie polynomials $\mathrm{Lie}_n^-(A_1)$ (defined below), the equality of left ideals $K\langle A_1\rangle\alpha^{-1}(\varphi_n^B(q))$ for all q then implies that $K\langle A_1\rangle\alpha^{-1}(\varphi_n^B(q)) = \mathrm{Lie}_n^-(A_1)$. In particular, $K\langle A_1\rangle\alpha^{-1}(\varphi_n^B(0)) = \mathrm{Lie}_n^-(A_1)$.

4.5 A Dynkin-like idempotent

Recall that the image of Ψ_n in the descent algebra Σ_n of $\mathfrak{S}_n$ is θ_n, whose action on words of length n is the standard left bracketing, i.e., if $w = x_1x_2\cdots x_n$, then

$$w\theta_n = [[[[x_1, x_2], x_3], \cdots], x_n].$$

The normalized element $n^{-1}\theta_n$ of Σ_n is an idempotent, called the Dynkin idempotent. See Garsia [10] for related results. We shall see that the image of $\Psi_n + \bar{\Psi}_n$ in the descent algebra ΣB_n also acts on words of length n by some sort of bracketing.

Let $K\langle A_1\rangle$, $\mathrm{Lie}(A_1)$, and the right action of $K[B_n]$ be as in section 4.3. Define an operation on A_1-words by

$$\overleftarrow{a_{i_1}a_{i_2}\cdots a_{i_n}} = \bar{a}_{i_n}\cdots\bar{a}_{i_2}\bar{a}_{i_1}$$

and extend linearly to all of $K\langle A_1\rangle$. It is clear that both $K\langle A_1\rangle$ and $K[B_n]$ are invariant under the involutory operation $\leftarrow$.

A Lie polynomial P is positive (resp., negative) if $\overleftarrow{P} = -P$ (resp., $\overleftarrow{P} = P$). Let $\mathrm{Lie}^+(A_1)$ and $\mathrm{Lie}^-(A_1)$ denote the set of positive and negative Lie polynomials, respectively. Then $\mathrm{Lie}(A_1)$ admits the decomposition

$$\mathrm{Lie}(A_1) = \mathrm{Lie}^+(A_1) \oplus \mathrm{Lie}^-(A_1).$$

Here, the positive and negative Lie polynomials are the same as those defined in [4], [3].

Recall the descent classes A_I defined in section 3.5. We now make the

DEFINITION 4.5.1. $\theta_n^B = \sum_{k=0}^{n-1}(-1)^k(A_{(0,1^k,n-k)} + A_{(0,n-k,1^k)} + A_{(1^k,n-k)} + A_{(n-k,1^k)})$.

PROPOSITION 4.5.2. *We have*

$$\theta_n^B = \sum_{\substack{\sigma\in B_n \\ \sigma_1<\cdots<\sigma_n}} \{[\sigma_1, \sigma_2, \dots, \sigma_n] + (-1)^{n-1}[\bar{\sigma}_1, \bar{\sigma}_2, \dots, \bar{\sigma}_n]\},$$

where $[x_1, x_2, \dots, x_n] = [[[[x_1, x_2], x_3], \cdots], x_n]$.

EXAMPLE 4.5.3. The increasing sequences of B_3 are 123, $\bar{1}23$, $\bar{2}13$, $\bar{3}12$, $\bar{2}\bar{1}3$, $\bar{3}\bar{1}2$, $\bar{3}\bar{2}1$, $\bar{3}\bar{2}\bar{1}$. So,

$$\begin{aligned}
&\sum_{k=0}^{2}(-1)^k(A_{(0,1^k,n-k)} + A_{(0,n-k,1^k)} + A_{(1^k,n-k)} + A_{(n-k,1^k)}) \\
&\quad = [1,2,3] + [\bar{1},2,3] + [\bar{2},1,3] + [\bar{3},1,2] + [\bar{2},\bar{1},3] + [\bar{3},\bar{1},2] + [\bar{3},\bar{2},1] + [\bar{3},\bar{2},\bar{1}] \\
&\quad + [\bar{1},\bar{2},\bar{3}] + [1,\bar{2},\bar{3}] + [2,\bar{1},\bar{3}] + [3,\bar{1},\bar{2}] + [2,1,\bar{3}] + [3,1,\bar{2}] + [3,2,\bar{1}] + [3,2,1]. \ \square
\end{aligned}$$

Let I be a pseudo-composition of n and $A_I = \sum_{C(\sigma)=I} \sigma$ be a descent class. Define refined descent classes $A_I^{\pm}$ of A_I as follows:

$$A_I^{\pm} = \sum_{\substack{C(\sigma)=I \\ \pm\sigma_n>0}} \sigma.$$

It is obvious that the descent class A_I admits the decomposition

$$A_I = A_I^+ + A_I^-.$$

The descent classes in the right hand side of Definition 4.5.1 pair up in the following manner.

LEMMA 4.5.4. *We have*

$$\overleftarrow{A_{(0,1^k,n-k)} + A_{(1^k,n-k)}} = A_{(0,n-k,1^k)} + A_{(n-k,1^k)},$$

and

$$\overleftarrow{A_{(0,n-k,1^k)} + A_{(n-k,1^k)}} = A_{(0,1^k,n-k)} + A_{(1^k,n-k)}.$$

Proof. Since $\sigma_1 > \cdots > \sigma_k > \sigma_{k+1} < \cdots < \sigma_n \iff \bar{\sigma}_n < \cdots < \bar{\sigma}_{k+1} > \bar{\sigma}_k > \cdots > \bar{\sigma}_1$, it follows that

$$\begin{aligned}
\overleftarrow{A_{(0,1^k,n-k)} + A_{(1^k,n-k)}} &= \overleftarrow{A^+_{(0,1^k,n-k)} + A^-_{(0,1^k,n-k)} + A^+_{(1^k,n-k)} + A^-_{(1^k,n-k)}} \\
&= A^+_{(0,n-k,1^k)} + A^+_{(n-k,1^k)} + A^-_{(0,n-k,1^k)} + A^-_{(n-k,1^k)} \\
&= A_{(0,n-k,1^k)} + A_{(n-k,1^k)}.
\end{aligned}$$

The remaining assertion follows from similar calculation. ■.

COROLLARY 4.5.5. *We have* $\overleftarrow{\theta_n^B} = \theta_n^B$.

LEMMA 4.5.6. *We have* $\overleftarrow{[\sigma_1, \ldots, \sigma_n]} = (-1)^{n-1}[\bar{\sigma}_1, \ldots, \bar{\sigma}_n]$.

Proof. Induction on n. When $n = 2$, $\overleftarrow{[\sigma_1, \sigma_2]} = \overleftarrow{\sigma_1\sigma_2 - \sigma_2\sigma_1} = \bar{\sigma}_2\bar{\sigma}_1 - \bar{\sigma}_1\bar{\sigma}_2 = -[\bar{\sigma}_1, \bar{\sigma}_2]$. Assume that it is true for k. Then by the definition of $[\sigma_1, \ldots, \sigma_{k+1}]$ and the induction hypothesis,

$$\begin{aligned}
\overleftarrow{[\sigma_1, \ldots, \sigma_k, \sigma_{k+1}]} &= \overleftarrow{[\sigma_1, \ldots, \sigma_k]\sigma_{k+1} - \sigma_{k+1}[\sigma_1, \ldots, \sigma_k]} \\
&= (-1)^{k-1}(\bar{\sigma}_{k+1}[\bar{\sigma}_1, \ldots, \bar{\sigma}_k] - [\bar{\sigma}_1, \ldots, \bar{\sigma}_k]\bar{\sigma}_{k+1}) \\
&= (-1)^k[\bar{\sigma}_1, \ldots, \bar{\sigma}_{k+1}]. \quad ■
\end{aligned}$$

Let $w = x_1 \cdots x_n$ be a word. Denote by $\tilde{w} = x_n \cdots x_1$ the reversal of w. When the letters $x_1, \ldots, x_n$ are distinct, two words u and v are called constituent subwords of w if w is a shuffle of u and v. We need one more result before proving Proposition 4.5.2.

LEMMA 4.5.7. *Let $\sigma_1 < \cdots < \sigma_n$ be given. Then*

$$[\sigma_1, \ldots, \sigma_n] = \sum (-1)^{d(\pi_1 \cdots \pi_n)} \pi_1 \cdots \pi_n,$$

where the sum ranges over all words $\pi_1 \cdots \pi_n$ obtained by concatenating $\tilde{w}_1$ and w_2, where w_1, w_2 are constituent subwords of σ, the first letter of w_2 being equal to σ_1, and $d(\pi_1 \cdots \pi_n) = \#\{i\colon \pi_i > \pi_{i+1}\}$ the number of descents of $\pi_1 \cdots \pi_n$.

Proof. The proof is by induction on n. The assertion clearly holds when $n = 2$. Assume that it is true for k. Then by the definition of $[\sigma_1, \ldots, \sigma_{k+1}]$ and the induction hypothesis,

$$\begin{aligned} [\sigma_1, \ldots, \sigma_k, \sigma_{k+1}] &= [\sigma_1, \ldots, \sigma_k]\sigma_{k+1} - \sigma_{k+1}[\sigma_1, \ldots, \sigma_k] \\ &= \sum_{\pi'} (-1)^{d(\pi'_1 \cdots \pi'_k)} \pi'_1 \cdots \pi'_k \sigma_{k+1} - \sum_{\pi''} (-1)^{d(\pi''_1 \cdots \pi''_k)} \sigma_{k+1} \pi''_1 \cdots \pi''_k \\ &= \sum_{\pi} (-1)^{d(\pi_1 \cdots \pi_{k+1})} \pi_1 \cdots \pi_{k+1}, \end{aligned}$$

because $d(\sigma_{k+1}\pi''_1 \cdots \pi''_k) = 1 + d(\pi''_1 \cdots \pi''_k)$. ■

EXAMPLE 4.5.8. Let $a < b < c$. Then $[a, b, c] = [[a, b], c] = abc - bac - cab + cba$, where in the right hand side, (w_1, w_2) are $(\varnothing, abc)$, (b, ac), (c, ab), and (bc, a), respectively. □

We are now ready to prove Proposition 4.5.2. We shall first prove

$$\sum_{\substack{\sigma \in B_n \\ \sigma_1 < \cdots < \sigma_n}} [\sigma_1, \ldots, \sigma_n] = \sum_{k=0}^{n-1} (-1)^k (A_{(1^k, n-k)} + A_{(0, 1^k, n-k)}).$$

Let $\sigma = \sigma_1 \cdots \sigma_n \in B_n$ be such that $\sigma_1 < \cdots < \sigma_n$. The latter condition is equivalent to saying that $\sigma_i < \sigma_j$ if and only if $i < j$. If $w_1 = \sigma_{j_1} \cdots \sigma_{j_k}$ and $w_2 = \sigma_1 \sigma_{i_2} \cdots \sigma_{i_{n-k}}$ are constituent subwords of σ, where $j_1 < \cdots < j_k$, and $1 < i_2 < \cdots < i_{n-k}$, then $\tilde{w}_1 w_2 = \sigma_{j_k} \cdots \sigma_{j_1} \sigma_1 \sigma_{i_1} \cdots \sigma_{i_{n-k}} = \pi$ has descent composition $C(\pi) = (1^k, n-k)$ or $(0, 1^k, n-k)$, depending on whether $\sigma_{j_k} > 0$ or not. Also, $d(\pi) = k$. Conversely, given a signed permutation $\pi \in B_n$ of descent composition either $(1^k, n-k)$ or $(0, 1^k, n-k)$, i.e., $\pi_1 > \cdots > \pi_k > \pi_{k+1} < \cdots < \pi_n$. There is exactly one way to interlace $\pi_k < \cdots < \pi_1$ and $\pi_{k+1} < \cdots < \pi_n$ to give an increasing word $\pi_{i_1} < \cdots < \pi_{i_n}$ of B_n. Thus, the correspondence just described is bijective. Summing over all $\sigma_1 < \cdots < \sigma_n$, the above assertion follows. Now, applying $\leftarrow$ to what we just proved, by virtue of Lemma 4.5.4 and Lemma 4.5.6, we have that

$$\sum_{\substack{\sigma \in B_n \\ \sigma_1 < \cdots < \sigma_n}} (-1)^{n-1}[\bar{\sigma}_1, \ldots, \bar{\sigma}_n] = \sum_{k=0}^{n-1} (-1)^k (A_{(n-k,1^k)} + A_{(0,n-k,1^k)}).$$

Summing the two displayed equations, Proposition 4.5.2 follows. ■

Now, let $K = \mathbb{Q}$, and $w = \sigma \in B_n$. Then $w\theta_n^B \in \mathrm{Lie}_n^-(A_1) \cap \mathbb{Q}[B_n]$, so that $\mathbb{Q}[B_n]((4n)^{-1}\theta_n^B) \subseteq \mathrm{Lie}_n^-(A_1) \cap \mathbb{Q}[B_n]$, as B_n-modules. To show that the inclusion is indeed an equality, we resort to a dimension argument. We have the following result from Reutenauer [29].

THEOREM 4.5.9. *Let K be a field, and G a finite group with identity element* 1. *Let $\rho = \sum_{\sigma \in G} a_\sigma \sigma$ be an idempotent of the group algebra KG. Then* $\dim KG\rho = |G|a_1$, *where a_1 is the coefficient of* 1 *in ρ.*

Take $G = B_n$. The coefficient of 1 in $(4n)^{-1}\theta_n^B$ is easily seen to be $(2n)^{-1}$. Therefore, $\dim \mathbb{Q}[B_n]((4n)^{-1}\theta_n^B) = 2^n n!(2n)^{-1} = 2^{n-1}(n-1)!$. But this is precisely $\dim \mathbb{Q}[B_n](\frac{1}{2}I_{(n)})$, as computed by Bergeron in [3].

4.6 A character formula

Let A_2 be the alphabet $\{1, 2, \ldots, n\}$, and $K\langle A_2 \rangle$ the free associative algebra generated by A_2. It is clear that the group algebra $K[\mathfrak{S}_n]$ is a vector subspace of $K\langle A_2 \rangle$. The right action of $\sigma \in \mathfrak{S}_n$ on the A_2-word $w = x_1 x_2 \cdots x_k$ is defined by

$$w\sigma = \begin{cases} x_{\sigma_1} x_{\sigma_2} \cdots x_{\sigma_n} & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases}$$

By linear extension this defines a right action of $\mathfrak{S}_n$ on $K\langle A_2\rangle$, and hence on $K[\mathfrak{S}_n]$.

Let ρ be a Lie idempotent of $\mathfrak{S}_n$, i.e., an idempotent in the group algebra $K[\mathfrak{S}_n]$ of $\mathfrak{S}_n$ such that

$$K[\mathfrak{S}_n]\rho = \mathrm{Lie}(A_2) \cap K[\mathfrak{S}_n], \tag{4.5}$$

where the right hand side is the multi-linear part of degree n of the free Lie algebra $\mathrm{Lie}(A_2)$ generated by A_2. The left action of $\mathfrak{S}_n$ on the left ideal $K[\mathfrak{S}_n]\rho$ is the Lie representation of $\mathfrak{S}_n$, the character of which [29] is given by

$$\chi_{\mathrm{Lie}} = \sum_{\sigma\in\mathfrak{S}_n} \sigma\rho\sigma^{-1}.$$

It is well-established that the Lie representation of $\mathfrak{S}_n$ can be obtained by inducing the one-dimensional faithful representation $\chi_\omega\colon \gamma^i \mapsto \omega^i$ of the cyclic subgroup C_n generated by the n-cycle $\gamma = (n\, n-1 \cdots 2\, 1)$ to $\mathfrak{S}_n$, where ω is a primitive nth root of unity, i.e.,

$$\chi_{\mathrm{Lie}} = \mathrm{ind}_{C_n}^{\mathfrak{S}_n} \chi_\omega = \frac{1}{|C_n|} \sum_{\pi\in\mathfrak{S}_n} \pi\chi_\omega\pi^{-1} = \frac{1}{n} \sum_{\pi\in\mathfrak{S}_n} \pi\chi_\omega\pi^{-1}.$$

See [10] or [29].

In view of Lemma 4.5.6, it is natural to consider the element of the group algebra of B_n, namely

$$\bar\rho = \frac{1}{2}(\rho + (-1)^{n-1}\bar\epsilon\rho),$$

where $\bar\epsilon = \bar 1\bar 2\cdots\bar n$, and ρ is a Lie idempotent of $\mathfrak{S}_n$ satisfying (4.5). Note that $\bar\epsilon$ commutes with all $\sigma \in B_n$, hence with ρ (because ρ is a linear combination of $\sigma \in B_n$).

Lemma 4.6.1. *The element $\bar\rho$ is an idempotent.*

Proof. Since $\bar{\epsilon}\rho = \rho\bar{\epsilon}$ and $\rho^2 = \rho$. We have

$$\begin{aligned}\bar{\rho}^2 = & \tfrac{1}{4}(\rho + (-1)^{n-1}\bar{\epsilon}\rho)(\rho + (-1)^{n-1}\bar{\epsilon}\rho) \\ = & \tfrac{1}{4}(\rho^2 + (-1)^{n-1}\rho\bar{\epsilon}\rho + (-1)^{n-1}\bar{\epsilon}\rho^2 + \bar{\epsilon}\rho\bar{\epsilon}\rho) \\ = & \tfrac{1}{2}(\rho + (-1)^{n-1}\bar{\epsilon}\rho) = \bar{\rho}. \quad \blacksquare\end{aligned}$$

It is a standard result in representation theory [29] that the character of the action of B_n on the left ideal $K[B_n]\bar{\rho}$ is given by

$$\hat{\rho} = \sum_{\pi \in B_n} \pi\bar{\rho}\pi^{-1}.$$

Since $K[B_n]\bar{\rho} = \mathrm{Lie}_n^-(A_1) \cap K[B_n]$, $\hat{\rho}$ is thus the character of a Lie-like representation of B_n. We now show that $\hat{\rho}$ is also obtained by inducing a one-dimensional representation of a subgroup of B_n. Let

$$\hat{C}_n = \langle \gamma, \bar{\epsilon} \rangle = \{\bar{\epsilon}^j\gamma^i \colon 0 \leqslant j \leqslant 1, 0 \leqslant i \leqslant n-1\}.$$

Define $\bar{\chi}_\omega \colon \hat{C}_n \to K$ by

$$\bar{\chi}_\omega(\bar{\epsilon}^j\gamma^i) = \begin{cases} \omega^i & \text{if } j = 0, \\ (-1)^{n-1}\omega^i & \text{if } j = 1. \end{cases}$$

It is straightforward to verify that $\bar{\chi}_\omega$ is a representation of $\hat{C}_n$.

Theorem 4.6.2. *We have* $\hat{\rho} = \mathrm{ind}_{\hat{C}_n}^{B_n} \bar{\chi}_\omega$.

Proof. Extend χ_ω to B_n by setting $\chi_\omega(\sigma) = \omega^i$ if $\sigma = \omega^i$ for some i, and 0 otherwise. Then $\bar{\chi}_\omega(\sigma) = \chi_\omega(\sigma) + (-1)^{n-1}\chi_\omega(\sigma\bar{\epsilon})$. Let $\tau_1, \tau_2, \ldots, \tau_{2^n}$ be a transversal of $\mathfrak{S}_n$ in B_n, i.e., $B_n = \tau_1\mathfrak{S}_n + \tau_2\mathfrak{S}_n + \cdots + \tau_{2^n}\mathfrak{S}_n$. We write $\pi\bar{\chi}_\omega\pi^{-1}(\sigma)$ to mean $\bar{\chi}_\omega(\pi\sigma\pi^{-1})$; the argument σ will be omitted for clarity. Since $\bar{\epsilon}$ commutes with all $\pi \in B_n$, we have

$$\begin{aligned}
\operatorname{ind}_{\hat{C}_n}^{B_n} \bar{\chi}_\omega &= \frac{1}{|\hat{C}_n|} \sum_{\pi \in B_n} \pi \bar{\chi}_\omega \pi^{-1} \\
&= \frac{1+(-1)^{n-1}\bar{\epsilon}}{2n} \sum_{\pi \in B_n} \pi \chi_\omega \pi^{-1} \\
&= \frac{1+(-1)^{n-1}\bar{\epsilon}}{2} \sum_i \tau_i \left(\frac{1}{n} \sum_{\sigma \in \mathfrak{S}_n} \sigma \chi_\omega \sigma^{-1} \right) \tau_i^{-1} \\
&= \frac{1+(-1)^{n-1}\bar{\epsilon}}{2} \sum_i \tau_i \chi_{\mathrm{Lie}} \tau_i^{-1} \\
&= \frac{1+(-1)^{n-1}\bar{\epsilon}}{2} \sum_i \tau_i \left(\sum_{\pi \in \mathfrak{S}_n} \pi \rho \pi^{-1} \right) \tau_i^{-1} \\
&= \sum_{\tau \in B_n} \tau \left(\frac{1+(-1)^{n-1}\bar{\epsilon}}{2} \right) \rho \tau^{-1} \\
&= \sum_{\tau \in B_n} \tau \bar{\rho} \tau^{-1} = \hat{\rho}. \quad \blacksquare
\end{aligned}$$

Denote by type(σ) the cycle type of a permutation σ of $\mathfrak{S}_n$. The character of the Lie representation of $\mathfrak{S}_n$ at σ is given by

$$\chi_{\mathrm{Lie}}(\sigma) = \operatorname{ind}_{C_n}^{\mathfrak{S}_n} \chi_\omega(\sigma) = \begin{cases} \frac{1}{n} p^{n/p} (\frac{n}{p})! \mu(p) & \text{if type}(\sigma) = p^{n/p} \\ 0 & \text{otherwise} \end{cases}, \tag{4.6}$$

where μ is the number-theoretic Möbius function.

Stembridge [35] studied the ordinary representations of B_n. Consider B_n as a finite Coxeter group having a set S of generators given by $S = \{s_0, s_1, s_2, \ldots, s_{n-1}\}$, where $s_i = (i\, i+1)$, $1 \leqslant i \leqslant n-1$, are the simple transpositions, and $s_0 = (1\,\bar{1})$. For any element π of B_n, π can be written as a product of cycles. Conjugating π by s_i, $1 \leqslant i \leqslant n-1$, preserves the sign inventory in each cycle, On the other hand, conjugating π by s_0 changes the sign of two entries in some cycle. Hence a conjugate class must have a prescribed number of cycles with even parity and odd parity. So, the cycle type of $\pi \in B_n$, denoted by type(π), consists of a pair of partitions (λ, μ) such that $|\lambda| + |\mu| = n$, where λ, μ denotes respectively the lengths of positive, negative cycles of π. The (λ, μ)-class has $2^n n!/(2^{l(\lambda)+l(\mu)} z_\lambda z_\mu)$ elements, where if $\lambda = 1^{m_1} 2^{m_2} \cdots$, then $z_\lambda = 1^{m_1} 2^{m_2} \cdots m_1! m_2! \cdots$.

EXAMPLE 4.6.3. The signed permutation $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \bar{3} & 1 & \bar{2} & \bar{5} & 4 \end{pmatrix}$ of B_5 has cycle type $(3^1, 2^1)$. $\square$

We have an analogous character formula for $\hat{\rho}$, as follows.

COROLLARY 4.6.4.

$$\hat{\rho}(\pi) = \frac{1}{2n} \begin{cases} [1 + (-1)^{n-1}](2p)^{n/p}(\frac{n}{p})!\mu(p) & \textit{if } \text{type}(\pi) = (p^{n/p}, \varnothing) \textit{ and } p \equiv 0 \pmod 2 \\ (2p)^{n/p}(\frac{n}{p})!\mu(p) & \textit{if } \text{type}(\pi) = (p^{n/p}, \varnothing) \textit{ and } p \equiv 1 \pmod 2 \\ (-1)^{n-1}(2p)^{n/p}(\frac{n}{p})!\mu(p) & \textit{if } \text{type}(\pi) = (\varnothing, p^{n/p}) \textit{ and } p \equiv 1 \pmod 2 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let $\pi \in B_n$. Then

$$\left(\sum_i \tau_i \chi_{\text{Lie}} \tau_i^{-1} \right)(\pi) = \begin{cases} \frac{1}{n}(2p)^{n/p}(\frac{n}{p})!\mu(p) & \text{if type}(\pi) = (p^{n/p}, \varnothing) \\ 0 & \text{otherwise,} \end{cases}$$

where the factor $2^{n/p}$ arises from sign changes. If $\text{type}(\pi) = (p^{n/p}, \varnothing)$ and $p \equiv 0 \pmod 2$, then $\text{type}(\pi\bar{\epsilon}) = (p^{n/p}, \varnothing)$ and

$$\hat{\rho}(\pi) = \frac{1}{2} \left(\sum_i \tau_i \chi_{\text{Lie}} \tau_i^{-1} \right)(\pi + (-1)^{n-1}\pi\bar{\epsilon}) = \frac{[1 + (-1)^{n-1}](2p)^{n/p}(\frac{n}{p})!\mu(p)}{2n}.$$

If $\text{type}(\pi) = (p^{n/p}, \varnothing)$ and $p \equiv 1 \pmod 2$, then

$$\hat{\rho}(\pi) = \frac{1}{2} \left(\sum_i \tau_i \chi_{\text{Lie}} \tau_i^{-1} \right)(\pi) = \frac{(2p)^{n/p}(\frac{n}{p})!\mu(p)}{2n}.$$

If $\text{type}(\pi) = (\varnothing, p^{n/p})$ and $p \equiv 1 \pmod 2$, then $\text{type}(\pi\bar{\epsilon}) = (p^{n/p}, \varnothing)$ so that

$$\hat{\rho}(\pi) = \frac{(-1)^{n-1}}{2} \left(\sum_i \tau_i \chi_{\text{Lie}} \tau_i^{-1} \right)(\pi\bar{\epsilon}) = \frac{(-1)^{n-1}(2p)^{n/p}(\frac{n}{p})!\mu(p)}{2n}. \quad \blacksquare$$

In [2], Bergeron considers a hyperoctahedral analogue of the free Lie algebra. Amongst other things, Bergeron proves an induction formula similar to Theorem 4.6.2, and gives a character formula similar to Corollary 4.6.4. The difference between Bergeron's results and ours lies in that the Lie polynomials in our case are invariant under signed reversal, whereas Bergeron's Lie polynomials are invariant under sign change only. The sign factor $(-1)^{n-1}$ in $\bar{\rho}$ reflects this little twist.

Chapter 5

Noncommutative Generalizations of Numbers of Euler

Two classical classes of numbers of Euler which are of combinatorial significance are the Eulerian numbers $A_{n,k}$, which count the number of permutations $\sigma \in \mathfrak{S}_n$ with k runs, and the Euler numbers E_n, which count the number of alternating permutations $\sigma \in \mathfrak{S}_n$. It is shown in [13] that these Euler's numbers admit noncommutative generalizations, defined as the image of the formal sum of permutations, prescribed by the number concerned, in **Sym**. The purpose of this chapter is to generalize further these noncommutative Euler's "numbers" to the hyperoctahedral groups of type B. In the first section, we give a noncommutative analog of the Eulerian numbers of type B. Several bases of the Eulerian subalgebra are also discussed. In the second section, noncommutative Euler numbers of type B are studied, and from which formulas for the Euler numbers of type B as well as refinements of them are derived.

5.1 Eulerian polynomials of type B

The noncommutative Eulerian "numbers" generalize readily to **BSym**. We may as well define the noncommutative Eulerian polynomials. The definitions are as natural as one would expect.

DEFINITION 5.1.1. The *Eulerian number* of type B is defined by

$$\tilde{A}(n,k) = \sum_{\substack{|I|=n \\ l(I)=k}} \tilde{R}_I, \quad n,k \geqslant 1,$$

and the *Eulerian polynomial* of type B by

$$\tilde{\mathcal{A}}_n(t) = \sum_{|I|=n} t^{l(I)-1} \tilde{R}_I = \sum_k \tilde{A}(n,k) t^{k-1}, \quad n \geqslant 1.$$

For convenience we set $\tilde{A}(0,k) = \tilde{A}(n,0) = 0$.

EXAMPLE 5.1.2. $\tilde{A}(3,2) = \tilde{R}_{(0,3)} + \tilde{R}_{(2,1)} + \tilde{R}_{(1,2)}$; $\tilde{\mathcal{A}}_3(t) = \tilde{R}_{(3)} + t(\tilde{R}_{(0,3)} + \tilde{R}_{(2,1)} + \tilde{R}_{(1,2)}) + t^2(\tilde{R}_{(0,2,1)} + \tilde{R}_{(0,1,2)} + \tilde{R}_{(1,1,1)}) + t^3 \tilde{R}_{(0,1,1,1)}$. $\square$

The exponential generating function for $\tilde{\mathcal{A}}_n(t)$ becomes

PROPOSITION 5.1.3. $\tilde{\sigma}(1)[1 - t\sigma(1)]^{-1} = \sum_I \dfrac{t^{l(I)-1}\tilde{R}_I}{(1-t)^{|I|+1}}$.

Proof. We have

$$\begin{aligned}
\sum_{|I|=n} t^{l(I)} \tilde{R}_I &= \sum_{|I|=n} t^{l(I)} \sum_{J \preccurlyeq I} (-1)^{l(I)-l(J)} \tilde{S}^J \\
&= \sum_{|J|=n} (-1)^{l(J)} \tilde{S}^J \sum_{J \preccurlyeq I} (-t)^{l(I)} \\
&= \sum_{|J|=n} (-1)^{l(J)} \tilde{S}^J \sum_{S(J) \subseteq T \subseteq [0,n-1]} (-t)^{\#T+1} \qquad (5.1) \\
&= \sum_{|J|=n} (-1)^{l(J)} \tilde{S}^J \sum_{k \geqslant 0} \binom{n+1-l(J)}{k} (-t)^{l(J)+k} \\
&= \sum_{|J|=n} (1-t)^{n+1-l(J)} t^{l(J)} \tilde{S}^J.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\tilde{\sigma}(1)[1-t\sigma(1)]^{-1} &= \frac{1}{1-t}\sum_{m\geqslant 0}\tilde{S}_m\left(1-\frac{t}{1-t}\sum_{j\geqslant 1}S_j\right)^{-1} \\
&= \sum_{m,k\geqslant 0}\frac{t^k}{(1-t)^{k+1}}\sum_{l(J)=k}\tilde{S}^{(m,J)} \\
&= \sum_{n,k\geqslant 0}\frac{t^k}{(1-t)^{k+1}}\sum_{|I|=n,\,l(I)=k+1}\tilde{S}^I \\
&= \sum_{n\geqslant 0}\frac{1}{(1-t)^{n+1}}\sum_{|I|=n}(1-t)^{n+1-l(I)}t^{l(I)-1}\tilde{S}^I \\
&= \sum_{n\geqslant 0}\frac{1}{(1-t)^{n+1}}\sum_{|I|=n}t^{l(I)-1}\tilde{R}_I \qquad \text{by (5.1)} \\
&= \sum_{I}\frac{t^{l(I)-1}\tilde{R}_I}{(1-t)^{|I|+1}}. \quad \blacksquare
\end{aligned}$$

We can obtain an explicit expression relating $\tilde{A}(n,k)$ and the Eulerian idempotent $\tilde{\sigma}(1)\sigma(1)^{(x-1)/2}$.

PROPOSITION 5.1.4. $\tilde{\sigma}(1)\sigma(1)^{(x-1)/2} = \sum_{n,k}\tilde{A}(n,k)\binom{\frac{x+1}{2}+n-k}{n}$.

Proof. The left hand side of Proposition 5.1.3 can be written as

$$\tilde{\sigma}(1)[1-t\sigma(1)]^{-1} = \sum_{m\geqslant 0}t^m\tilde{\sigma}(1)\sigma(1)^m,$$

whereas the right hand side can be expanded as

$$\begin{aligned}
\sum_{I}\frac{t^{l(I)-1}\tilde{R}_I}{(1-t)^{|I|+1}} &= \sum_{n\geqslant 0}\frac{1}{(1-t)^{n+1}}\sum_{k}\tilde{A}(n,k)t^{k-1} \\
&= \sum_{n\geqslant 0}\sum_{l\geqslant 0}\binom{n+l}{l}t^l\sum_{k}\tilde{A}(n,k)t^{k-1} \\
&= \sum_{m\geqslant 0}t^m\sum_{n,k}\tilde{A}(n,k)\binom{n+m+1-k}{n}.
\end{aligned}$$

Equating the coefficients of t^m, we get

$$\tilde{\sigma}(1)\sigma(1)^m = \sum_{n,k} \tilde{A}(n,k)\binom{n+m+1-k}{n}. \tag{5.2}$$

Since the coefficients of $\tilde{A}(n,k)$ in the right hand side are polynomials in m, (5.2) still holds for any m. Replacing m by $(x-1)/2$, the proposition follows. ■

The next corollary, the B_n noncommutative analogue of Worpitzky's identity [7], is an immediate consequence of Proposition 5.1.4.

COROLLARY 5.1.5. $$\sum_{m\geqslant 0} x^m \tilde{E}_n^{[m]} = \sum_{k\geqslant 0} \tilde{A}(n,k)\binom{\frac{x+1}{2}+n-k}{n}.$$

Proof. On noting Definition 4.2.3 and taking components of weight n on both sides of Proposition 5.1.4, the result follows. ■

The element $\phi(m)$ defined by (2.4), which after renaming m by x, and r by n, can be rewritten as

$$\phi(x) = \sum_{k\geqslant 0}\binom{\frac{x-1}{2}+n-k}{n}\sum_{d(\pi)=k}\pi = \sum_{k\geqslant 0}\binom{\frac{x+1}{2}+n-k}{n}\sum_{l(C(\pi))=k}\pi.$$

But the right hand is easily seen to be the image of that of Corollary 5.1.5 in ΣB_n. In other words, $\phi(m)$ is nothing but a disguised form of the Worpitzky's identity, and the elements e_i defined preceding Theorem 2.4.3 are the images of $\tilde{E}_n^{[i]}$ in ΣB_n.

Besides $\tilde{A}(n,k)$, there are other bases for the Eulerian subalgebra $\mathbf{E}_n$. One of which is the following:

DEFINITION 5.1.6. $$\tilde{M}_n^{[k]} = \sum_{\substack{|I|=n\\ l(I)=k}} \tilde{S}^I.$$

PROPOSITION 5.1.7. $$\sum_k x^{k-1}(1+x)^{n+1-k}\tilde{A}(n,k) = \sum_k x^k \tilde{M}_n^{[k]}.$$

Proof. Setting $t = x/(1+x)$ in (5.1), we get

$$\sum_k \tilde{A}(n,k)\left(\frac{x}{1+x}\right)^k = \sum_{|I|=n} \frac{x^{l(I)}\tilde{S}^I}{(1+x)^{n+1}}.$$

Multiplying both sides by $(1+x)^{n+1}$ and rearranging the sum on the right, we obtain

$$\sum_k \tilde{A}(n,k)(1+x)^{n+1-k}x^k = \sum_k x^k \sum_{\substack{|I|=n \\ l(I)=k}} \tilde{S}^I = \sum_k x^k \tilde{M}_n^{[k]}. \quad \blacksquare$$

We can also derive a relation between $\tilde{E}_n^{[m]}$ and $\tilde{M}_n^{[k]}$.

PROPOSITION 5.1.8. $\displaystyle\sum_m x^m \tilde{E}_n^{[m]} = \sum_k \binom{\frac{x-1}{2}}{k-1} \tilde{M}_n^{[k]}.$

Proof.

$$\sum_{n,m} x^m \tilde{E}_n^{[m]} = \tilde{\sigma}(1)\sigma(1)^{(x-1)/2} = \sum_{n,m} \binom{\frac{x-1}{2}}{m-1} \sum_{\substack{|I|=n \\ l(I)=m}} \tilde{S}^I = \sum_{n,m} \binom{\frac{x-1}{2}}{m-1} \tilde{M}_n^{[m]}$$

Taking components of weight n on both sides, the proposition follows. $\blacksquare$

Yet another basis for the Eulerian subalgebra can be given. Define $\tilde{S}_n^{[p]}$ to be the term of weight n in $\tilde{\sigma}(1)\sigma(1)^{p-1}$. More precisely,

LEMMA 5.1.9. *We have* $\displaystyle\tilde{S}_n^{[p]} = \sum_{k\geqslant 1} \binom{p-1}{k-1} \tilde{M}_n^{[k]}.$

Proof. Setting $(x-1)/2 = p-1$ in Proposition 5.1.8, the result follows. $\blacksquare$

The relationship between $\tilde{S}_n^{[p]}$ and $\tilde{A}(n,k)$ is clarified by

PROPOSITION 5.1.10. *We have* $\displaystyle\tilde{S}_n^{[p]} = \sum_k \tilde{A}(n,k)\binom{n+p-k}{n}.$

Proof. Upon setting $p = (x+1)/2$, Proposition 5.1.4 becomes

$$\tilde{\sigma}(1)\sigma(1)^{p-1} = \sum_{n,k} \tilde{A}(n,k) \binom{n+p-k}{n}.$$

Taking the term of weight n on both side, the proposition follows. ■.

Proposition 5.1.10 expresses $\tilde{S}_n^{[p]}$ as a linear combination of $\tilde{A}(n,k)$. The inverse relation is given in the next proposition.

PROPOSITION 5.1.11. *We have* $\tilde{A}(n,k+1) = \sum_{i=0}^{k} (-1)^i \binom{n+1}{i} \tilde{S}_n^{[k+1-i]}$.

Proof. Taking the component of weight n of

$$\sum_{n\geqslant 1} \frac{\tilde{\mathcal{A}}_n(t)}{(1-t)^{n+1}} = \tilde{\sigma}(1)[1-t\sigma(1)]^{-1} = \sum_{p\geqslant 1} t^{p-1}\tilde{\sigma}(1)\sigma(1)^{p-1},$$

we have

$$\sum_{n\geqslant 1} \frac{\tilde{\mathcal{A}}_n(t)}{(1-t)^{n+1}} = \sum_{p\geqslant 1} t^{p-1}\tilde{S}_n^{[p]}.$$

Multiplying both sides by $(1-t)^{n+1}$ and expanding by the binomial theorem, we get

$$\begin{aligned}\sum_{k\geqslant 0} t^k \tilde{A}(n,k+1) = & \sum_{p\geqslant 1} t^{p-1}\tilde{S}_n^{[p]} \sum_{r=0}^{n+1} \binom{n+1}{r} (-1)^r t^r \\ = & \sum_{k\geqslant 0} t^k \sum_{r=0}^{k} (-1)^r \binom{n+1}{r} \tilde{S}_n^{[k+1-r]}.\end{aligned}$$

The proposition now follows by equating the coefficients of t^k. ■

In [13], a similar quantity $S_n^{[p]}$ for the Eulerian subalgebra of type A, is defined as the term of weight n in $\sigma(1)^p$. A formula of $S_n^{[p]}$ is also given, namely

$$S_n^{[p]} = \sum_{\substack{|I|=n \\ l(I)\leqslant p}} S^I.$$

This formula, as stated, is misleading. For example, if $n = 3$ and $p = 2$, then according to the formula, $S_3^{[2]} = S^{(1,2)} + S^{(2,1)} + S^{(3)}$. However, in $\sigma(1)^2 = (1 + S_1 + S_2 + \cdots)(1 + S_1 + S_2 + \cdots)$, $S_3^{[2]} = 2S^{(3)} + S^{(1,2)} + S^{(2,1)}$. A more precise formula for $S_n^{[p]}$ is

$$S_n^{[p]} = \sum_{k \geqslant 1} \binom{p}{k} \sum_{\substack{|I|=n \\ l(I)=k}} S^I.$$

The commutative Eulerian polynomials of type B are well-known. See Reiner [28] and Steingrímsson [34]. The exponential generating function for the commutative Eulerian polynomial $\tilde{A}_n(t)$, as given in Steingrímsson [34], is

$$\sum_{n \geqslant 0} \tilde{A}_n(t) \frac{x^n}{n!} = \frac{(1-t)e^{x(1-t)}}{1 - te^{2x(1-t)}}.$$

By expanding the denominator as a power series in $te^{2x(1-t)}$, followed by expanding the exponentials, and then equating the coefficient of x^n on both sides, we obtain

$$\tilde{A}_n(t) = (1-t)^{n+1} \sum_{k \geqslant 0} (2k+1)^n t^k. \tag{5.3}$$

Expanding $(1 - t)^{n+1}$ by binomial theorem we have the following formula for the Eulerian numbers $\tilde{A}_{n,k}$ of type B.

$$\tilde{A}_{n,k} = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (2k - 2j + 1)^n, \quad k = 0, 1, 2, \ldots, n.$$

The first seven $\tilde{A}_n(t)$ are listed in Table 5.1.

The Eulerian numbers $\tilde{A}(n,k)$ and polynomials $\tilde{\mathcal{A}}_n(t)$ are the noncommutative analogues of the Eulerian numbers $\tilde{A}_{n,k}$ and polynomials $\tilde{A}_n(t)$ for the hyperoctahedral groups B_n.

In the case of **Sym**, the "normalized" Eulerian number $A_{n,k}/n!$ can be obtained by the specialization $S_n \mapsto 1/n!$, which was adapted from Foulkes [9] for commutative

n	$\tilde{A}_n(t)$
1	$1+t$
2	$1+6t+t^2$
3	$1+23t+23t^2+t^3$
4	$1+76t+230t^2+76t^3+t^4$
5	$1+237t+1682t^2+1682t^3+237t^4+t^5$
6	$1+722t+10543t^2+23548t^3+10543t^4+722t^5+t^6$
7	$1+2179t+60657t^2+259723t^3+259723t^4+60657t^5+2179t^6+t^7$

Table 5.1: Eulerian polynomials $\tilde{A}_n(t)$ of type B, for $n = 1, 2, \ldots, 7$

symmetric functions. A similar specialization can be given for **BSym**.

DEFINITION 5.1.12. Define φ: **BSym** $\to K$ by $\varphi(\tilde{S}_n) = 1/n!$, $\varphi(S_n) = 2^n/n!$, and extend algebraically to all of **BSym**.

Call a signed permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in B_n$ increasing if $\sigma_1 < \sigma_2 < \cdots < \sigma_n$.

LEMMA 5.1.13. *There are 2^n increasing sequences in B_n.*

Proof. Let w_n be an increasing sequence in B_n. Then w_n is of the form either $w_{n-1}n$ or $\bar{n}w_{n-1}$, where w_{n-1} is an increasing sequence in B_{n-1}. Thus, if I_n is the number of increasing sequences in B_n, then $I_n = 2I_{n-1}$. It is clear that $I_1 = 2$. Hence, $I_n = 2^n$. ∎

The purpose of Lemma 5.1.13 is to pave the way to proving

PROPOSITION 5.1.14. *Let $I = (i_1, i_2, \ldots, i_k)$ be a pseudo-composition of n. Then $n!\varphi(\tilde{S}^I)$ is the number of signed permutations σ of B_n with descent composition $C(\sigma) \preccurlyeq I$.*

Proof. By the definition of φ,

$$n!\varphi(\tilde{S}^I) = n!\varphi(\tilde{S}_{i_1})\varphi(S_{i_2})\cdots\varphi(S_{i_k}) = \frac{2^{i_2+\cdots+i_k}n!}{i_1!i_2!\cdots i_k!} = 2^{i_2+\cdots+i_k}\binom{n}{i_1, i_2, \ldots, i_k}.$$

There are $\begin{pmatrix} n \\ i_1, i_2, \ldots, i_k \end{pmatrix}$ ways of choosing disjoint subsets $S_1, S_2, \ldots, S_k$ of $[n]$ whose union is $[n]$, with $\#S_j = i_j$. There is only one increasing sequence in $\mathfrak{S}_{S_1}$ ($\mathfrak{S}_\varnothing$ consists of the empty sequence only); there are 2^{i_j} increasing sequences in B_{S_j}, $2 \leqslant j \leqslant k$. By concatenating these increasing sequences, we obtain signed permutations whose descent composition $\preccurlyeq I$. But, from the above equation, the number of signed permutations so-formed is precisely $n!\varphi(\tilde{S}^I)$, proving the proposition. ■

EXAMPLE 5.1.15. For $I = (2, 1)$, the pair of subsets (S_1, S_2) of $[3]$ can be either $(\{1,2\},\{3\})$, $(\{1,3\},\{2\})$, or $(\{2,3\},\{1\})$; the sequences formed as in the proof of preceding proposition are 123, $12\bar{3}$, 132, $13\bar{2}$, 231, and $23\bar{1}$, which have descent compositions $\preccurlyeq I$. □

By inclusion-exclusion, we have

COROLLARY 5.1.16. *Let $I = (i_1, i_2, \ldots, i_k)$ be a pseudo-composition of n. Then $n!\varphi(\tilde{R}_I)$ is the number of signed permutations σ of B_n with descent composition $C(\sigma) = I$.*

As an immediate consequence of Corollary 5.1.16, we have

PROPOSITION 5.1.17. *The image of $\tilde{A}(n,k)$ under φ is $(n!)^{-1}$ times the Eulerian number $\tilde{A}_{n,k}$ for B_n.*

Proof. The proposition follows from

$$\varphi(\tilde{A}(n,k)) = \sum_{\substack{|I|=n \\ l(I)=k}} \varphi(\tilde{R}_I) = \sum_{\substack{|I|=n \\ l(I)=k}} \frac{\#\{\sigma \in B_n \colon C(\sigma) = I\}}{n!} = \frac{\tilde{A}_{n,k}}{n!}. \quad ■$$

5.2 Euler numbers of type B

A permutation σ in $\mathfrak{S}_n$ is said to be *alternating* (or *up-down*) if $\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \cdots$, and *reverse alternating* (or *down-up*) if $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \cdots$. The number E_n of alternating permutations has a nice generating function [1], namely,

$$\sum_{n\geqslant 0} E_n \frac{x^n}{n!} = \tan x + \sec x. \tag{5.4}$$

Steingrímsson [34] gave a generalization of this result to the so-called indexed permutations, a disguised form of the wreath product, which was also studied by Poirier [26], a special case of which is the hyperoctahedral group B_n.

We shall define alternating and reverse alternating permutations σ in B_n in the same way as in $\mathfrak{S}_n$. Here, the sign of σ_1 is unrestricted. It is easy to see that alternating signed permutations and reverse alternating signed permutations of B_n are equi-numerous, as the map $\sigma = \sigma_1 \cdots \sigma_n \mapsto \bar{\sigma} = \bar{\sigma}_1 \cdots \bar{\sigma}_n$, where $\bar{\sigma}_i = -\sigma_i$, is clearly a bijection sending an alternating element to a reverse alternating one, and vice versa.

Denote by $\tilde{E}_n$ the number of alternating signed permutations of B_n. The first ten values of $\tilde{E}_n$ are given in Table 5.2. It is convenient to record $\tilde{E}_n$ by means of a generating function, namely

$$\sum_{n\geqslant 0} \tilde{E}_n \frac{x^n}{n!} = \tan 2x + \sec 2x. \tag{5.5}$$

This result has a noncommutative generalization, as we shall see shortly.

Noncommutative sine and cosine functions are defined in [13] as

$$COS = \sum_{i\geqslant 0} (-1)^i S_{2i}, \quad SIN = \sum_{i\geqslant 0} (-1)^i S_{2i+1},$$

and other noncommutative trigonometric functions are then defined in terms of them. For example, the noncommutative secant $SEC = (COS)^{-1}$, and the noncommutative tangents $TAN_l = (COS)^{-1} \cdot SIN$, and $TAN_r = SIN \cdot (COS)^{-1}$. Motivated by (5.4), the noncommutative Euler "numbers" are then defined by

$$T_{2n} = R_{(2^n)}, \quad T_{2n+1}^{(r)} = R_{(1,2^n)}, \quad T_{2n+1}^{(l)} = R_{(2^n,1)}. \tag{5.6}$$

This choice of symbols by the authors of [13] is defective, in the sense that the letter T apparently stands for *T*angent numbers; but the secant number counterpart is unfortunately missing, as the letter S is already in use to denote the homogeneous functions.

The noncommutative generalization of (5.4) is given in Proposition 5.23 of [13], recollected below.

PROPOSITION 5.2.1. *One has the following identities:*

$$SEC = 1 + \sum_{n \geqslant 1} T_{2n},$$
$$TAN_l = \sum_{n \geqslant 0} T_{2n+1}^{(l)}, \quad TAN_r = \sum_{n \geqslant 0} T_{2n+1}^{(r)}.$$

LEMMA 5.2.2. *We have*

(i) $SEC = \sum_I (-1)^{|I|-l(I)} S^{2I}$, *where* $2I = (2i_1, \ldots, 2i_k)$ *if* $I = (i_1, \ldots, i_k)$,

(ii) $TAN_l = \sum_I (-1)^{|I|-l(I)} S^{2I} \sum_{j \geqslant 0} (-1)^j S_{2j+1}$,

(iii) $TAN_r = \sum_{j \geqslant 0} (-1)^j S_{2j+1} \sum_I (-1)^{|I|-l(I)} S^{2I}$,

with sums range over all compositions I.

Proof. (i) follows from

$$SEC = \left(1 - \sum_{i \geqslant 1} (-1)^{i-1} S_{2i}\right)^{-1} = \sum_{n \geqslant 0} \left(\sum_{i \geqslant 1} (-1)^{i-1} S_{2i}\right)^n$$
$$= \sum_{n \geqslant 0} \sum_{l(I)=n} (-1)^{|I|-l(I)} S^{2I} = \sum_I (-1)^{|I|-l(I)} S^{2I}.$$

(ii), (iii) follow from (i), $TAN_l = SEC \cdot SIN$, and $TAN_r = SIN \cdot SEC$. ■

We already have the trigonometric functions in S_i in place. It is natural to introduce their $\tilde{S}_i$ counterparts, as follows.

DEFINITION 5.2.3. The noncommutative trigonometric functions of type B are defined by

$$\widetilde{COS} = \sum_{i\geqslant 0}(-1)^i \tilde{S}_{2i}, \quad \widetilde{SIN} = \sum_{i\geqslant 0}(-1)^i \tilde{S}_{2i+1}.$$

We shall not introduce the B-analog of $\tilde{T}$. Instead, we use the ribbon Schur functions $\tilde{R}_I$ with I of the form $(0, 2^i)$, (2^i), $(0, 2^i, 1)$, $(2^i, 1)$, $(0, 1, 2^i)$, or $(1, 2^i)$, to denote the corresponding noncommutative Euler number. The leading 0 in I, if any, indicates the presence of descent at position 0, thus offering a refinement of the $\tilde{E}_n$ in (5.5). The generating functions for the Euler numbers $\tilde{R}_I$ can be expressed in terms of trigonometric functions in S_i and $\tilde{S}_i$. This is reported in the

PROPOSITION 5.2.4. *We have*

(i) $\sum_{i\geqslant 1} \tilde{R}_{(2^i)} = (1 - \widetilde{COS})\, SEC.$

(ii) $\sum_{i\geqslant 1} \tilde{R}_{(0,2^i)} = \widetilde{COS} \cdot SEC - 1.$

(iii) $\sum_{i\geqslant 0} \tilde{R}_{(2^i,1)} = \widetilde{SIN} + (1 - \widetilde{COS})\, TAN_l.$

(iv) $\sum_{i\geqslant 0} \tilde{R}_{(0,2^i,1)} = \widetilde{COS} \cdot TAN_l - \widetilde{SIN}.$

(v) $\sum_{i\geqslant 0} \tilde{R}_{(1,2^i)} = \widetilde{SIN} \cdot SEC.$

(vi) $\sum_{i\geqslant 0} \tilde{R}_{(0,1,2^i)} = (SIN - \widetilde{SIN})\, SEC.$

Proof. (i) By inclusion-exclusion,

$$\sum_{i\geqslant 1}\tilde{R}_{(2^i)}=\sum_{i\geqslant 1}\sum_{|I|=i}(-1)^{i-l(I)}\tilde{S}^{2I}.$$

If $I=(i_1,\dots,i_k)$, then let $I=J\cdot K$, where $J=(i_1)$, $i_1>0$, and $K=(i_2,\dots,i_k)$, possibly empty. It is obvious that $l(I)=1+l(K)$ so that $i-l(I)=i_1-1+|K|-l(K)$. By Lemma 5.2.2 (i), the above equation becomes

$$\begin{aligned}\textstyle\sum_{i\geqslant 1}\tilde{R}_{(2^i)} &= \sum_{i\geqslant 1}\sum_{\substack{i_1+|K|=i\\ i_1>0}}(-1)^{|J|+|K|-l(J)-l(K)}\tilde{S}^{2J}S^{2K}\\ &= -\sum_{i_1>0}(-1)^{i_1}\tilde{S}_{2i_1}\sum_{K}(-1)^{|K|-l(K)}S^{2K}\\ &= (1-\widetilde{COS})\,SEC\,.\end{aligned}$$

Thus, (i) holds. Since $T_{2n}=R_{(2^i)}=\tilde{R}_{(0,2^i)}+\tilde{R}_{(2^i)}$. (ii) follows from (i) and the first identity of Proposition 5.2.1. For (iii), by inclusion-exclusion,

$$\sum_{i\geqslant 0}\tilde{R}_{(2^i,1)}=\sum_{i\geqslant 0}\sum_{I\preccurlyeq(2^i,1)}(-1)^{i+1-l(I)}\tilde{S}^{I}.$$

Note that if $I=(i_1,\dots,i_k)\preccurlyeq(2^i,1)$, then either $I=(2j+1)$, $j\geqslant 0$, or $I=J\cdot 2K\cdot(2j+1)$ for some composition J,K, with $J=(2k)$, $k>0$, K possibly empty, and $j\geqslant 0$. It is easy to see that $2i+1=|I|=2|J|+2|K|+2j+1$ and $l(I)=l(J)+l(K)+1$ imply that $i+1-l(I)=|J|-l(J)+|K|-l(K)+j$. Incorporating these changes in the previous line, and using Lemma 5.2.2 (ii), we conclude that

$$\begin{aligned}\sum_{i\geqslant 0}\tilde{R}_{(2^i,1)} &= \sum_{j\geqslant 0}(-1)^j\tilde{S}_{2j+1}+\sum_{k>0}(-1)^{k-1}\tilde{S}_{2k}\sum_{K}(-1)^{|K|-l(K)}S^{2K}\sum_{j\geqslant 0}(-1)^jS_{2j+1}\\ &= \widetilde{SIN}+(1-\widetilde{COS})\;TAN_l,\end{aligned}$$

which is (iii). For (iv), using (iii) and the second equality of Proposition 5.2.1. To

n	$\tilde{E}_n$
1	2
2	4
3	16
4	80
5	512
6	3904
7	34816
8	354560
9	4063232
10	51733504

Table 5.2: Euler numbers $\tilde{E}_n$ of type B, for $n = 1, 2, \ldots, 10$.

prove (v), we invoke the inclusion-exclusion again, obtaining

$$\sum_{i \geqslant 0} \tilde{R}_{(1,2^i)} = \sum_{i \geqslant 0} \sum_{I \preccurlyeq (1,2^i)} (-1)^{i+1-l(I)} \tilde{S}^I.$$

We can write $I = (2j+1) \cdot 2K$, where $j \geqslant 0$, K a composition, possibly empty. The conditions $|I| = 2i+1 = 2j+1+2|K|$ and $l(I) = 1 + l(K)$ yield that $i + 1 - l(I) = j + |K| - l(K)$, so that the preceding equation becomes

$$\sum_{i \geqslant 0} \tilde{R}_{(1,2^i)} = \sum_{j \geqslant 0} (-1)^j \tilde{S}_{2j+1} \sum_{K} (-1)^{|K|-l(K)} S^{2K} = \widetilde{SIN} \cdot SEC,$$

where the last equality holds by Lemma 5.2.2 (i). Finally, (v) and the last equality of Proposition 5.2.1 together give (vi). ■

Recall the map φ as in Definition 5.1.12. Let z be an indeterminate commuting with **BSym**. We can extend φ to a homomorphism $\varphi_z\colon$ **BSym** $\to K[z]$ by defining $\varphi_z(\tilde{S}_n) = z^n/n!$ and $\varphi_z(S_n) = 2^n z^n/n!$, and extending algebraically to all of **BSym**. It is clear that $\varphi_z(\tilde{S}^I) = \varphi(\tilde{S}^I) z^{|I|}$. Clearly, $\varphi_z(\widetilde{COS}) = \cos z$, $\varphi_z(\widetilde{SIN}) = \sin z$, $\varphi_z(COS) = \cos 2z$, $\varphi_z(SIN) = \sin 2z$, and $\varphi_z(TAN_l) = \varphi_z(TAN_r) = \tan 2z$. So,

Corollary 5.2.5. *We have*

(i) $\sum_{i\geqslant 1}\varphi(\tilde{R}_{(2^i)})z^{2i} = (1-\cos z)\sec 2z.$

(ii) $\sum_{i\geqslant 1}\varphi(\tilde{R}_{(0,2^i)})z^{2i} = \cos z\sec 2z - 1.$

(iii) $\sum_{i\geqslant 0}\varphi(\tilde{R}_{(2^i,1)})z^{2i+1} = \sin z + (1-\cos z)\tan 2z.$

(iv) $\sum_{i\geqslant 0}\varphi(\tilde{R}_{(0,2^i,1)})z^{2i+1} = \cos z\tan 2z - \sin z.$

(v) $\sum_{i\geqslant 0}\varphi(\tilde{R}_{(1,2^i)})z^{2i+1} = \sin z\sec 2z.$

(vi) $\sum_{i\geqslant 0}\varphi(\tilde{R}_{(0,1,2^i)})z^{2i+1} = (\sin 2z - \sin z)\sec 2z.$

Simple algebra will show that the right sides of (iii) and (vi) are equal, and so are those of (iv) and (v). The sum of (i)–(iv) of Corollary 5.2.5 is

$$\sum_{i\geqslant 0}\{[\varphi(\tilde{R}_{(2^i)})+\varphi(\tilde{R}_{(0,2^i)})]z^{2i}+[\varphi(\tilde{R}_{(2^i,1)})+\varphi(\tilde{R}_{(0,2^i,1)})]z^{2i+1}\} = \sec 2z + \tan 2z,$$

the right hand side of which is the same as that of (5.5). Thus,

$$\varphi(\tilde{R}_{(2^i)})+\varphi(\tilde{R}_{(0,2^i)}) = \frac{\tilde{E}_{2i}}{(2i)!},\quad \varphi(\tilde{R}_{(2^i,1)})+\varphi(\tilde{R}_{(0,2^i,1)}) = \frac{\tilde{E}_{2i+1}}{(2i+1)!},\quad i=0,1,2,\dots,$$

showing that $R_{(2^i)} = \tilde{R}_{(2^i)} + \tilde{R}_{(0,2^i)}$ and $R_{(2^i,1)} = \tilde{R}_{(2^i,1)} + \tilde{R}_{(0,2^i,1)}$ are the noncommutative generalizations of $\tilde{E}_n/n!$. Since TAN_l and TAN_r have the same image under φ_z, we may as well form the sum (i)+(ii)+(v)+(vi), which gives the same result.

Example 5.2.6. We have

$$\varphi(\tilde{R}_{(2,2,2)}+\tilde{R}_{(0,2,2,2)}) = \varphi(\tilde{S}^{(0,2,2,2)} - \tilde{S}^{(0,4,2)} - \tilde{S}^{(0,2,4)} + \tilde{S}^{(0,6)}) = \frac{3904}{6!} = \frac{\tilde{E}_6}{6!}. \quad \square$$

EXAMPLE 5.2.7. We can obtain the number of alternating signed permutations of B_n in which 0 is a descent, denoted $\tilde{E}_{0,n}$, from Corollary 5.2.5. For example, the series expansion of the right side of Corollary 5.2.5 (i) is

$$\frac{1}{2!}z^2 + \frac{23}{4!}z^4 + \frac{1141}{6!}z^6 + \frac{103823}{8!}z^8 + \frac{15151981}{10!}z^{10} + O(z^{12}),$$

from which we obtain that $\tilde{E}_{0,2} = 1$, $\tilde{E}_{0,4} = 23$, $\tilde{E}_{0,6} = 1141$, etc. □

Formulas for $\varphi(\tilde{R}_{(2^i)})$, $\varphi(\tilde{R}_{(0,2^i)})$, $\varphi(\tilde{R}_{(1,2^i)})$, and $\varphi(\tilde{R}_{(0,1,2^i)})$ can also be given. Let us first give two useful formulas, adapted from Sachkov [30], namely

$$\begin{aligned}\sec 2x &= \sum_{k\geqslant 0} \tilde{E}_{2k}\frac{x^{2k}}{(2k)!},\\ \tan 2x &= \sum_{k\geqslant 1}(-1)^{k-1}B_{2k}2^{4k-1}(2^{2k}-1)\frac{x^{2k-1}}{(2k)!},\end{aligned}$$

where B_k is the kth Bernoulli number, and $\tilde{E}_k$ is the kth Euler number of type B. By equating the coefficients of $x^{2k}/(2k)!$ in the series expansion of $1 = \sec 2x \cos 2x$, we have the recursive formula for $\tilde{E}_{2k}$:

$$\tilde{E}_{2k} = \sum_{i=1}^{k}(-1)^{i+1}2^{2i}\binom{2k}{2i}\tilde{E}_{2(k-i)}, \quad k \geqslant 1,$$

where $\tilde{E}_0 = 1$.

PROPOSITION 5.2.8. *We have*

(i) $\varphi(\tilde{R}_{(2^i)}) = \sum_{k=0}^{i}(-1)^{i-k}\binom{2i}{2k}\tilde{E}_{2k}$,

(ii) $\varphi(\tilde{R}_{(0,2^i)}) = \sum_{k=0}^{i-1}(-1)^{i-k+1}\binom{2i}{2k}\tilde{E}_{2k}$,

(iii) $\varphi(\tilde{R}_{(1,2^i)}) = \varphi(\tilde{R}_{(0,2^i,1)}) = \sum_{k=0}^{i}(-1)^{i-k}\binom{2i+1}{2k}\tilde{E}_{2k}$,

(vi) $\varphi(\tilde{R}_{(0,1,2^i)}) = \varphi(\tilde{R}_{(2^i,1)}) = \sum_{k=0}^{i}(-1)^{i-k}(2^{2(i-k)+1}-1)\binom{2i+1}{2k}\tilde{E}_{2k}.$

Proof. Since the proofs of (i)–(iv) are of the same spirit, we only prove (i) and omit others. The right hand side of Corollary 5.2.5 (i) can be expanded as

$$\begin{aligned}
(1-\cos z)\sec 2z - 1 &= \sum_{l\geqslant 1}(-1)^{l+1}\frac{z^{2l}}{(2l)!}\sum_{k\geqslant 0}\tilde{E}_{2k}\frac{z^{2k}}{(2k)!} - 1 \\
&= \sum_{i\geqslant 0} z^{2i}\sum_{k+l=i}\frac{(-1)^{l+1}\tilde{E}_{2k}}{(2k)!(2l)!} - 1 \\
&= \sum_{i\geqslant 1}\frac{z^{2i}}{(2i)!}\sum_{k=0}^{i}(-1)^{i-k+1}\binom{2i}{2k}\tilde{E}_{2k}.
\end{aligned}$$

Equating the coefficients of z^{2i}, (i) follows. ■

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