Combinatorial and Umbral Methods for Orthogonal Polynomials

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Abstract

Combinatorial and Umbral Methods
for Orthogonal Polynomials

A dissertation presented to the Faculty of the
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This thesis gives generating functions for various classical orthogonal polynomials. The particular generating functions obtained are: a generating function for $H_{3n}(u)$ where $H_n(u)$ is the Hermite polynomial, a multilinear generating function for the Hermite polynomials of two variables, and a bilinear and multilinear generating function for Charlier polynomials. These generating functions are obtained using combinatorial and umbral methods which are different from the analytical methods used previously for the study of these polynomials.

For the combinatorial method, the polynomials are shown to represent certain combinatorial objects which are usually certain graphs. The main idea of the combinatorial proofs is to show that both sides of an identity enumerate the same set of graphs.

Contents

Introduction

The classical orthogonal polynomials such as Hermite, Laguerre, Jacobi, etc. have been extensively studied for many years and one of the treatises on the subject is Szegő's book [**17**]. There are several identities involving these polynomials and analytical methods involving the integral representations of these polynomials have been used to prove the identities. However, these polynomials can be shown to represent certain combinatorial objects and thus we can come up with new identities and generating functions, the validity of which can be proved using the combinatorial interpretation. The combinatorial objects are usually certain graphs and the main idea of the combinatorial proofs is to show that both sides of an identity enumerate the same set of graphs. These proofs give an interpretation to the identities and also help to count certain objects which are otherwise diffficult to count. For these reasons, combinatorial methods have been effectively used in the past two decades to study these polynomials.

In Chapter 1, we take a look at the computational models used in the combinatorial methods and then discuss some of the early combinatorial proofs of identities for Hermite polynomials. Then we go on to prove a new identity for Hermite polynomials in Chapter 2 using the combinatorial interpretation developed in the previous chapter. In the same chapter we also introduce another method of proof called the umbral calculus and give a second proof of the same identity.

We discuss Hermite polynomials of two variables in Chapter 3 and prove a multilinear extension of a bilinear generating function provided by Carlitz [**3**]. In Chapter 4, we turn to Charlier polynomials. We give a bilinear generating function for these polynomials and also extend it to a multilinear generating function.

CHAPTER 1

Exponential generating function formulas and their uses

1.1. Exponential Generating Functions

In this section, we rely on Chapter 5 from Stanley's book [**16**] for notation and basic properties of exponential generating functions. The exponential generating function $E_f(x)$ of the function $f\colon \mathbf{N} \to K$, where K is a field of characteristic 0, is the power series

$$E_f(x) = \sum_{n\geq 0} f(n)\frac{x^n}{n!}\,.$$

The exponential generating function is usually used when the function f counts "labeled objects". An example of a "labeled object" is a graph with numbered vertices.

There are some formulas which help us to find the exponential generating function of a new function which is defined in terms of functions whose exponential generating functions we know. We state two such formulas here which we will use later.

Product Formula. *Fix $k \in \mathbf{P}$ and let $f_1, \ldots, f_k\colon \mathbf{N} \to K$ be functions. Define a new function $h\colon \mathbf{N} \to K$ by*

$$h(n) = \sum_{(T_1,\ldots,T_k)} f_1(|T_1|)\cdots f_k(|T_k|) \tag{1}$$

where the sum runs over all ordered partitions $(T_1, \ldots, T_k)$ of $[n]$ into k blocks. Then $E_h(x) = \prod_{i=1}^{k} E_{f_i}(x)$.

1.1.1. Combinatorial significance of the Product Formula.

Suppose there are k types of structures, $\alpha_1, \ldots, \alpha_k$ which can be put on a finite set and $w_i(\Gamma)$ is the weight assigned to a structure Γ of type α_i. A "combined" structure of type "$\cup_i \alpha_i$" can be put on a finite set by taking an ordered partition $(T_1, \ldots, T_k)$ of the finite set and placing a structure of type α_i on T_i. The weight assigned to such a combined structure is defined to be the product of the weights of each part. If $f_i(n)$ denotes the sum of the weights of all structures of type α_i on $[n]$, then the equation (1) tells us that $h(n)$ is the sum of the weights of all structures of type $\cup_i \alpha_i$ on $[n]$. Thus the Product Formula can be used to find the exponential generating function of h if we know the exponential generating functions of the functions f_i; i.e., it helps us to count structures of type $\cup_i \alpha_i$ if we know how to count structures of type α_i.

We will use this formula in the following context : Let G be a set of a certain kind of weighted graphs. We assume that a graph is in G if and only if all its connected components are in G. The weight of a graph is equal to the product of the weights of its components, and isomorphic graphs have the same weight. Suppose $h(n)$ denotes the sum of the weights of the graphs in G on n vertices. Graphs are made up of connected components. Suppose there are k types of connected components, $\beta_1, \dots, \beta_k$ in the graphs in G. Define a graph of type α_i on $[n]$ to be a graph in G on n vertices whose connected components are only of type β_i. Let $f_i(n)$ denote the sum of the weights of all graphs of type α_i. Since any graph in G on n vertices can be obtained by taking an ordered partition $(T_1, \dots, T_k)$ of $[n]$ and placing a graph of type α_i on T_i, it is clear that h and f_i satisfy equation (1). Then the Product Formula tells us that the exponential generating function of the graphs in G is the product of the exponential generating functions of the f_i. So to find the exponential generating function of G, it is enough to find for each i, the exponential generating function of graphs with only β_i type connected components.

Exponential Formula. *Given a function $f\colon \mathbf{N} \to K$, define a new function $h\colon \mathbf{N} \to K$ by*

$$
\begin{aligned}
h(0) &= 1 \\
h(n) &= \sum_{\{B_1,\dots,B_k\} \in \Pi(n)} f(|B_1|) \cdots f(|B_k|), \quad \textit{if } n > 0,
\end{aligned}
\tag{2}
$$

where $\Pi(n)$ is the set of all partitions of $[n]$. Then $E_h(x) = \exp(E_f(x))$.

1.1.2. Combinatorial significance of the Exponential Formula. Many structures are made up of connected components. Such a structure can be put on a finite set by choosing a partition of the set and placing a connected structure on each block of the partition. If $f(n)$ denotes the sum of the weights of all connected structures on $[n]$, then the equation (2) tells us that $h(n)$ is the sum of the weights of all structures on $[n]$. Thus the Exponential Formula helps us to count structures if we know how to count the connected components of the structures. Again we will use this formula in the context of graphs. The formula tells us that to find the exponential generating function of graphs in a set, it is enough to know the exponential generating function of the connected graphs in that set. Here also we make the same assumptions for the set of graphs as we did for the product formula.

1.2. Mehler formula

One of the earliest papers which used the combinatorial method for orthogonal polynomials is [**4**] in which Foata gives a combinatorial proof of the Mehler formula involving the Hermite polynomials, using the Exponential Formula we saw in the previous section. The usual generating function for the Hermite polynomials $H_n(u)$ is

$$\sum_{n=0}^{\infty} H_n(u)\frac{z^n}{n!} = e^{2uz-z^2}, \tag{3}$$

which yields the following formula:

$$H_n(u) = \sum_{0\leq 2k\leq n} \frac{n!}{2^k\, k!\,(n-2k)!}(-2)^k(2u)^{n-2k}.$$

1.2.1. Combinatorial interpretation. Let $[n]$ denote the set $\{1, 2, \ldots, n\}$. Then the quotient in the above expression is the number of involutions of $[n]$ with k transpositions (cycles of length 2) and $n-2k$ fixed points. If we attach the weight -2 to each transposition and the weight $2u$ to each fixed point, it is clear that $H_n(u)$ is the generating function for the number of fixed points over the set of involutions of $[n]$. If σ is an involution of $[n]$, then $\sigma(i) = j$ if and only if $\sigma(j) = i$. So σ can be represented by a graph on n labeled vertices in which there is an edge joining vertices i and j if and only if $\sigma(i) = j$ and there is a loop at i if i is a fixed point of σ. Thus an involution of $[n]$ can be represented by a graph with n labeled vertices (with loops allowed) such that the degree of every vertex is one, where a loop contributes one to the degree of a vertex. Such a graph is called a **matching** as each vertex is either matched with one other vertex or with itself. Then the Hermite polynomial can be viewed as the generating function for the number of loops over the set of all matchings on n vertices.

Foata used this combinatorial model to prove the bilinear generating function for Hermite polynomials, known as the **Mehler formula**:

$$\sum_{n=0}^{\infty} H_n(u)H_n(v)\frac{z^n}{n!} = (1-4z^2)^{-1/2}\exp\left(\frac{4uvz-4(u^2+v^2)z^2}{1-4z^2}\right).$$

1.2.2. Foata's proof. The idea was to construct the same graphs in two ways: as superpositions of two matchings and as sets of their connected components and to show that this gives the two sides of the formula. The left-hand side was shown to be the exponential generating function for bicolored involutionary graphs. A bicolored involutionary graph is a graph with labeled vertices in which the edges and loops are colored with two colors, say blue and red, such that each vertex is incident to one blue edge or loop and one red edge or loop. Thus a bicolored involutionary graph on n vertices is simply the graphs of two involutions of $[n]$ superimposed on the same set of vertices. All the blue edges represent one involution and the red edges represent the other involution. Since we have the product of two Hermite polynomials in the left-hand side, it is clear that the left-hand side gives the exponential generating function for bicolored involutionary graphs.

The connected components of these graphs were shown to be of the following types:

1. Cycles of even length in which alternate edges are colored red and blue.

2. Paths with alternate edges colored red and blue and a red loop at one end and a blue loop at the other end. These paths are of even length.

3. Paths with alternate edges colored red and blue and a red loop at each end. These paths are of odd length.

4. Paths with alternate edges colored red and blue and a blue loop at each end. These paths are of odd length.

The following picture shows a representative of each type of connected component (the solid line represents the red color and the dotted line represents the blue color):

The right-hand side of the formula can be written as

$$\exp\left(\frac{1}{2}\log(1-4z^2)^{-1} + \frac{4uvz}{1-4z^2} - \frac{4u^2z^2}{1-4z^2} - \frac{4v^2z^2}{1-4z^2}\right),$$

which can be further rewritten as

$$\exp\left(\sum_{n=1}^{\infty}\frac{(2z)^{2n}}{2n} + \sum_{n=0}^{\infty}2uv(2z)^{2n+1} + \sum_{n=0}^{\infty}(-u^2)(2z)^{2n+2} + \sum_{n=0}^{\infty}(-v^2)(2z)^{2n+2}\right).$$

Each sum in the above expression was shown to be the exponential generating function for one type of connected bicolored involutionary graphs. Let us see how this was done for the cycle components. For a cycle, the number of vertices is even and each edge has a weight of -2 since it is a transposition in the corresponding involution. The number of such cycles on $2n$ vertices is $(2n-1)!$. So the weighted exponential generating function for the cycles is

$$\sum_{n=1}^{\infty}(2n-1)!(-2)^{2n}\frac{z^{2n}}{(2n)!} = \sum_{n=1}^{\infty}\frac{(2z)^{2n}}{2n}.$$

The exponential generating function for each of the other type of connected components was found on similar lines. Then the Exponential Formula (stated in section 1.1) was used to get the generating function for bicolored involutionary graphs.

1.3. Other formulas involving Hermite polynomials

Multilinear extensions of the Mehler formula were found by Kibble [**9**], Slepian [**14**] and Louck [**10**] and these were proved using various analytic methods. In [**6**], Foata and Garsia gave a combinatorial proof of the following formula of Slepian:

$$\sum_{(n_{ij})} \frac{\prod\limits_{1\le i<j\le k} {x_{ij}}^{n_{ij}}}{\prod\limits_{1\le i<j\le k} n_{ij}!} H_{n_1}(u_1)\cdots H_{n_k}(u_k) \tag{4}$$

$$= (detX)^{-1/2} \exp\left(\frac{-1}{2}\sum_{1\le i,j\le k} ({x_{ij}}^{-1} - \delta_{ij}) u_i\, u_j\right),$$

where the first sum runs over all $k \times k$ symmetric matrices (n_{ij}) with non-negative integral entries and with diagonal entries zero, and $n_i = \sum\limits_{j=1}^{k} n_{ij}$ for $1 \le i \le k$. $X = (x_{ij})$ is a $k \times k$ real symmetric matrix with all diagonal elements 1 and $X^{-1} = ({x_{ij}}^{-1})$ is the inverse of X and δij is the Kronecker symbol.

They used the same idea developed by Foata for the Mehler formula. But in this case, they had to count more elaborate graphs which they called n-involutionary graphs. The left-hand side was shown to be the exponential generating function of these graphs and the right-hand side was shown to be the exponential of the generating function for connected n-involutionary graphs. We use a similar method in Chapter 3 to get the multilinear extension of the bilinear generating function for Hermite polynomials of two variables.

In [**5**], Foata showed that the above mentioned multilinear extensions are equivalent. He also gave a combinatorial proof of the classical **Doetsch identity** which can be stated as:

$$\sum_{n=0}^{\infty} H_n(u) \frac{z^n}{\lfloor n/2 \rfloor !} = (1+4z^2)^{-3/2}(1+2uz+4z^2)\exp\left(\frac{4u^2z^2}{1+4z^2}\right).$$

Clearly this is the sum of the following two identities:

$$\sum_{n=0}^{\infty} H_{2n}(u) \frac{z^{2n}}{n!} = (1+4z^2)^{-1/2}\exp\left(\frac{4u^2z^2}{1+4z^2}\right). \tag{5}$$

$$\sum_{n=0}^{\infty} H_{2n+1}(u) \frac{z^{2n+1}}{n!} = (1+4z^2)^{-3/2} 2uz \exp\left(\frac{4u^2z^2}{1+4z^2}\right). \tag{6}$$

Both of these identities are consequences of the Mehler formula and so Foata used the combinatorial proof of the Mehler formula to prove the Doetsch identity.

If we replace z^2 by z in formula (5) we see that the Doetsch identity gives a generating function for $H_{2n}(u)$. In the next chapter we obtain a generating function for $H_{3n}(u)$.

CHAPTER 2

Hermite polynomials

2.1. Generating function for $H_{3n}(u)$

We normalize the generating function for Hermite polynomials given in (3) differently to get rid of the minus sign and obtain the following generating function:

$$\sum_{n=0}^{\infty} h_n(u)\frac{z^n}{n!} = e^{uz+z^2/2},$$

where $h_n(u) = \dfrac{i^n}{2^{n/2}}H_n\left(\dfrac{-iu}{\sqrt{2}}\right)$. Using this normalization, and replacing z^2 by z, formula (5) can be written as

$$\sum_{n=0}^{\infty} h_{2n}(u)\frac{z^n}{n!} = (1-2z)^{-1/2}\exp\left(\frac{u^2z}{1-2z}\right). \tag{7}$$

We go one step further and prove the following formula for $h_{3n}(u)$ by two methods — umbral and combinatorial:

$$\sum_{n=0}^{\infty} h_{3n}(u)\frac{z^n}{n!} = \frac{\exp\left(w^3z-9w^4z^2/2\right)}{\sqrt{1-6wz}}\sum_{n=0}^{\infty}\frac{(6n)!}{2^{3n}(3n)!(1-6wz)^{3n}}\frac{z^{2n}}{(2n)!}, \tag{8}$$

where $w = (1-\sqrt{1-12uz})/6z = uC(3uz)$ and $C(x) = (1-\sqrt{1-4x})/(2x)$ is the Catalan generating function. Note that the formula can be written in terms of hypergeometric series as follows:

$$\sum_{n=0}^{\infty} h_{3n}(u)\frac{z^n}{n!} = \frac{\exp\left(w^3z-9w^4z^2/2\right)}{\sqrt{1-6wz}}\,{}_2F_0\left(\frac{1}{6},\frac{5}{6};-;\frac{54z^2}{(1-6wz)^3}\right),$$

where ${}_2F_0(a,b,-;z) = \sum_{n\geq 0}(a)_n\,(b)_n\dfrac{z^n}{n\,!}$, is the usual hypergeometric series and $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ is the rising factorial.

In terms of the usual Hermite polynomials, the formula (8) can be written as:

$$\sum_{n=0}^{\infty} H_{3n}(u)\frac{z^n}{n!} = \frac{\exp(8v^3z+144v^4z^2)}{(1+48uz)^{1/4}}\sum_{n=0}^{\infty}\frac{(-1)^n(6n)!}{(3n)!(1+48uz)^{3n/2}}\frac{z^{2n}}{(2n)!},$$

where $v = uC(-12uz)$. However, we will use (8) in our proofs.

For the umbral proof, we define an umbra and study its properties that are useful in the calculations which lead to the formula. In the combinatorial proof we show that both sides of the formula enumerate the same set of graphs. We begin with an explanation of umbral calculus and then proceed to the proofs.

2.2. The Umbral calculus

The computations involving a sequence of numbers (a_n) are in some cases simplified if the subscripts are written as powers. Mathematicians like Blissard and Riordan had used this method in some proofs, but without much explanation as to why and how it works. Rota was the first one to point out explicitly in [**13**] that the shift from powers to subscripts can be explained through the introduction of a linear functional on the vector space of polynomials in the variable or 'umbra' a. However, just a linear functional was not enough to justify all the calculations done with the umbral method. So years later, Rota and Taylor in [**11**] laid a rigorous foundation for this method, which has come to be called the umbral calculus. They considered the vector space of polynomials in several variables or 'umbrae' and defined the linear functional eval on it. A sequence (a_n) is represented by an umbra A if $\text{eval}(A^n) = a_n$ for all n. In practice, the word eval is usually dropped and we simply write $A^n = a_n$ with the understanding that a functional has been applied. To justify various steps in umbral calculations, Rota and Taylor defined equivalence relations on umbral polynomials and demonstrated the umbral method with examples in the same paper and also in [**12**].

We do not need all the machinery developed by Rota and Taylor. The main point we wish to emphasize is that when we write $f(A) = g(A)$ for certain formal power series, we mean $\text{eval}(f(A)) = \text{eval}(g(A))$. We consider formal power series $f(t)$ with coefficients in a ring of formal power series $R = \mathbf{Q}[[x, y, z, \dots]]$.

Definition 2.2.1. A formal power series $f(t) = \sum_{n=0}^{\infty} f_n t^n$ is **admissible** if for every monomial $x^i y^j z^k \cdots$ in R, the coefficient of $x^i y^j z^k \cdots$ in f_n is nonzero for only finitely many values of n.

So for example, $f(t) = e^{xt}$ is an admissible formal power series, while $f(t) = e^t$ is not.

2.3. The Umbra and its properties

We define the umbra M by $e^{Mz} = e^{z^2/2}$ so that

$$M^n = \begin{cases} \dfrac{(2k)!}{2^k k!}, & \text{if } n = 2k \\ 0, & \text{if } n \text{ is odd.} \end{cases} \tag{9}$$

The following two formulas hold for M.

Lemma 2.3.1. *(i) For any admissible formal power series f,*

$$e^{Mz} f(M) = e^{z^2/2} f(M+z).$$

(ii) $e^{M^2 z} = \dfrac{1}{\sqrt{1-2z}}$.

Proof. (i) It is sufficient to prove that the formula holds for $f(y) = e^{yx}$. If the formula is true for $f(y) = e^{yx}$, then comparing coefficients of $x^n/n!$ on both sides we get that the formula holds for $f(y) = y^n$. But this implies that the formula is true for all admissible formal power series f, since the formula is linear in f. So we prove the formula for $f(y) = e^{yx}$. Then we have

$$e^{Mz} e^{Mx} = e^{M(z+x)} = e^{z^2/2+zx+x^2/2} = e^{z^2/2} e^{zx} e^{Mx} = e^{z^2/2} e^{(M+z)x}.$$

(ii) We have

$$e^{M^2 z} = \sum_{n=0}^{\infty} M^{2n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^n n!} \frac{z^n}{n!} = \frac{1}{\sqrt{1-2z}}.$$

□

Corollary 2.3.2. *For any admissible formal power series f,*

$$e^{M^2 z} f(M) = \frac{1}{\sqrt{1-2z}} f\left(\frac{M}{\sqrt{1-2z}}\right).$$

Proof. As in the lemma, it is sufficient to prove the formula for $f(y) = e^{yx}$. Then $e^{M^2 z} f(M) = e^{M^2 z + Mx}$. If we apply Lemma 2.3.1 (i) at this stage with $f(M) = e^{M^2 z}$, we cannot do anything with the linear term in M to get the desired result. So we introduce a new variable t and rewrite $e^{M^2 z + Mx}$ as $e^{Mt} e^{M^2 z + M(x-t)}$. We choose the value of t later. Now applying Lemma 2.3.1 (i) we get

$$e^{Mt}e^{M^2z+M(x-t)} = e^{t^2/2}e^{(M+t)^2z+(M+t)(x-t)} = e^{M^2z+M(x-(1-2z)t)+zt^2+xt-t^2/2}.$$

Now we choose the value of t to eliminate the term in M on the right; i.e., we set $x-(1-2z)t = 0$ and get $t = x/(1-2z)$. Putting this value of t in the above expression and simplifying, we obtain that $e^{M^2z+Mx} = e^{M^2z}e^{x^2/2(1-2z)}$. By Lemma 2.3.1 (ii), we get $e^{M^2z+Mx} = \dfrac{1}{\sqrt{1-2z}}e^{x^2/2(1-2z)}$. Finally using the definition of the umbra we get the desired result. □

Applying the corollary to $f(M) = e^{M^3x}$ gives the following result:

$$e^{M^2z+M^3x} = \frac{\exp\left(\dfrac{M^3x}{(1-2z)^{3/2}}\right)}{\sqrt{1-2z}}. \tag{10}$$

2.4. The umbral proof

To prove formula (8), we first obtain an umbral expression for $h_n(u)$ in terms of the umbra M. We have

$$\sum_{n=0}^{\infty} h_n(u)\frac{z^n}{n!} = e^{uz+z^2/2} = e^{(u+M)z}.$$

Comparing the coefficients of $z^n/n!$ on both sides, we get $h_n(u) = (u+M)^n$. Using this, we get

$$\sum_{n=0}^{\infty} h_{3n}(u)\frac{z^n}{n!} = \sum_{n=0}^{\infty}(u+M)^{3n}\frac{z^n}{n!} = e^{(u+M)^3z} = e^{(u^3+3Mu^2+3M^2u+M^3)z}$$

We follow the same procedure as in the proof of the Corollary 2.3.2. We introduce a new variable t and rewrite the last expression as $e^{Mt}e^{u^3z+M(3u^2z-t)+3M^2uz+M^3z}$. Now applying Lemma 2.3.1 (i) we get

$$e^{Mt}e^{u^3z+M(3u^2z-t)+3M^2uz+M^3z} = e^{t^2/2}e^{u^3z+(M+t)(3u^2z-t)+3(M+t)^2uz+(M+t)^3z}$$

$$= e^{zt^3+3uzt^2-t^2/2+3u^2zt+u^3z+M(3zt^2+(6uz-1)t+3u^2z)+M^2(3zt+3uz)+M^3z}. \tag{11}$$

In order to apply formula (10), we need to eliminate the term in M. So we choose a value of t that makes the coefficient of M equal to zero. Setting the coefficient of M equal to zero, we get $t = (1-\sqrt{1-12uz})/6z - u$. We know that the Catalan generating function $C(x)$ is given by $(1-\sqrt{1-4x})/2x$. In terms of this generating function $t = uC(3uz) - u = w - u$ where $w = uC(3uz)$. Using this value of t to simplify expression (11), we get that

$$\sum_{n=0}^{\infty} h_{3n}(u)\frac{z^n}{n!} = \exp(w^3z - (w-u)^2/2 + 3M^2wz + M^3z).$$

$C(x)$ satisfies the equation $C(x) = 1 + x(C(x))^2$. Substituting $x = 3uz$ and using the fact that $C(3uz) = w/u$, we obtain $w = u + 3w^2z$; i.e., $w - u = 3w^2z$. This gives us

$$\sum_{n=0}^{\infty} h_{3n}(u)\frac{z^n}{n!} = e^{w^3z-9w^4z^2/2}e^{3M^2wz+M^3z}.$$

Finally using Corollary 2.3.2 with $f(M) = M^3z$, we get that

$$\sum_{n=0}^{\infty} h_{3n}(u)\frac{z^n}{n!} = \frac{\exp\left(w^3z - 9w^4\frac{z^2}{2}\right)}{\sqrt{1-6wz}}\exp\left(\frac{M^3z}{(1-6wz)^{3/2}}\right).$$

Then writing the second exponential function as a series and using (9), we obtain the following formula:

$$\sum_{n=0}^{\infty} h_{3n}(u)\frac{z^n}{n!} = \frac{\exp\left(w^3z - 9w^4\frac{z^2}{2}\right)}{\sqrt{1-6wz}}\sum_{n=0}^{\infty}\frac{(6n)!}{2^{3n}(3n)!(1-6wz)^{3n}}\frac{z^{2n}}{(2n)!}.$$

We will give a combinatorial interpretation to this formula. To do so, first we take a look at the structures counted by w.

2.5. w-trees

We know that the Catalan generating function $C(z)$ is the generating function for unlabeled rooted binary trees with internal vertices weighted by z. By a rooted binary tree, we mean a rooted tree in which every vertex has zero or two children. A vertex with zero children is called a leaf while a vertex with two children is called an internal vertex. Thus every internal vertex (except the root) has degree three whereas a leaf has degree one. We will think of an edge as being made up of two **'half-edges'**, each half-edge incident with one of the end-vertices. For our purpose (which will become clear in the next section) we attach one extra half-edge to the root. Now every internal vertex has degree three. The picture below shows such a tree.

Note that in an unlabeled rooted binary tree, the order of the chidren at any internal vertex matters; i.e.,

So if $C(z) = \sum_{n=0}^{\infty} c_n z^n$, then c_n is the number of rooted binary trees (as shown in the picture) with n internal vertices.

Definition 2.5.1. A w-tree is a rooted binary tree with labeled internal vertices in which the three half-edges incident with every internal vertex are marked 1, 2 or 3. Moreover, all the marks at each internal vertex are distinct.

The picture below shows a w-tree.

A special case of a w-tree is the one with no internal vertices and no marks. It is simply a leaf with a half-edge attached to it. This is shown in the following picture.

Note that in a w-tree, the order of the children does not make a difference; i.e.,

2.5.1. Generating function for w-trees. We will find the exponential generating function in z for w-trees with each internal vertex weighted with z and each leaf weighted with u. To count w-trees with n internal vertices, we will construct such a w-tree starting with a rooted binary tree with n internal vertices. Then there are $n!$ ways to label the internal vertices. Now we need to put in the marks on the half-edges at the internal vertices. In order to avoid counting a w-tree twice, we will take the canonical representation of a w-tree to be the representation in which the two half-edges at an internal vertex connected to the two children are marked in an increasing order from left to right. Then in the above picture, the canonical representation is the one on the left. Thus while marking the half-edges, there are 3 choices to mark the half-edge at an internal vertex going to the parent (in case of the root, it is the extra half-edge) and then the other marks get decided automatically. Hence the number of w-trees with n internal vertices is $n!\, c_n 3^n$. Since a w-tree with n internal vertices has $n+1$ leaves, the exponential generating function for w-trees with the appropriate weights is

$$\sum_{n=0}^{\infty} n!\, c_n 3^n u^{n+1} \frac{z^n}{n!} = uC(3uz) = w.$$

Thus $w = uC(3uz)$ is the exponential generating function for weighted w-trees.

2.6. The combinatorial proof

To prove formula (8) combinatorially, we show that both sides of the formula count the same weighted structures. We have discussed the combinatorial interpretation of the Hermite polynomials in section 1.2. With the different normalization in this chapter, the interpretation stays the same; i.e., $h_n(u)$ is the generating polynomial for the number of fixed points over the set of involutions of $[n]$. The only difference is that now the weight associated with a transposition is 1 and the weight of a fixed point is u. Also in this section, we will not draw a loop at a fixed point in the matching associated to an involution. So a matching will be a graph with degree of every vertex at most one. The following picture shows a matching on 6 vertices.

Hence $h_n(u)$ can be considered as the generating polynomial for the number of vertices of degree zero over the set of all matchings on n vertices, where each vertex of degree zero is assigned the weight u.

2.6.1. Graphs counted by the left-hand side. The left-hand side of the formula (8) is an exponential generating function for $h_{3n}(u)$. From the interpretation of $h_n(u)$ given above, it is clear that this function counts certain graphs. Let us see how we can describe them. For a particular value of n, we start with n labels say $1, 2, \ldots, n$. For each label i, we construct a graph like this:

So the trivalent vertex gets the label i and each of the other three vertices is marked 1, 2 or 3. The marks have to be distinct. Then we have a graph with $3n$ marked vertices on which we construct the matching. Thus we get a graph in which every vertex has degree one (monovalent vertex), degree two (bivalent vertex), or degree three (trivalent vertex). A monovalent vertex (which is a fixed point of the matching) is always joined to a trivalent vertex and has weight u. A bivalent vertex is joined to a trivalent vertex and to another bivalent vertex. A trivalent vertex is joined to monovalent or bivalent vertices. The monovalent and bivalent vertices constitute a matching. Figure 1 shows such a graph.

Now we know the graphs that are counted by the left-hand side of the formula. Let G be the set of these graphs. We will use the Product Formula of section 1.1 to complete the proof. For this we need to figure out the various types of connected components of the graphs in G. Then for each type we will find the exponential generating functions for graphs that contain only that type of connected component.

2.6.2. Connected components of graphs in G. To start with, we eliminate all bivalent vertices from the graphs in G. A bivalent vertex is connected to one trivalent vertex and to another bivalent vertex. We transfer the mark on the bivalent vertex to the half-edge at the trivalent vertex to which it is connected. We also transfer the mark on a monovalent vertex to the half-edge at the trivalent vertex to which it is connected. Figure 2 shows the graph in Figure 1 with all bivalent vertices eliminated and all the marks transferred to the half-edges.

We see from the figure that this elimination may create some loops at some of the vertices. We think of a loop as contributing two to the degree of a vertex since a loop is an edge made up of two half-edges and each half-edge contributes one to the degree of the vertex. So every vertex in the new graph is monovalent or trivalent. Each of the three edges coming out of a trivalent vertex is incident to another trivalent vertex or to the same vertex or to a monovalent vertex. Each half-edge at a trivalent vertex has a mark and hence an edge joining two trivalent vertices or a loop has two marks on its two half-edges. We can easily recover the original graph by adding a vertex at each half-edge of such an edge or loop. Thus the graphs in G are reduced to graphs (with loops) which have only trivalent or monovalent vertices. Each trivalent vertex is weighted z and has marked half-edges incident to it while each monovalent vertex is weighted u.

Figure 1

Figure 2

The connected components of a graph in G are of two types: the ones which do not contain any cycles (acyclic) and the ones which contain at least one cycle. Note that a loop is considered to be a cycle.

2.6.3. Graphs with acyclic components. An acyclic connected component is a tree. The figure below shows such a tree.

We wish to count such unrooted trees. If we choose any trivalent vertex as a root, then the three half-edges at this vertex are joined to w-trees. So the exponential generating function for such rooted trees is $w^3 z$. We can edge-root a tree by choosing one special edge. If we root the tree at an edge joining two trivalent vertices, then with the help of the figure below we see that the exponential generating function for such edge-rooted trees is $9w^4z^2/2$.

An unrooted tree with n trivalent vertices gets counted n times in the generating function for rooted trees as there are n choices for the root. Since there are $n-1$ choices to choose an edge-root, it gets counted $n-1$ times in the generating function for edge-rooted trees. In order to get the generating function for unrooted trees, we subtract one function from the other so that each tree is counted once. Hence the exponential generating function for unrooted trees is $w^3z - 9w^4z^2/2$. Then the exponential generating function for graphs whose connected components are only trees is $\exp(w^3z - 9w^4z^2/2)$.

Now let us look at the connected components with at least one cycle. The two components besides the tree in the graph in Figure 2 are such components. First we reduce these components by shrinking all the w-trees present in them. The picture below illustrates this process.

As we see in the picture, the process of shrinking w-trees leaves behind vertices weighted with z which are now bivalent. We further reduce the components by eliminating these newly created bivalent vertices. Figure 3 shows what we get after this further reduction. Thus we get either circles with no vertices or components (with multiple edges and loops) in which each vertex is trivalent. We will consider the two cases separately.

2.6.4. Components with one cycle. If the reduction of a component results in a circle, this means the original component has exactly one cycle and each trivalent vertex on the cycle has a w-tree attached at its third half-edge; i.e, there is a cycle of

FIGURE 3

w-trees. We can see this if we trace the circle component in Figure 3 back to its original component in Figure 2. The exponential generating function for directed cycles with at least three vertices is $\sum_{n=3}^{\infty} z^n/n$ and hence for undirected cycles is $(1/2)\sum_{n=3}^{\infty} z^n/n$. However, in this case we have a cycle of vertices with a w-tree attached to each vertex which indicates that we need to replace z with wz in the generating function. Moreover, the half-edges at each vertex have distinct marks and this marking can be done in $3! = 6$ ways. So in fact, we need to replace z with $6wz$. Thus the exponential generating function for undirected cycles of w-trees is $(1/2)\sum_{n=3}^{\infty} (6wz)^n/n$. Note that the sum begins with $n = 3$ because three vertices are needed to form a cycle when no loops and multiple edges are allowed. But the graphs we are counting do have loops and multiple edges. So there can be a cycle with one or two vertices.

A cycle with only one vertex will look as follows:

The exponential generating function for such cycles is $3wz$ as there are 3 ways to mark the half-edge which goes to the w-tree and the choice of marks for the other two half-edges does not matter since they are connected to each other.

The picture for a cycle with two vertices is shown below:

The marks for each of the half-edges going to the w-trees can be chosen in 3 ways and then there are two ways to form the cycle. Hence the exponential generating function for these cycles is $18w^2z^2/2!$. Thus the exponential generating function for cycles of w-trees is

$$\frac{1}{2}\sum_{n=3}^{\infty}\frac{(6wz)^n}{n}+3wz+18w^2\frac{z^2}{2!}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(6wz)^n}{n}=\log{(1-6wz)}^{-1/2}.$$

Then the exponential generating function for graphs whose connected components are only cycles of w-trees is $\exp{(\log{(1-6wz)}^{-1/2})}=(1-6wz)^{-1/2}$.

2.6.5. Components with more than one cycle. The components that reduce to components with only trivalent vertices are the ones with more than one cycle. Let H be the set of graphs in G all of whose components have more than one cycle. Starting with any graph in H, we get a reduced graph by shrinking all the w-trees and eliminating all the newly created bivalent vertices. As we have seen through our example, all the components of a reduced graph have only trivalent vertices. Consider a reduced graph with m vertices. For the purposes of counting, we add a new vertex at each half-edge of a trivalent vertex. Then the graph looks as follows:

It has $3m$ additional vertices and we can think of it as a complete matching on $3m$ vertices. A **complete matching** is a graph with labeled vertices in which every vertex has degree one; i.e, every vertex is matched with exactly one other vertex. Then it is clear that in a complete matching the number of vertices is even. For $3m$ to be even, m has to be even. So $m = 2n$ for some n. Then the number of complete matchings on $6n$ vertices is $(6n)!/(2^{3n}(3n)!)$ and so the exponential generating function for reduced graphs is

$$\sum_{n=0}^{\infty} \frac{(6n)!}{2^{3n}(3n)!} \frac{z^{2n}}{(2n)!}.$$

Note that a connected reduced graph does have at least two cycles. This follows from the fact that in a connected graph with m vertices and e edges, any spanning tree has $m-1$ edges and for each of the $e-m+1$ edges not in a spanning tree there is a cycle containing this edge and some edges from the tree. So there are at least $e-m+1$ different cycles. A connected reduced graph on $m = 2n$ vertices has $e = 3n$ edges as it can be thought of as a complete matching on $6n$ vertices and for the purposes of counting we count the edges that constitute the matching. So the number of cycles is $3n - 2n + 1 = n + 1 \geq 2$. Thus every reduced graph is in H and any graph in H can be obtained from a reduced graph by introducing an ordered sequence of vertices at each edge. The vertices which are introduced are trivalent vertices with marked half-edges and with a w-tree attached to one of the half-edges. We can see this for the component with trivalent vertices in Figure 3 by tracing it back to its original component in Figure 2. So in the exponential generating function we get a term $\sum_{k=0}^{\infty} (6wz)^k = 1/(1-6wz)$ for each of the edges in a reduced graph. We have seen in subsection 2.6.4 how we get $6wz$ when we have a trivalent vertex with a w-tree attached to it. Hence the exponential generating function for graphs in H; i.e., for graphs all of whose components have more than one cycle is

$$\sum_{n=0}^{\infty} \frac{(6n)!}{2^{3n}(3n)!(1-6wz)^{3n}} \frac{z^{2n}}{(2n)!}.$$

We know that a connected component of any graph in G is either acyclic or contains exactly one cycle or has more than one cycle. In each case, we have found the exponential generating function for graphs whose connected components are only of that type. Then by the Product Formula in section 1.1, the exponential generating function for the graphs in G is the product of these generating functions which is

$$\exp(w^3z - 9w^4z^2/2) \frac{1}{(1-6wz)^{1/2}} \sum_{n=0}^{\infty} \frac{(6n)!}{2^{3n}(3n)!(1-6wz)^{3n}} \frac{z^{2n}}{(2n)!},$$

and this is the right-hand side of formula (8).

CHAPTER 3

Hermite polynomials of two variables

3.1. Definition and Generating Functions

The Hermite polynomials of two variables are defined as follows:

$$H_{m,n}(x,y) = \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} k!\, x^{m-k} y^{n-k}.$$

Carlitz proved in [**3**] that the bilinear generating function for the Hermite polynomials of two variables is given by

$$\sum_{m,\,n=0}^{\infty} H_{m,n}(x_1,y_1)\, H_{m,n}(x_2,y_2) \frac{\alpha^m \beta^n}{m!\, n!} = (1-\alpha\beta)^{-1} \exp\left(\frac{x_1x_2\alpha + y_1y_2\beta + (x_1y_1 + x_2y_2)\alpha\beta}{1-\alpha\beta} \right). \tag{12}$$

We obtain the following multilinear extension of this formula, similar to the multilinear expansion of the Mehler formula found by Slepian:

$$\begin{aligned}\sum_{(p_{ij})}\sum_{(q_{ij})} \frac{\prod\limits_{1\le i<j\le r} \alpha_{ij}^{p_{ij}} \beta_{ij}^{q_{ij}}}{\prod\limits_{1\le i<j\le r} p_{ij}!\, q_{ij}!} & H_{m_1,n_1}(x_1,y_1)\cdots H_{m_r,n_r}(x_r,y_r) \\ = (\det(I-\alpha\beta))^{-\frac{1}{2}} \exp\Bigg(\sum_{i,j=1}^{r} \frac{1}{2} \big[& x_i x_j ((I-\alpha\beta)^{-1}\alpha)_{ij} \\ & + y_i y_j (\beta (I-\alpha\beta)^{-1})_{ij} \big] + x_i y_j (\alpha\beta(I-\alpha\beta)^{-1})_{ij} \Bigg).\end{aligned} \tag{13}$$

In the above formula, $\alpha = (\alpha_{ij})$ and $\beta = (\beta_{ij})$ are matrices of indeterminates such that $\alpha_{ij} = \alpha ji$, $\beta_{ij} = \beta_{ji}$, and $\alpha ii = 0 = \beta ii$, I is the $r \times r$ identity matrix, the matrices (p_{ij}) and (q_{ij}) run over the set of all symmetric $r \times r$ matrices with non-negative integral entries and diagonal entries zero, and for $1 \leq i \leq r$, $m_i = \sum_{1 \leq j \leq r} p_{ij}$ and $n_i = \sum_{1 \leq j \leq r} q_{ij}$; i.e., m_i and n_i are the ith row sums of (p_{ij}) and (q_{ij}) respectively.

Recently Zeilberger (in [**18**]) defined the 'straight Hermite polynomials' by the following formula:

$$\widetilde{H}_{m,n}(x) = \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} k!\, x^k .$$

He obtained the following bilinear generating function for these polynomials:

$$\sum_{m,\,n=0}^{\infty} \widetilde{H}_{m,n}(x)\, \widetilde{H}_{m,n}(y) \frac{t^m\, s^n}{m!\, n!} = (1 - xyts)^{-1} \exp\left(\frac{t + s + xts + yts}{1 - xyts}\right). \tag{14}$$

These polynomials are related to the Hermite polynomials of two variables as follows: $H_{m,n}(x, y) = x^m\, y^n \widetilde{H}_{m,n}(1/xy)$ and so formula (14) can be obtained from formula (12). However, Zeilberger proved formula (14) using combinatorial methods which was not the case in the proof of formula (12). We will use the combinatorial interpretation of Zeilberger in our proof of formula (21).

3.2. Combinatorial Interpretation

Definition 3.2.1. An (m, n)-graph is a graph together with a partition (V_1, V_2) of the vertex set with $|V_1| = m$ and $|V_2| = n$, such that there is a non-loop edge between two vertices only if one vertex belongs to V_1 and the other to V_2.

So an (m, n)-graph can have loops at vertices in V_1 or V_2. In this chapter, we consider that a loop contributes one to the degree of a vertex. Then the straight Hermite polynomial can be viewed as the generating function for the number of non-loop edges over the set of (m, n)-graphs with every vertex of degree one, where each non-loop edge is assigned the weight x. Figure 1 shows a $(5, 4)$-graph with every vertex of degree one. This graph is similar to a matching (defined in Chapter 1) except that the vertex set is partitioned into two subsets and vertices from one subset can be matched either with themselves or with vertices from the other subset. We will call an (m, n)-graph with every vertex of degree one, an $(\mathbf{m}, \mathbf{n})$**-matching**.

FIGURE 1

Zeilberger used this interpretation of the straight Hermite polynomial for his proof of formula (14). The computational model he used was the **Bipartite Exponential Formula**, which is simply the adaptation of the Exponential Formula of section 1.1 to the case of two variables. For more details see [**18**]. We will use the Bipartite Exponential Formula in the same way as we used the Exponential Formula. We will find the bivariate exponential generating function of graphs in a set by computing the bivariate exponential generating function of the connected graphs in that set and then taking the exponential of this function.

We modify Zeilberger's combinatorial interpretation slightly to get an interpretation of the Hermite polynomial of two variables. In an (m, n)-matching, instead of attaching a weight x to an edge, we attach weights x, y to the loops at vertices in V_1 and V_2 respectively. For an (m, n)-matching G, let $l_m(G)$ denote the number of loops in V_1 and let $l_n(G)$ denote the number of loops in V_2. Then

$$H_{m,n}(x, y) = \sum x^{l_m(G)} y^{l_n(G)}, \tag{15}$$

where the sum extends over all (m, n)-matchings. Thus the Hermite polynomial of two variables is the generating function for the number of loops over the set of all (m, n)-matchings.

3.2.1. Generating function for Hermite polynomials of two variables. The connected components of an (m,n)-matching are loops in V_1, loops in V_2, and non-loop edges. Then the exponential generating function for connected components is $x\alpha+y\beta+\alpha\beta$. So by the Bipartite Exponential Formula, the exponential generating function for Hermite polynomials of two variables is

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{H_{m,n}(x,y)\alpha^m\beta^n}{m!\,n!} = e^{x\alpha+y\beta+\alpha\beta}\,.$$

In our proof of formula (21), we shall show that both sides enumerate certain kinds of (m,n)-graphs.

3.3. Graphs counted by the left-hand side.

The left-hand side of (21) can be rewritten as follows

$$\sum_{m,\,n=0}^{\infty}\frac{1}{m!}\frac{1}{n!}\sum\frac{m!}{\prod\limits_{1\le i<j\le r} p_{ij}!}\frac{n!}{\prod\limits_{1\le i<j\le r} q_{ij}!}\prod_{1\le i<j\le r}\alpha_{ij}^{p_{ij}}\beta_{ij}^{q_{ij}}\widetilde{H}_{m_1,n_1}(x_1,y_1)\cdots\widetilde{H}_{m_r,n_r}(x_r,y_r)\,.$$

where the inner sum runs over all the matrices (p_{ij}) and (q_{ij}) with $\sum\limits_{i<j} p_{ij} = m$ and $\sum\limits_{i<j} q_{ij} = n$.

Let $(P_{ij})_{1\le i<j\le r}$ and $(Q_{ij})_{1\le i<j\le r}$ be ordered partitions of $[m]$ and $[n]$ respectively such that $|P_{ij}| = p_{ij}$ and $|Q_{ij}| = q_{ij}$. For $j>i$, let $P_{ji} = P_{ij}$ and $Q_{ji} = Q_{ij}$. Let $M_i = \cup_{j\ne i}P_{ij}$ and $N_i = \cup_{j\ne i}Q_{ij}$. Then from the definition, it follows that for $i\ne j$, $M_i\cap M_j = P_{ij}$ and $N_i\cap N_j = Q_{ij}$. Then $|M_i| = m_i$ and $|N_i| = n_i$, where m_i and n_i are defined as in formula (21).

Let $G_{m,n}$ be the set of all ordered tuples $((P_{ij})_{i<j},(Q_{ij})_{i<j},G_1,\ldots,G_r)$ such that

(1) $(P_{ij})_{1\le i<j\le r}$ and $(Q_{ij})_{1\le i<j\le r}$ are ordered partitions of $[m]$ and $[n]$ respectively with the above properties.

(2) Each G_i is an (m_i,n_i)-matching with vertex bipartition (M_i,N_i), where M_i, N_i are as given above.

Each point of $[m]$ and $[n]$ is in exactly two of the matchings. This follows from the fact that each point of $[m]$ is in exactly one P_{ij} and $M_i\cap M_j = P_{ij}$ and each point of $[n]$ is in exactly one Q_{ij} and $N_i \cap N_j = Q_{ij}$. To each tuple $((P_{ij})_{i<j}, (Q_{ij})_{i<j}, G_1, \ldots, G_r)$, we associate an (m, n)-graph with the vertex set partitioned into (V_1, V_2) such that $V_1 = [m]$ and $V_2 = [n]$; this graph is obtained by superimposing all the G_i on the vertex set (V_1, V_2). Then $G_{m,n}$ is the set of all such (m, n)-graphs. For $r = 3$, Figure 2 shows a graph G in $G_{6,6}$. $G = ((P_{ij})_{i<j}, (Q_{ij})_{i<j}, G_1, G_2, G_3)$ where $P_{12} = \{1,\ 2,\ 3,\ 6\}$, $P_{13} = \varnothing$, $P_{23} = \{4,\ 5\}$.

and $Q_{12} = \{2,\ 3\}$, $Q_{13} = \{5\}$, $Q_{23} = \{1,\ 4,\ 6\}$. The solid lines represent G_1, the dashed lines represent G_2, and the dotted lines represent G_3. Here G_1 is a

FIGURE 2

$(4, 3)$-matching, G_2 is a $(6, 5)$-matching, and G_3 is a $(2, 4)$-matching. These matchings are shown in Figure 3.

Since $m!/\prod_{i<j} p_{ij}!$ counts the number of ordered partitions of $[m]$ into $(P_{ij})_{i<j}$, and $n!/\prod_{i<j} q_{ij}!$ counts the number of ordered partitions of $[n]$ into $(Q_{ij})_{i<j}$, the

FIGURE 3

left-hand side of (21) can be further rewritten as

$$\sum_{m,\,n=0}^{\infty} \frac{1}{m!}\frac{1}{n!} \sum \prod_{1\le i<j\le r} \alpha_{ij}^{p_{ij}} \beta_{ij}^{q_{ij}} \prod_{1\le i\le r} x_i^{l_{m_i}(G_i)}\, y_i^{l_{n_i}(G_i)},$$

where the second sum extends over all the graphs in $G_{m,n}$. Let each point of P_{ij} be weighted by α_{ij} and each point of Q_{ij} be weighted by β_{ij}. We define the weight of a graph $G = ((P_{ij})_{i<j}, (Q_{ij})_{i<j}, G_1, \ldots, G_r)$ in $G_{m,n}$ by

$$w(G) = \prod_{1\le i<j\le r} \alpha_{ij}^{p_{ij}} \beta_{ij}^{q_{ij}} \prod_{1\le i\le r} x_i^{l_{m_i}(G_i)}\, y_i^{l_{n_i}(G_i)}\,.$$

Then the left-hand side of (21) is

$$\sum_{m\ge 0}\sum_{n\ge 0} \frac{1}{m!}\frac{1}{n!} \sum_{G\in G_{m,n}} w(G)\,. \tag{16}$$

If we replace each α_{ij} with $t\alpha_{ij}$ and each β_{ij} with $s\beta_{ij}$ in (21), then the above expression will be

$$\sum_{m\ge 0}\sum_{n\ge 0} \frac{t^m}{m!}\frac{s^n}{n!} \sum_{G\in G_{m,n}} w(G)\,,$$

and we can see that this is the bivariate exponential generating function for $G_{m,n}$. However, the variables t and s do not play any role in the proof. To avoid keeping track of these extra variales, we will think of (16) as the bivariate exponential generating function for $G_{m,n}$ keeping in mind that we can introduce the extra variables to see this.

3.4. Connected Graphs in $G_{m,n}$

Now we try to enumerate the graphs in $G_{m,n}$ using their connected components. There are two kinds of connected graphs in $G_{m,n}$: cycles and paths.

3.4.1. Cycles. For a cycle in $G_{m,n}$, $m = n$ and the number of edges is $2m$, where $m \geq 1$, and there are no loops. Since a cycle has at least two edges, we have $m \geq 1$. A loop is not considered a cycle. Figure 4 shows a cycle.

FIGURE 4

Then we get

$$\sum_{m,n=1}^{\infty} \frac{1}{m!}\frac{1}{n!} \sum_{G \text{ a cycle in } G_{m,n}} w(G) = \sum_{m\geq 1} \frac{1}{(m!)^2} \sum w(G)\,,$$

where $w(G) = \prod_{1\leq i<j\leq r} \alpha_{ij}^{p_{ij}} \beta_{ij}^{q_{ij}}$ (since G has no loops.) The weight of the cycle in Figure 4 is $\alpha_{12}\alpha_{23}\beta_{12}\beta_{23}$. Suppose we fix the vertex 1 on the left as a root of the cycle and direct the cycle by tracing it starting with the root and then going to vertex 2 on the right and so on. Then starting with the root we can write the sequence of weights corresponding to this rooted directed cycle as $\alpha_{12}\beta_{23}\alpha_{32}\beta_{21}$. In general, the sequence

of weights corresponding to a rooted directed cycle (where the root is in V_1) is given by $\alpha_{k_1k_2}\beta_{k_2k_3}\alpha_{k_3k_4}\cdots\beta_{k_{2m}k_1}$, for some $1 \le k_1, k_2, \ldots, k_{2m} \le r$. Since the labels on the vertices in a cycle in $G_{m,n}$ can be permuted in $m!\,m!$ ways,

$$m!\,m!\sum_{i=1}^{r}\sum_{k_1=i,k_2,\ldots,k_{2m}}\alpha_{k_1k_2}\beta_{k_2k_3}\alpha_{k_3k_4}\cdots\beta_{k_{2m}k_1}$$

counts all rooted directed cycles, each of which corresponds to a cycle in $G_{m,n}$. In fact, each cycle in $G_{m,n}$ occurs $2m$ times in the above expression since there are m choices for the root and 2 ways to direct a rooted cycle. Hence

$$\begin{aligned}\sum_{G \text{ a cycle in } G_{m,n}} w(G) &= \frac{m!\,m!}{2m}\sum_{i=1}^{r}\sum_{k_2,\ldots,k_{2m}}\alpha_{ik_2}\beta_{k_2k_3}\alpha_{k_3k_4}\cdots\beta_{k_{2m}i}\\ &= \frac{m!\,m!}{2m}\sum_{i=1}^{r}(\alpha\beta)^m_{ii} = \frac{m!\,m!}{2m}\operatorname{Tr}(\alpha\beta)^m,\end{aligned}$$

where $\operatorname{Tr}(\alpha\beta)^m$ denotes the trace of the matrix $(\alpha\beta)^m$.

Then

$$\sum_{m\ge 1}\frac{1}{(m!)^2}\sum_{G \text{ a cycle in } G_{m,n}} w(G) = \sum_{m\ge 1}\frac{\operatorname{Tr}(\alpha\beta)^m}{2m} = \frac{1}{2}\operatorname{Tr}(\log(I-\alpha\beta)^{-1}).$$

But $\operatorname{Tr}(\log(I-\alpha\beta)^{-1}) = \log(\det(I-\alpha\beta))^{-1}$ (see [**8**]). Hence the exponential generating function for cycles is

$$\sum_{m\ge 1}\frac{1}{(m!)^2}\sum_{G \text{ a cycle in } G_{m,n}} w(G) = \frac{1}{2}\log(\det(I-\alpha\beta))^{-1} = \log(\det(I-\alpha\beta))^{-\frac{1}{2}}.$$

Remember that every graph in $G_{m,n}$ has a vertex bipartition (V_1, V_2). So a path in $G_{m,n}$ is one of the following three types:

(1) Both the endpoints are in V_1. Let P_1 be the set of all such paths.
(2) Both the endpoints are in V_2. Let P_2 be the set of all such paths.
(3) One endpoint is in V_1 and the other is in V_2. Let P_3 be the set of all such paths.

We will enumerate each of these types separately.

3.4.2. Paths in P_1. For a path in P_1, $m = n+1$ and the number of loops is 2 and the number of edges is $2n$, where $n \ge 0$. For $n = 0$, we get a path with just two loops at a vertex in V_1. Figure 5 shows a path in P_1.

Figure 5

Then we get

$$\sum_{m,\,n=0}^{\infty} \frac{1}{m!}\frac{1}{n!} \sum_{G\in P_1} w(G) = \sum_{n\geq 0} \frac{1}{n!\,(n+1)!} \sum w(G)\,,$$

where $w(G) = x_k x_l \prod_{1\leq i<j\leq r} \alpha_{ij}^{p_{ij}} \beta_{ij}^{q_{ij}}$ for some k and l (x_k and x_l are the weights of the two loops). The weight of the path in Figure 5 is $x_1 x_2 \alpha_{12}^2 \beta_{12}$. If we direct the path starting at one of the loops, say the loop at 1, then we can write the sequence of weights corresponding to this directed path as $x_1\alpha_{12}\beta_{21}\alpha_{12}x_2$. In general, the sequence of weights corresponding to a directed path is given by $x_i\alpha_{ik_2}\beta_{k_2k_3}\alpha_{k_3k_4}\cdots\beta_{k_{2n}j}\alpha_{jk}x_k$, for some $1\leq i,j,k,k_2,\ldots,k_{2n}\leq r$. Since the labels on the vertices in a path in P_1 can be permuted in $n!\,(n+1)!$ ways,

$$n!\,(n+1)! \sum_{i,j,k=1}^{r} \sum_{k_2,\ldots,k_{2n}} x_i\alpha_{ik_2}\beta_{k_2k_3}\cdots\beta_{k_{2n}j}\alpha_{jk}x_k$$

counts all directed paths, each of which corresponds to a path in P_1. Each path in P_1 can be directed in two ways and so we get

$$\sum_{G\in P_1} w(G) = \frac{n!\,(n+1)!}{2} \sum_{i,j,k=1}^{r} \sum_{k_2,\ldots,k_{2n}} x_i\alpha_{ik_2}\beta_{k_2k_3}\cdots\beta_{k_{2n}j}\alpha_{jk}x_k \tag{17}$$

$$= \frac{n!\,(n+1)!}{2} \sum_{i,j,k=1}^{r} x_i x_k (\alpha\beta)_{ij}^n \alpha_{jk}\,.$$

Hence the exponential generating function for the paths in P_1 is

$$\sum_{n\geq 0} \frac{1}{n!\,(n+1)!} \sum_{G\in P_1} w(G) = \frac{1}{2} \sum_{n\geq 0} \sum_{i,j,k=1}^{r} x_i x_k (\alpha\beta)_{ij}^n \alpha_{jk} = \frac{1}{2} \sum_{i,j,k=1}^{r} x_i x_k (I-\alpha\beta)_{ij}^{-1} \alpha_{jk}$$

$$= \frac{1}{2} \sum_{i,k=1}^{r} x_i x_k ((I - \alpha\beta)^{-1}\alpha)_{ik} \,,$$

where $(\alpha\beta)^n_{ij}$ denotes the (i,j) entry in the matrix $(\alpha\beta)^n$ and $(I-\alpha\beta)^{-1}_{ij}$ denotes the (i,j) entry in the matrix $(I-\alpha\beta)^{-1}$.

3.4.3. Paths in P_2. For a path in P_2, $n = m+1$, the number of loops is 2 and the number of edges is $2m$, where $m \geq 0$. Figure 6 shows such a path. This case is

FIGURE 6

analogous to the previous case. So proceeding on similar lines as before, we get

$$\sum_{G \in P_2} w(G) = \frac{m!\,(m+1)!}{2} \sum_{i,j,k=1}^{r} \sum_{k_2,\dots,k_{2m}} y_i \beta_{ij} \alpha_{jk_2} \beta_{k_2k_3} \alpha_{k_3k_4} \cdots \beta_{k_{2m}k} y_k \tag{18}$$

$$= \frac{m!\,(m+1)!}{2} \sum_{i,j,k=1}^{r} y_i y_k \beta_{ij} (\alpha\beta)^m_{jk} \,.$$

Hence the exponential generating function for the paths in P_2 is

$$\sum_{m \geq 0} \frac{1}{m!\,(m+1)!} \sum_{G \in P_2} w(G) = \frac{1}{2} \sum_{m \geq 0} \sum_{i,j,k=1}^{r} y_i y_k \beta_{ij} (\alpha\beta)^m_{jk} = \frac{1}{2} \sum_{i,j,k=1}^{r} y_i y_k \beta_{ij} (I - \alpha\beta)^{-1}_{jk}$$

$$= \frac{1}{2} \sum_{i,k=1}^{r} y_i y_k (\beta (I - \alpha\beta)^{-1})_{ik} \,.$$

3.4.4. Paths in P_3. For a path in P_3, $n = m$, the number of loops is 2 and the number of edges is $2m-1$, where $m \geq 1$. A path of this type is shown in Figure 7. Then we get

$$\sum_{m \geq 1} \sum_{n \geq 1} \frac{1}{m!} \frac{1}{n!} \sum_{G \in P_3} w(G) = \sum_{m \geq 1} \frac{1}{(m!)^2} \sum w(G) \,,$$

FIGURE 7

where $w(G) = x_k y_l \prod_{1 \le i < j \le r} \alpha_{ij}^{p_{ij}} \beta_{ij}^{q_{ij}}$ for some k and l. The weight of the path in Figure 7 is $x_2 y_3 \alpha_{12}\alpha_{23}\beta_{13}\beta_{23}$ which can be rewritten as $x_2\alpha_{23}\beta_{32}\alpha_{21}\beta_{13}y_3$. In general, the sequence of weights corresponding to such a path is given by $x_i\alpha_{ik_2}\beta_{k_2k_3}\alpha_{k_3k_4}\cdots\beta_{k_{2m}j}y_j$, for some $1 \le i, j, k_2, \ldots, k_{2m} \le r$. Note that in this case, each sequence gives a unique undirected, unlabeled path. Since the labels on the vertices can be permuted in $m!\,m!$ ways, we get

$$\sum_{G \in P_3} w(G) = m!\,m! \sum_{i,j=1}^{r} \sum_{k_2,\ldots,k_{2m}} x_i\alpha_{ik_2}\beta_{k_2k_3}\cdots\beta_{k_{2m}j}y_j = m!\,m! \sum_{i,j=1}^{r} x_i y_j (\alpha\beta)_{ij}^m .$$

Hence the exponential generating function for the paths in P_3 is

$$\begin{aligned}\sum_{m\ge 1}\frac{1}{(m!)^2}\sum_{G\in P_3} w(G) &= \sum_{m\ge 1}\sum_{i,j=1}^{r} x_i y_j (\alpha\beta)_{ij}^m = \sum_{i,j=1}^{r} x_i y_j \sum_{n\ge 0}(\alpha\beta(\alpha\beta)^n)_{ij} \\ &= \sum_{i,j=1}^{r} x_i y_j (\alpha\beta(I-\alpha\beta)^{-1})_{ij} .\end{aligned}$$

Thus the exponential generating function for the connected graphs in $G_{m,n}$ is

$$\begin{aligned}&\log(\det(I-\alpha\beta))^{-\frac{1}{2}} \\ &\quad + \sum_{i,j=1}^{r}\frac{1}{2}\big[x_i x_j((I-\alpha\beta)^{-1}\alpha)_{ij} + y_i y_j(\beta(I-\alpha\beta)^{-1})_{ij}\big] + x_i y_j(\alpha\beta(I-\alpha\beta)^{-1})_{ij} .\end{aligned}$$

Hence by the Bipartite Exponential formula, the exponential generating function for $G_{m,n}$ is

$$\begin{aligned}&\exp\Bigg(\log(\det(I-\alpha\beta))^{-\frac{1}{2}} \\ &\quad + \sum_{i,j=1}^{r}\frac{1}{2}\big[x_i x_j((I-\alpha\beta)^{-1}\alpha)_{ij} + y_i y_j(\beta(I-\alpha\beta)^{-1})_{ij}\big] + x_i y_j(\alpha\beta(I-\alpha\beta)^{-1})_{ij}\Bigg)\end{aligned}$$

$$= (\det(I - \alpha\beta))^{-\frac{1}{2}}$$
$$\cdot \exp\left(\sum_{i,j=1}^{r} \frac{1}{2}\left[x_i x_j((I-\alpha\beta)^{-1}\alpha)_{ij} + y_i y_j(\beta(I-\alpha\beta)^{-1})_{ij}\right] + x_i y_j(\alpha\beta(I-\alpha\beta)^{-1})_{ij}\right),$$

which gives the right-hand side of (21).

For $r = 2$, formula (21) reduces to the bilinear generating function (12).

CHAPTER 4

Charlier polynomials

4.1. Definition and Generating Functions

The Charlier polynomials are usually defined by the following formula:

$$c_n(a,r) = {}_2F_0(-n,-a;-;-r^{-1}).$$

For the purpose of combinatorial interpretation, we will normalize them as follows:

$$C_n(a,r) = r^n c_n(-a,r) = r^n\, {}_2F_0(-n,a;-;-r^{-1}) = \sum_{k=0}^{n} \binom{n}{k} (a)_k r^{n-k}\,.$$

We will prove the following bilinear formula using combinatorial methods:

$$\sum_{n\geq 0} C_n(a,r)C_n(b,s)\frac{x^n}{n!} = e^{rsx}\sum_{n\geq 0} \frac{(a)_n(b)_n}{(1-sx)^{n+a}\,(1-rx)^{n+b}}\frac{x^n}{n\,!}\,. \tag{19}$$

Formula (2.47) in Askey's book [**1**] is equivalent to the case of (19) in which a and b are negative integers and the general case is easily derived from this. Note that a and b are switched in the right-hand side of the formula in the book. The book does not give a combinatorial proof of the formula.

This generalizes to the following multilinear formula:

$$\sum_{(n_{ij})} \frac{\prod\limits_{1\leq i<j\leq k} {x_{ij}}^{n_{ij}}}{\prod\limits_{1\leq i<j\leq k} n_{ij}!} C_{n_1}(a_1,r_1)\cdots C_{n_k}(a_k,r_k) \tag{20}$$

$$= \prod_{1\leq i<j\leq k} e^{r_i r_j x_{ij}} \sum_{(n_{ij})} \prod_{1\leq i\leq k} \frac{(a_i)_{n_i}}{(1-\sum_{j\neq i} r_j x_{ij})^{n_i+a_i}} \frac{\prod\limits_{1\leq i<j\leq k} {x_{ij}}^{n_{ij}}}{\prod\limits_{1\leq i<j\leq k} n_{ij}!}\,,$$

where each sum runs over all $k\times k$ symmetric matrices (n_{ij}) with non-negative integral entries and with diagonal entries zero, and $n_i = \sum\limits_{j=1}^{k} n_{ij}$ for $1\leq i\leq k$.

To prove these identities combinatorially, first we show that the Charlier polynomial is the generating polynomial for some structure on a set of n elements. To do so, we need the lemmas and the theory in the following two sections.

4.2. Some Lemmas

Let σ be a permutation of $[n]$. Define the weight of σ to be $w_1(\alpha, \beta; \sigma) = \alpha^{\mathrm{cyc}(\sigma)} \beta^n$, where $\mathrm{cyc}(\sigma)$ is the number of cycles of σ, i.e., each cycle of σ is assigned a weight Al and each point is assigned a weight β.

Lemma 4.2.1. *$(\alpha)_n \beta^n$ is the generating polynomial for all the permutations of $[n]$, with the weight w_1.*

Proof. $\sum_{\sigma \in S_n} w_1(\alpha, \beta; \sigma) = \sum_{\sigma \in S_n} \alpha^{\mathrm{cyc}(\sigma)} \beta^n = \sum_{k=1}^{n} \left(\sum \alpha^k \beta^n\right)$, where the inner sum is over all permutations of $[n]$ with exactly k cycles. Let $c(n,k)$ be the number of such permutations. Then $\sum_{\sigma \in S_n} w_1(\alpha, \beta; \sigma) = \sum_{k=1}^{n} c(n,k)\alpha^k \beta^n = \beta^n \left(\sum_{k=1}^{n} c(n,k)\alpha^k\right)$. In Stanley's book [**15**] on page 19, it is shown that $\sum_{k=1}^{n} c(n,k)\alpha^k = (\alpha)_n$. So we get $\sum_{\sigma \in S_n} w_1(\alpha, \beta; \sigma) = \beta^n (\alpha)_n$. □

Lemma 4.2.2. *$(1 - \beta x)^{-\alpha}$ is the exponential generating function for all permutations, with the weight w_1.*

Proof. From lemma 4.2.1, we get that the exponential generating function for all permutations is $\sum_{n \geq 0} (\alpha)_n \beta^n \frac{x^n}{n!}$. Using the definition of $(\alpha)_n$ and the binomial theorem, we get that this is equal to $(1 - \beta x)^{-\alpha}$. □

4.3. Charlier Configurations

Definition 4.3.1. A Charlier configuration on $[n]$ is defined to be a pair $\Phi = ((A, \sigma), B)$, where (A, B) is an ordered partition of $[n]$ and σ is a permutation of A.

The configuration Φ can be represented by a digraph (that is, a directed graph) with vertex set $[n]$ and with an edge going from i to j if and only if $\sigma(i) = j$. The following figure shows a Charlier configuration on [10].

Here $A = \{1, 3, 4, 5, 6, 7, 9, 10\}$, $B = \{2, 8\}$, and $\sigma = (7)\ (4\ 9\ 6)\ (1\ 5\ 10\ 3)$ where the permutation is written as a product of disjoint cycles.

4.3.1. Combinatorial interpretation of Charlier polynomials. We assign a weight to a Charlier configuration Φ by assigning a weight a to each cycle of σ and a weight r to each point of B. Then Φ is weighted by $w_2(a, r; \Phi) = w_1(a, 1; \sigma)\, r^{|B|} = a^{\mathrm{cyc}(\sigma)} r^{|B|}$. Let C_n denote the set of Charlier configurations on $[n]$. Then letting $\alpha = a, \beta = 1$ in Lemma 4.2.1, we get $\displaystyle\sum_{\Phi \in C_n} w_2(a, r; \Phi) = \sum_{i=0}^{n} \binom{n}{i} (a)_i r^{n-i}$ which is the generating polynomial for all Charlier configurations on $[n]$ with the weight w_2. But

$$C_n(a, r) = \sum_{k=0}^{n} \binom{n}{k} (a)_k r^{n-k}\,.$$

Thus $C_n(a, r)$ is the generating polynomial for all Charlier configurations on $[n]$.

4.3.2. Generating function for Charlier polynomials. Let $\Phi = ((A, \sigma), B) \in C_n$. Then the connected components of the digraph of Φ are cycles of σ and vertices which are not in σ. Each vertex, which is not in σ, is in B and so is weighted by r. Hence the exponential generating function for digraphs with only this type of connected components is e^{rx}. Each cycle of σ is weighted by a. Then it follows from Lemma 4.2.2 that the exponential generating function for digraphs with only this type of connected components is $(1 - x)^{-a}$. Now it follows from the Product Formula of section 1.1 that $\displaystyle\sum_{n \geq 0} C_n(a, r) \frac{x^n}{n\,!} = \frac{e^{rx}}{(1 - x)^a}$.

4.4. Proof of Bilinear Formula

4.4.1. Structures counted by the left-hand side. Since $C_n(a,r)$ is the generating polynomial for all Charlier configurations on $[n]$, it is clear that the left-hand side of (19) counts ordered pairs of Charlier configurations, (Φ, Ψ) with Φ weighted by $w_2(a,r;\Phi)$ and Ψ weighted by $w_2(b,s;\Psi)$. We can represent an ordered pair (Φ,Ψ) by a digraph obtained by superimposing the digraphs of Φ and Ψ on the same set of labeled vertices. Let G be the set of these digraphs. Figure 1 shows a digraph in G representing the pair (Φ,Ψ) in which $\Phi = ((A,\sigma),B)$ where $A = \{1,4,7,9,11,13,14,16\}$, $B = \{2,3,5,6,8,10,12,15,17,18\}$, $\sigma = (7)\ (4\ 9\ 14)\ (1\ 13\ 11\ 16)$, and $\Psi = ((C,\pi),D)$ where $C = \{3,5,6,7,10,11,12,15,16,17,18\}$, $D = \{1,2,4,8,9,13,14\}$, $\pi = (3)\ (7\ 10)\ (11\ 18\ 16)\ (5\ 15\ 6\ 12\ 17)$. The permutation σ is shown by the solid lines while π is represented by the dashed lines.

FIGURE 1

Then the left-hand side of (19) is the generating function for the digraphs in G. We will show that the right-hand side also counts the same graphs by using the Product Formula. For this we need to find the connected components of the digraphs in G.

4.4.2. Connected components of digraphs in G. A digraph in G is obtained by superimposing the two Charlier configurations $\Phi = ((A, \sigma), B)$ and $\Psi = ((C, \pi), D)$ on a set of n labeled vertices. Each vertex of the graph is in both of the configurations. The connected components of the digraph are of the following types :

1. Vertices which are neither in π nor in σ. Vertices 2 and 8 in Figure 1 are such vertices. These vertices belong to $B \cap D$. So each vertex is weighted by r and s. Hence the exponential generating function for digraphs whose connected components are only of this type is e^{rsx}.

2. Cycles of σ in which no vertex is in π. The vertices of these cycles belong to $A \cap D$. The cycle (4 9 14) in Figure 1 is of this type. Each cycle is weighted by a and each vertex is weighted by s. The digraphs whose connected components are only of this type can be considered as permutations weighted by w_1. So from Lemma 4.2.2, it follows that the exponential generating function for such digraphs is $(1 - sx)^{-a}$.

3. Cycles of π in which no vertex is in σ. The vertices of these cycles belong to $B \cap C$. The cycles (3) and (5 15 6 12 17) in Figure 1 are of this type. As in the previous case, we get that the exponential generating function for digraphs whose connected components are only of this type is $(1 - rx)^{-b}$.

4. Connected digraphs consisting of cycles of σ and π, such that a cycle of σ has at least one vertex of π and vice-versa. These vertices are in either both π and σ or only in π or only in σ. The connected digraphs of this type from Figure 1 are shown in Figure 2. Consider digraphs whose connected components are only of this type. Let us say that such a digraph is reduced if every vertex is in both π and σ. Figure 3 shows a reduced digraph. Each cycle of π is weighted by b and each cycle of σ is weighted by a. So the exponential generating function for reduced digraphs is $\displaystyle\sum_{n \geq 0} (a)_n (b)_n \frac{x^n}{n!}$.

Any digraph of this type can be obtained from a reduced digraph as follows: Each vertex in a reduced digraph has two outgoing edges, one belonging to σ and the other belonging to π. Suppose (i, j) and (i, k) are the two outgoing edges at

Figure 2

Figure 3

a vertex i. Suppose (i,j) is in σ. Replace (i,j) by a sequence of directed edges $(i,i_1),(i_1,i_2),\ldots,(i_{m_i-1},i_{m_i}),(i_{m_i},j)$, such that each new vertex is in σ, but not in π. So each new vertex is weighted by s. Similarly replace (i,k) by a sequence of l_i+1 edges, such that each new vertex is in π, but not in σ. So each new vertex is weighted by r. We can do this at each vertex i. Then the weight of such a digraph is $a^{\mathrm{cyc}(\sigma)}\, b^{\mathrm{cyc}(\pi)}\, s^{m_1+m_2+\cdots+m_n}\, r^{l_1+l_2+\cdots+l_n}$, where each $m_i \geq 0$ and each $l_i \geq 0$. Thus starting with a reduced digraph, we obtain a digraph by replacing each outgoing edge at every vertex by a sequence of ordered edges whose vertices are either only in σ or only in π. This can be seen by comparing the reduced digraph in Figure 3 to the connected components in Figure 2. So in the exponential generating function for reduced digraphs, we get additional terms $\sum_{m=0}^{\infty} s^m = 1/(1-sx)$ and $\sum_{l=0}^{\infty} r^l = 1/(1-rx)$ for each vertex. Hence the exponential generating function for such digraphs is $\displaystyle\sum_{n\geq 0} \frac{(a)_n(b)_n}{(1-sx)^n\,(1-rx)^n}\frac{x^n}{n\,!}$.

Now it follows from the Product Formula that the exponential generating function of the digraphs in G is the product of the above four generating functions. This product is equal to

$$e^{rsx}\frac{1}{(1-sx)^a}\frac{1}{(1-rx)^b}\sum_{n\geq 0}\frac{(a)_n(b)_n}{(1-sx)^n\,(1-rx)^n}\frac{x^n}{n\,!}\,,$$

which is the same as

$$e^{rsx}\sum_{n\geq 0}\frac{(a)_n(b)_n}{(1-sx)^{n+a}\,(1-rx)^{n+b}}\frac{x^n}{n\,!}\,.$$

This is the right hand side of (19).

4.5. Multilinear Formula

One way to prove the multilinear formula is to think of it as a multivariate exponential generating function. In this case we have to keep track of different sets of labels. In order to use only one set of labels, we rewrite the formula by replacing x_{ij} with zx_{ij}. Now we can think of the formula as an exponential generating fuction in only one variable, namely z. The formula now looks as follows :

$$\sum_{(n_{ij})}\frac{\prod\limits_{1\leq i<j\leq k} {x_{ij}}^{n_{ij}}}{\prod\limits_{1\leq i<j\leq k} n_{ij}!}C_{n_1}(a_1,r_1)\cdots C_{n_k}(a_k,r_k)z^{\sum\, n_{ij}} = \tag{21}$$

$$\prod_{1\leq i<j\leq k} e^{r_ir_jx_{ij}z}\prod_{1\leq i\leq k}\frac{1}{\left(1-z\sum_{j\neq i}r_jx_{ij}\right)^{a_i}}\sum_{(n_{ij})}\prod_{1\leq i\leq k}\frac{(a_i)_{n_i}}{\left(1-z\sum_{j\neq i}r_jx_{ij}\right)^{n_i}}\frac{\prod\limits_{1\leq i<j\leq k} {x_{ij}}^{n_{ij}}}{\prod\limits_{1\leq i<j\leq k} n_{ij}!}z^{\sum\, n_{ij}}\,.$$

We will prove this formula, which is equivalent to (20).

4.5.1. Structures counted by the left-hand side. We can rewrite the left-hand side of (20) as follows:

$$\sum_{n\geq 0}\frac{z^n}{n!}\sum_{\substack{(n_{ij})\\ \sum_{i<j} n_{ij}=n}}\frac{n!}{\prod\limits_{1\leq i<j\leq k} n_{ij}!}\prod_{1\leq i<j\leq k} {x_{ij}}^{n_{ij}}C_{n_1}(a_1,r_1)\cdots C_{n_k}(a_k,r_k)\,.$$

Let $(N_{ij})_{1\le i<j\le k}$ be an ordered partition of $[n]$ such that $|N_{ij}| = n_{ij}$. For $j > i$, let $N_{ji} = N_{ij}$. Let $N_i = \cup_{j\ne i} N_{ij}$. Then from the definition, it follows that $N_i \cap N_j = N_{ij}$. Since $n_i = \sum_j n_{ij}$, $|N_i| = n_i$. Let H be the set of all ordered tuples $((N_{ij}), \Phi_1, \dots, \Phi_k)$ such that

(1) (N_{ij}) is an ordered partition of $[n]$ with the above properties.
(2) each Φ_i is a Charlier configuration on N_i, i.e., $\Phi_i \in C_{n_i}$.

Each point of $[n]$ is in exactly two configurations. This follows from the fact that each point is in exactly one N_{ij} and $N_i \cap N_j = N_{ij}$. Let each Φ_i be weighted by $w_2(a_i, r_i; \Phi_i)$ and each point of N_{ij} be weighted by x_{ij} . If we first fix the n_{ij}, then $\dfrac{n!}{\prod\limits_{1\le i<j\le k} n_{ij}!}$ counts the number of ordered partitions of $[n]$ into (N_{ij}) and $C_{n_i}(a_i, r_i)$ is the generating polynomial of all Charlier configurations on $[n_i]$, we see that the left hand side is the exponential generating function for H with these weights.

We associate a digraph to a tuple $((N_{ij}), \Phi_1, \dots, \Phi_k)$ in H by superimposing the digraphs of these k Charlier configurations on a set of n labeled vertices, such that each Φ_i is on n_i of these vertices. Figure 4 shows such a digraph for $k = 3$. The configurations Φ_1, Φ_2, Φ_3 are respectively represented by solid lines, dashed lines and dotted lines. Each vertex is in exactly two configurations and this is indicated by the two different circles around each vertex. The tuple and the configurations corresponding to the figure are:

$N_{12} = \{5, 8, 9, 11, 14, 16, 17\}$, $N_{13} = \{1, 2, 6, 12, 15\}$ and $N_{23} = \{3, 4, 7, 10, 13, 18\}$.

$\Phi_1 = ((A_1, \sigma_1), B_1)$ where $A_1 = \{1, 8, 9, 11, 12, 15, 16\}$, $B_1 = \{2, 5, 6, 14, 17\}$ and $\sigma_1 = (9\ 16\ 15)\ (1\ 12\ 11\ 8)$.

$\Phi_2 = ((A_2, \sigma_2), B_2)$ where $A_2 = \{3, 4, 5, 7, 8, 10, 11, 17, 18\}$, $B_2 = \{9, 13, 14, 16\}$ and $\sigma_2 = (5\ 17\ 8\ 11)\ (3\ 10\ 7\ 4\ 18)$.

$\Phi_3 = ((A_3, \sigma_3), B_3)$ where $A_3 = \{1, 4, 6, 7, 10, 13, 18\}$, $B_3 = \{2, 3, 12, 15\}$ and $\sigma_3 = (6\ 13)\ (1\ 10\ 7\ 18\ 4)$.

FIGURE 4

Now we can think of H as the set of these digraphs and then the left-hand side is the exponential generating function for these digraphs. As we did for the bilinear formula, we will find the connected components of the digraphs in H and then use the Product Formula to show that the right-hand side is also a generating function for the same digraphs.

4.5.2. Connected components of digraphs in H. Let $((N_{ij}), \Phi_1, \dots, \Phi_k)$ be a tuple in H, where $\Phi_i = ((A_i, \sigma_i), B_i) \in C_{n_i}$, for $1 \leq i \leq k$. The connected components of the digraph representing this tuple are of the following types:

1. For $i < j$, vertices which are in Φ_i and Φ_j but are not in σ_i and σ_j. Vertices 2 and 14 in Figure 4 are of this type. These vertices belong to $B_i \cap B_j$. Each vertex is weighted by r_i, r_j and x_{ij}. The digraphs, whose connected components are only

of this type, are simply points with the appropriate weights. Hence the exponential generating function for such digraphs is $e^{r_i r_j x_{ij} z}$.

2. For each i, cycles of σ_i in which no vertex is in any other σ_j. In Figure 4, the cycles of this type are (6 13) and (9 16 15). Each cycle is weighted by a_i. Each vertex is in some B_j and so is weighted by r_j and x_{ij}. The digraphs in which every connected component is of this type can be considered as permutations in which every cycle is weighted by a_i and every vertex is weighted by some r_j and x_{ij}. By a slight modification of Lemma 4.2.2 it follows that the exponential generating function for such digraphs is $(1 - z \sum_{j \neq i} r_j x_{ij})^{-a_i}$.

3. Connected digraphs in which for each i, cycles of σ_i have at least one vertex in σ_j for some $j \neq i$. The connected component of this type from Figure 4 are shown in Figure 5. Consider digraphs whose connected components are only of this type.

FIGURE 5

Let us say that such a digraph is reduced if every vertex is in two permutations. Thus, a reduced digraph on n vertices represents an ordered tuple $((N_{ij}), \Phi_1, \dots, \Phi_k)$ where each $\Phi_i = ((A_i, \sigma_i), \varnothing)$; i.e., each Φ_i is simply a permutation σ_i on n_i vertices. Figure 6 shows a reduced digraph. A cycle of σ_i is weighted by a_i. So the exponential

FIGURE 6

generating function for reduced digraphs is

$$\sum_{n\geq 0}\frac{z^n}{n!}\sum_{\substack{(n_{ij})\\ \sum\limits_{i<j} n_{ij}=n}}\frac{n!}{\prod\limits_{1\leq i<j\leq k} n_{ij}!}\prod_{1\leq i\leq k}(a_i)_{n_i}\prod_{1\leq i<j\leq k}{x_{ij}}^{n_{ij}}\,.$$

Each vertex in a reduced digraph has two outgoing edges belonging to two different permutations.

As in the proof of the bilinear formula, we see that any digraph of this type can be obtained from a reduced digraph by replacing each outgoing edge at every vertex by a sequence of ordered edges. An outgoing edge in σ_i is replaced by a sequence of edges, such that each new vertex is in σ_i, but not in any other σ_j. So each new vertex is weighted by some r_j and x_{ij} . Hence the exponential generating function for such digraphs is

$$\sum_{n\geq 0}\frac{z^n}{n!}\sum_{\substack{(n_{ij})\\ \sum\limits_{i<j} n_{ij}=n}}\frac{n!}{\prod\limits_{1\leq i<j\leq k} n_{ij}!}\prod_{1\leq i\leq k}\frac{(a_i)_{n_i}}{(1-z\sum\limits_{j\neq i} r_j x_{ij})^{n_i}}\prod_{1\leq i<j\leq k}{x_{ij}}^{n_{ij}}\,.$$

Again it follows from the Product formula of section 1.1 that the exponential generating function for digraphs in H is the product of all the above generating functions which is

$$\prod_{1\le i<j\le k} e^{r_i r_j x_{ij} z} \frac{1}{(1 - z\sum_{j\ne i} r_j x_{ij})^{a_i}}$$

$$\cdot \sum_{n\ge 0} \frac{z^n}{n!} \sum_{\substack{(n_{ij}) \\ \sum_{i<j} n_{ij}=n}} \frac{n!}{\prod_{1\le i<j\le k} n_{ij}!} \prod_{1\le i\le k} \frac{(a_i)_{n_i}}{(1 - z\sum_{j\ne i} r_j x_{ij})^{n_i}} \prod_{1\le i<j\le k} {x_{ij}}^{n_{ij}} .$$

This is the same as

$$\prod_{1\le i<j\le k} e^{r_i r_j x_{ij} z} \prod_{1\le i\le k} \frac{1}{(1 - z\sum_{j\ne i} r_j x_{ij})^{a_i}} \sum_{(n_{ij})} \prod_{1\le i\le k} \frac{(a_i)_{n_i}}{(1 - z\sum_{j\ne i} r_j x_{ij})^{n_i}} \frac{\prod_{1\le i<j\le k} {x_{ij}}^{n_{ij}}}{\prod_{1\le i<j\le k} n_{ij}!} z^{\sum n_{ij}} ,$$

which is the right-hand side of (21).

For $k = 2$, the multilinear formula reduces to the bilinear formula.

4.6. Related Formulas

Our approach is similar to the one taken by Gessel in [**7**] in which he gave a generating function for $3\times n$ Latin rectangles. A $k\times n$ **Latin rectangle** is a $k\times n$ array of numbers such that each row is a permutation of $[n]$ and each column has distinct numbers. Gessel identified a $3 \times n$ Latin rectangle with a pair of permutations (π, σ) of $[n]$ such that π, σ and $\pi\sigma^{-1}$ are derangements. A **derangement** is a permutation with no fixed points. In his proof, he obtained the generating function for $3 \times n$ Latin rectangles as a specialization of a generating function for pairs of derangements. He also obtained the following specialization of the generating function for pairs of derangements:

$$\sum_{n=0}^{\infty} D_n(\alpha) D_n(\beta) \frac{x^n}{n!} = e^{\alpha\beta x} \sum_{n=0}^{\infty} \frac{(\alpha)_n\,(\beta)_n}{(1+\beta x)^{n+\alpha}\,(1+\alpha x)^{n+\beta}} \frac{x^n}{n\,!} \tag{22}$$

where $D_n(\alpha) = \sum_{\pi} \alpha^{\mathrm{cyc}(\pi)}$ and the sum is over all derangements π of $[n]$. Gessel mentioned that this formula is a special case of the bilinear generating function for Charlier polynomials and we will show how it can be obtained from the bilinear formula (19).

4.6.1. Special case of the bilinear formula. We have seen that the Charlier polynomial $C_n(a,r)$ is the generating polynomial for all Charlier configurations on $[n]$. A Charlier configuration $\Phi = ((A,\sigma),B)$ on $[n]$ can be identified with a permutation π of $[n]$ such that $\pi = \sigma$ on the set A and the points of B are fixed points of π. So the cycles of σ of length one and the points of B are all the fixed points of π. Then

$$C_n(a,r) = \sum_{\pi \in S_n} a^{\mathrm{cyc}_{>1}(\pi)} (a+r)^{\mathrm{fix}(\pi)},$$

where S_n is the set of all permutations of $[n]$, $\mathrm{cyc}_{>1}(\pi)$ is the number of cycles of π with length greater than one and $\mathrm{fix}(\pi)$ is the number of fixed points of π. Setting $r = b - a$, we get

$$C_n(a, b-a) = \sum_{\pi \in S_n} a^{\mathrm{cyc}_{>1}(\pi)} b^{\mathrm{fix}(\pi)}.$$

Then putting $b = 0$ gives $C_n(a,-a) = \sum_\pi a^{\mathrm{cyc}(\pi)}$ where the sum is over all derangements π of $[n]$; i.e., $C_n(a,-a) = D_n(a)$. Hence setting $a = \alpha$, $r = -\alpha$, $b = \beta$ and $s = -\beta$ in the bilinear formula (19) yields formula (22).

4.6.2. Special case of the multilinear formula. In [**2**], Carlitz obtained some formulas which later turned out to be special cases of Slepian's multilinear extension of the Mehler formula. On similar lines, we obtain a special case of the multilinear formula. We look at the case $k = 3$. Furthermore, we set $n_{12} = 0,\ n_{13} = m,\ n_{23} = n,\ x_{12} = 0,\ x_{13} = x,\ x_{23} = y$ in formula (20). Then we have $n_1 = m,\ n_2 = n$ and $n_3 = m + n$. Hence we get the following formula:

$$(23) \quad \sum_{m,n=0}^{\infty} C_m(a,r)\, C_n(b,s)\, C_{m+n}(d,t)\, \frac{x^m\, y^n}{m!\, n!}$$
$$= e^{(rx+sy)t} \sum_{m,n=0}^{\infty} \frac{(a)_m\, (b)_n\, (d)_{m+n}}{(1-tx)^{m+a}\,(1-ty)^{n+b}\,(1-rx-sy)^{m+n+d}} \frac{x^m\, y^n}{m!\, n!}.$$

Note that a_1 has been replaced by a, r_1 has been replaced by r and so on. From our proof of the multilinear formula, we know that each side of formula (23) counts tuples consisting of three Charlier configurations. For particular values of m and n, $n_{12} = 0$ means that the first and second configurations have no points in common and each point is in either the first and third or the second and third configurations.

In the previous subsection, we saw that $C_n(a,-a) = D_n(a)$. In particular, setting $a = 1$ we get $C_n(1,-1) = D_n(1) = D_n$ where D_n is the number of derangements of the set $[n]$. So setting $a = b = d = 1$ and $r = s = t = -1$ in formula (23) we obtain the following formula for derangement numbers:

$$\sum_{m,n=0}^{\infty} D_m\, D_n\, D_{m+n}\, \frac{x^m\, y^n}{m!\, n!} = e^{(x+y)} \sum_{m,n=0}^{\infty} \frac{(m+n)!}{(1+x)^{m+1}\,(1+y)^{n+1}\,(1+x+y)^{m+n+1}} x^m\, y^n\,.$$

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