Polytopes, Permutations, and the Bin Packing Problems

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ABSTRACT

Polytopes, Permutations, and the Bin Packing Problems

A dissertation presented to the Faculty of the
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by Marguerite Ann Eisenstein-Taylor

We consider the NEXT FIT and DUAL NEXT FIT algorithms for the one-dimensional bin and dual bin packing problems. We introduce polytopes derived from these algorithms in such a way that each polytope corresponds to an arrangement of objects into bins produced by one of the bin packing algorithms. In the case of a continuous uniform distribution on object sizes, the volume of this polytope is the probability that the corresponding arrangement occurs.

In the dual bin case, we calculate the expectation and variance for the number of bins used and calculate a (three-variable) generating function for the probability that a given number of objects uses a particular number of bins. We furthermore introduce a map on the unit hypercube that transforms the problem into a question of permutation enumeration. In the process, we generalize the hook decomposition [**5**] of Gessel and provide a combinatorial explanation of why the generating function for the dual bin case specializes to the generating function for derangements.

We proceed to calculate the continuous and discrete generating functions in the threshold k dual bin packing problem, and generalize the restatement as a question of permutation enumeration, introducing the k-hook decomposition of permutations.

The techniques we have introduced for the dual bin case can be modified for the bin packing problem. We do this and reduce computation of polytope volume to enumeration of permutations according to their numbers of maximal runs or peaks and valleys. As a result, we directly recover the expectation and variance results of Hofri [**10, 9**] as well as Hofri's generating function for the capacity 1 bin case.

We introduce the notion of a "k"-marked permutation shape in order to generalize the combinatorics for the capacity 1 bin case to the capacity k situation. As a result, we are able to explicitly calculate the capacity 2 generating function.

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CHAPTER 1

Introduction and Preliminaries

1.1. Introduction

Previous considerations of the one-dimensional bin and dual bin packing problems ([**10, 9, 1**]) have concentrated on examining the efficacy of different packing algorithms, analyzing them via probabilistic methods. In this dissertation we examine the inequalities derived from the simplest algorithms, NEXT FIT and DUAL NEXT FIT. We use these inequalities to define polytopes, then calculate generating functions, expectations, and variances by examining the volumes of (or number of lattice points in) these polytopes. To do so, we make use of transformations similar to those discussed by Stanley in [**14**] and [**13**], thereby either expressing the volumes in terms of Eulerian numbers or converting the problem into one of permutation enumeration. We then apply enumeration techniques developed in [**2**], [**6**], [**12**], and [**8**].

We first directly calculate the generating function for the threshold 1 continuous and discrete dual bin packing problems. We accomplish this by calculating the volume of the polytopes corresponding to single bins, and then considering a dual bin arrangement to be an element of a free monoid whose primes are full bins, followed by a leftover bin. A similar method gives the generating function when the object sizes follow a discrete distribution.

We use the generating function to derive a closed form for the generating function for the expectation of how many bins a sequence fills. We give an explicit formula for the expectation as well as an asymptotic approximation together with a bound on the error. We use the techniques of Wilf [**15**] to find an asymptotic approximation for the variance, as well as a bound for its error term.

We then convert the problem into one of permutation enumeration. We introduce the hook-shape decomposition set, generalizing a construction of Gessel in [**5**]. We then define a function that bijectively takes polytopes corresponding to dual bin arrangements to polytopes corresponding to hook-shape decomposition sets.

We next consider the threshold k case of the dual bin problem. We calculate the continuous distribution generating function via manipulations involving the Eulerian numbers, and the discrete distribution generating function by counting the number of compositions that we can view as a certain dual bin arrangement. We generalize the threshold 1 situation by defining k-hook-shape decomposition sets, and relating polytopes corresponding to threshold k dual bin arrangements to polytopes corresponding to these sets. We briefly explore a few identites involving the number of permutations whose shapes are k-hook-shapes.

We analyze the continous bin packing problem by converting it into one of permutation enumeration; instead of finding the volumes of polytopes corresponding to bin arrangements, we merely need to count permutations according to their numbers of maximal runs or peaks and valleys. This interpretation leads to a direct method of calculating the expectation and variance. Similarly, this approach allows us to explicitly restate the discrete bin case as yet another enumeration problem.

We provide three different methods of deriving the continuous bin generating function found in by Hofri in [**10**] by probabilistic methods. (In [**9**], he uses this function to determine the expectation and variance.) First we use methods of Goulden and Jackson describe in [**8**], then we count certain barred permutations motivated by work in Gessel [**6**]. Finally, we combine and apply techniques discussed in [**12**, pp. 241–243], [**6**, chs. 4, 6], and [**2**] to count permutations whose patterns of maximal increasing run lengths are encoded as paths in a digraph with weighted edges.

For the capacity k case of the bin packing problem, we "k-mark" permutation shapes. The volume of a particular capacity k bin arrangement is the number of permutations whose k-marked shapes have marked letters occurring in certain positions. Finally, we explicitly calculate the generating function for the capacity 2 case using the digraph method developed for the capacity 1 situation, and we set up the corresponding calculation for capacity 3.

1.2. Polytopes and the Bin Packing Problems

In (one dimensional) *bin packing* problems a collection of objects (pickles, rolled oats, freight) of varying sizes is placed into bins (pickle jars, oat boxes, freight containers) of a fixed capacity. In essence, the problem looks at how to fit the objects into the bins. Considerations include finding the most efficient algorithms for bin packing, determining whether the algorithms are practical for real-time applications, maximizing the number of bins a collection of objects can fill, or minimizing the number of bins they will fit in. The distributions of the sizes of the objects can be varied, as well as the capacity of the bin relative to the object sizes.

We will always assume that the size of our objects follows a uniform distribution, sometimes continuous, sometimes discrete.

1.2.1. The Bin Packing Problem. In the (ordinary) bin packing problem, one wishes to minimize the number of bins that the objects fill. So the constraining factor is the maximum amount that each bin can hold.

We formalize this as follows. The objects correspond to a set of items $\{1, 2, \dots, m\}$, each of which is assigned a numerical size. We let x_i be the size of i and require that $0 < x_i \leq 1$. The bins correspond to sets B into which $\{1, 2, \dots, m\}$ is partitioned. Each bin has the same capacity, which we will take to be some positive integer k. Since the constraining factor is the maximum that each bin can hole, we require that for any bin B, $\sum_{i \in B} x_i \leq k$.

Definition 1.1. A bin arrangement of size m is an ordered partition of $\{1, 2, \dots, m\}$.

For example, the following shows two different bin arrangements of five objects into two bins.

$$A : \qquad B_1 = \boxed{1\ \ 2\ \ 3} \qquad B_2 = \boxed{4\ \ 5}$$

$$A' : \qquad B_1 = \boxed{1\ \ 2} \qquad B_2 = \boxed{3\ \ 4\ \ 5}$$

If we take $k = 1$, then arrangement A implies that the inequalities $x_1 + x_2 + x_3 \leq 1$ and $x_4 + x_5 \leq 1$ hold. Similarly, arrangement A' implies that the inequalities $x_1 + x_2 \leq 1$

and $x_3 + x_4 + x_5 \leq 1$ hold. Note that the sequence of objects having weights $(.25, .25, .25, .5, .1)$ is compatible with either bin arrangement.

Perhaps the easiest (real-time) algorithm for placing a sequence of objects into bins is the NEXT FIT algorithm which we describe following [**10**].

Definition 1.2. The NEXT FIT bin packing algorithm is as follows.

Set $i, j := 1$

Place $i, i+1, \ldots, l \in B_j$ as long as $x_i + \cdots + x_l \leq k$

If $x_i + \cdots + x_{l+1} > k$ then

Reset $i := l + 1$, $j := j + 1$, and continue with the second step.

Remark 1.1. The algorithm NEXT FIT places the objects in order into the current bin as long as possible, then starts on the next bin. It sequentially splits the objects into bins; if i is in the jth bin B_j, then $i + 1$ must be either in B_j or in the $j + 1$st bin B_{j+1}. Hence we can consider each NEXT FIT-compatible bin arrangement with m objects and n bins as a composition of m with n parts.

If the bin arrangements A and A' shown above were obtained using NEXT FIT, then arrangement A would require the additional inequality $x_1 + x_2 + x_3 + x_4 > 1$, and A' would require that $x_1 + x_2 + x_3 > 1$ holds. The sequence $(.25, .25, .25, .5, .1)$ mentioned above would then under NEXT FIT only yield bin arrangement A. In particular, given a sequence of objects, NEXT FIT will determine a unique bin arrangement it satisfies. Henceforth we will assume that all bin arrangements are obtained from applying NEXT FIT to some sequence of objects.

1.2.2. The Dual Bin Packing Problem. A variation of the ordinary bin packing problem is what [**1**] calls the *dual bin packing* problem. Instead of minimizing the bins necessary, we wish to maximize the number of bins that the objects can fill, while still requiring that each bin contains at least a certain amount k. We call this amount the threshold of the bin. For example, given a sequence of pickles, we may wish to fill as many jars as possible while still making sure that each jar contains at least 10 oz. of pickles.

We require that the object sizes x_i satisfy $0 \leq x_i < 1$. We wish the amount in each bin to exceed the threshold. Hence in order for a bin to be full, $\sum_{i \in B} x_i \geq k$. This motivates the following definition.

Definition 1.3. Let B be a dual bin with threshold k. Then we call B a full bin if $\sum_{i \in B} x_i \geq k$ and a leftover bin if $\sum_{i \in B} x_i < k$.

Note that, since the object size is strictly less than the threshold of a bin, a full bin must necessarily contain at least two objects.

Definition 1.4. A dual bin arrangement is a partition of labeled objects into labeled sets, with the added information of whether the sums of the weights of the objects in the last set exceeds the threshold of the bin. If so, we call it a full dual bin arrangement. If not, we call it a leftover dual bin arrangement.

Note that in a dual bin arrangement, all the bins are full bins except the last bin of a leftover arrangement.

We call the dual bin problem's equivalent of the NEXT FIT algorithm DUAL NEXT FIT.

Definition 1.5. We define the DUAL NEXT FIT algorithm as follows.

Set $i, j := 1$

Place $i, i+1, \dots, l \in B_j$ as long as $x_i + \cdots + x_l < k$

If $x_i + \cdots + x_{l+1} \geq k$ then

Place x_{l+1} into B_j and

Reset $i := l+2$, $j := j+1$, and continue with the second step.

Remark 1.2. Similar to NEXT FIT, DUAL NEXT FIT places objects sequentially into the current bin until the threshold is reached, placing the subsequent object in a new bin. We thus can consider a DUAL NEXT FIT-compatible full dual bin arrangement of m elements into n bins to be a composition of m with n parts, each part at least two. A leftover dual bin arrangement is a composition of m with n parts, the first $n-1$ of which must be at least two.

Henceforth we consider all dual bin arrangements to arise from applying DUAL NEXT FIT to a sequence of objects. We also will frequently refer to a leftover dual bin arrangement with n bins as instead having $n-1$ bins plus one leftover bin.

For example, consider the two sequences of objects $v = (.5, .75, .125, .125, .25)$ and $w = (.5, .75, .75, .125, .25)$. If we take $k = 1$, DUAL NEXT FIT will partition both of them into the sets

$$\boxed{1\,2} \qquad \boxed{3\,4\,5}\,.$$

Since the last three object sizes in v add to $.125 + .125 + .25 = .5 < 1$, if we append another object onto v it would still fit in the second bin. Since the last three object sizes in w add to $.75 + .125 + .25 = 1.125 \geq 1$, appending another object to w would necessarily result in starting a third bin. Thus the resulting dual bin arrangements are in fact different ones: v yields the leftover arrangement $\quad \boxed{1\,2} \quad \boxed{3\,4\,5}\ L$ and w the full arrangement $\quad \boxed{1\,2} \quad \boxed{3\,4\,5}\ F$.

Furthermore, the two arrangements satisfy different inequalities. In the leftover arrangement arising from v, the inequalities $x_1 + x_2 \geq 1$ and $x_3 + x_4 + x_5 < 1$ hold. In the full dual bin arrangement arising from w, the inequalities $x_1 + x_2 \geq 1$, $x_3 + x_4 < 1$, and $x_3 + x_4 + x_5 \geq 1$ hold.

To each bin arrangement or dual bin arrangement, we will form a polytope based on such inequalities.

Definition 1.6. A *polytope* is a bounded finite intersection of closed and open half spaces.

Proposition 1.1. *Each sequence of m objects with i having weight x_i corresponds to the m-tuple $(x_1, x_2, \ldots, x_m)$ and hence to a point in the unit m-hypercube. Thus for any m,* NEXT FIT *and* DUAL NEXT FIT *partition the unit m-hypercube into polytopes corresponding to the different bin arrangements.*

Proof. The relevent inequalities determining the polytopes are just those arising from the bin arrangements and dual bin arrangements under NEXT FIT and DUAL NEXT FIT. $\square$

Remark 1.3. Most definitions of polytope require that they be closed. Since NEXT FIT and DUAL NEXT FIT generate both strict and weak inequalities, we modify the definition of polytope accordingly. Note that this means the polytopes fit together to form $[0,1)^m$ or $(0,1]^m$ with no overlap.

Definition 1.7. If A is a dual bin arrangement with m elements, we define $P_A \subset [0,1)^m$ to be the polytope whose inequalities are those derived from A under DUAL NEXT FIT. If A is a bin arrangement with m elements, we define $P_A \subset (0,1]^m$ to be the polytope whose inequalities are those derived from A under NEXT FIT.

We will consider the volumes of these polytopes. Note that, if we assume the objects exhibit a continuous uniform distribution, the volume of a polytope is precisely the probability that a random sequence of objects will yield the corresponding bin arrangement (dual bin arrangement) under (DUAL) NEXT FIT. In the discrete case, we consider the integer lattice points in h times these polytopes, for h an integer greater than 2.

Finally, we introduce the following notation.

Notation. We let $[m]$ be the set $\{1,2,\ldots,m\}$. We let S_m be the symmetric group on m elements, or the set of permutations of $[m]$. We denote $\binom{h+m-1}{m}$ by $\left(\!\binom{h}{m}\!\right)$.

CHAPTER 2

The Threshold 1 Case of the Dual Bin Problem

We examine the dual bin problem first, since it is mathematically simpler than the bin problem. In the dual bin problem, it is clear when we have finished with a bin; once we've exceeded the threshold then the next object automatically starts a new bin. In the bin problem, it is not so clear. We do not *a priori* know whether an object will fit in the current bin or not; each time we must explicitly check to see if it will fit. Only then can we say whether it fits in the old bin or starts a new one.

As a result, the inequalities arising from the dual bin problem are simpler. In particular, if l is the last object in a dual bin, then x_{l+1} will appear only in the inequalities corresponding to the next bin. If l is the last object in an ordinary bin, then x_{l+1} appears both in the inequalities for that bin as well as in those corresponding to the next bin.

2.1. Generating Functions for the Continuous and Discrete Cases

We start by letting the object sizes be uniformly continuously distributed between $[0, 1)$ and by taking the threshold $k = 1$.

We will consider the free monoid whose primes are full bins. For each $m \geq 2$, there is one full bin with m objects. A full dual bin arrangement is an element of this free monoid. A leftover dual bin arrangement is an element of this free monoid followed by a leftover bin. For each $m \geq 1$, there is one leftover bin with m objects.

We thus consider the full and leftover bins with m objects.

Definition 2.1. Let A_{F_m} be the dual bin arrangement that consists of one full bin with m elements. Let F_m be the corresponding polytope in $[0,1)^m$, and f_m its volume. Let A_{L_m} be the leftover dual bin arrangement that consists of one leftover bin with m elements. Let L_m be its corresponding polytope in $[0,1)^m$, and l_m its volume.

Lemma 2.1. *We have* $f_m = \dfrac{1}{(m-1)!} - \dfrac{1}{m!}$ *and* $l_m = \dfrac{1}{m!}$.

Proof. Consider the linear map ψ_m as follows

$$\psi_m : \Big\{(x_1, x_2 \dots, x_m) \mid x_1 + \cdots + x_m < 1 \text{ and } 0 \leq x_i < 1\Big\} \to \Big\{(y_1, y_2 \dots, y_m) \mid 0 \leq y_1 \leq \cdots \leq y_m < 1\Big\} .$$

Explicitly, ψ_m takes $(x_1, \dots, x_m)$ to $(y_1, \dots, y_m)$ where $y_i = x_1 + \cdots + x_i$. Since its matrix is lower triangular, consisting of 1's everywhere on and below the main diagonal, its determinant is 1 and ψ_m is volume preserving. Its inverse ψ_m^{-1} takes $(y_1, \dots, y_m)$ to $(x_1, \dots, x_m)$ where $x_i = y_i - y_{i-1}$ (letting $y_0 = 0$). Hence $\psi_m(A_{L_m})$ has volume $\frac{1}{m!}$ and thus $l_m = \frac{1}{m!}$.

We can write F_m as a set by

$$\Big\{(x_1, x_2 \dots, x_m) \mid x_1 + \cdots + x_{m-1} < 1 \text{ and } 0 \leq x_i < 1\Big\} \Big\backslash \Big\{(x_1, x_2 \dots, x_m) \mid x_1 + \cdots + x_m < 1 \text{ and } 0 \leq x_i < 1\Big\} .$$

We apply $\psi_{m-1} \times I$ (where I is the identity map) to the first set. The volume of its image is $\frac{1}{(m-1)!}$. As noted before, the second set has volume $\frac{1}{m!}$. Hence $f_m = \frac{1}{(m-1)!} - \frac{1}{m!}$. □

This map is similar to the ones Stanley considers in [**13**] and [**14**] .

Remark 2.1. Suppose we consider a sequence of m objects, the first m_1 of which fill a bin. The polytope $F_{m_1} \subset \mathbf{R}^{m_1}$ is not the same as the polytope

$$\left\{(x_1, \dots, x_m) \mid x_1 + \cdots + x_{m_1 - 1} < 1 \text{ and } x_1 + \cdots + x_{m_1} \geq 1\right\} = \\ = F_{m_1} \times [0,1)^{m-m_1} \subset [0,1)^m \quad .$$

Nevertheless, they have the same volume. We will refer to the two interchangeably when convenient. Similarly, if any subsequence of m_i consecutive objects fill a bin, say starting with the $m' + 1$st, we will often refer to F_{m_i} rather than to the polytope $[0,1)^{m'} \times F_{m_i} \times [0,1)^{m-m_i-m'}$. Note that we may do so because if i is in a bin B, then x_i will only appear in the inequalities that determine that particular bin B.

Lemma 2.2. *The number of integer lattice points in* hL_m *is* $\left(\!\binom{h}{m}\!\right) = \binom{h+m-1}{m}$. *The number in* hF_m *is* $h\left(\!\binom{h}{m-1}\!\right) - \left(\!\binom{h}{m}\!\right) = (m-1)\left(\!\binom{h-1}{m}\!\right)$.

Proof. Similar to Lemma 2.1, we define the function ψ_m^h by

$$\psi_m^h : \left\{(x_1, x_2 \dots, x_m) \mid x_1 + \cdots + x_m < h \text{ and } 0 \leq x_i < h\right\} \to \\ \left\{(y_1, y_2 \dots, y_m) \mid 0 \leq y_1 \leq \cdots \leq y_m < h\right\} .$$

Its action is identical to that of ψ_m. It takes integer lattice points to integer lattice points: integer solutions to $x_1 + \cdots + x_m < h$ and $0 \leq x_i < h$ give integer solutions to $0 \leq y_1 \leq \cdots \leq y_m < h$ and vice versa. The number of integer solutions to the latter is $\left(\!\binom{h}{m}\!\right) = \binom{h+m-1}{m}$, giving the number of integer lattice points of hL_m. For counting the lattice points of hF_m, we want to count the number of integers satisfying $0 \leq y_1 \leq \cdots \leq y_{m-1} < h$ minus those satisfying $0 \leq y_1 \leq \cdots \leq y_m < h$. Since y_m can be any of the h choices in the first set of inequalities, this gives us $\left(\!\binom{h}{m-1}\!\right) h - \left(\!\binom{h}{m}\!\right)$. Finally, a conceptual proof of $h\left(\!\binom{h}{m-1}\!\right) - \left(\!\binom{h}{m}\!\right) = (m-1)\left(\!\binom{h-1}{m}\!\right)$ follows from the map in Section 2.3. □

Remark 2.2. Suppose, given an integer $h \geq 2$, our object sizes are chosen uniformly at random from the set $\frac{0}{h}, \frac{1}{h}, \ldots, \frac{h-1}{h}$. Then Lemma 2.2 gives the number of sequences of such objects that yield the dual bin arrangements A_{F_m} and A_{L_m}.

Definition 2.2. Weight each point of the unit m-hypercube by $x^m y^n z^p$, where n and p are the number of full and leftover bins in the dual bin arrangement obtained from applying DUAL NEXT FIT to the m-tuple describing that point. Let $b_{m,n,p}$ be the volume of the region whose points are given the weight $x^m y^n z^p$. Set $A(x, y, z) = \sum b_{m,n,p} x^m y^n z^p$. Then $A(x, y, z)$ is the generating function for the continuous dual bin threshold 1 problem.

In this notation, $b_{m,0,1} = l_m$ and $b_{m,1,0} = f_m$.

Note that the only possible values for p are 0 and 1. Note also that points with different dual bin arrangements can have the same weight: for example, any points giving the arrangements $\boxed{1\,2}\ \boxed{3\,4\,5}\,L\quad \boxed{1\,2\,3}\ \boxed{4\,5}\,L$ or $\boxed{1\,2\,3\,4}\ \boxed{5}\,L$

will have the same weight.

Theorem 2.3.

$$A(x,y,z) = \frac{1+z(e^x-1)}{1-y[e^x(x-1)+1]} \ .$$

Proof. We first split the leftover and full dual bin arrangements, writing

$$\sum b_{m,n,p}\ x^m y^n z^p = \sum b_{m,n,0}\ x^m y^n + \sum b_{m,n,1}\ x^m y^n z \ .$$

Note that a leftover arrangement is a full arrangement followed by a leftover bin, so

$$b_{m,n,1} = \sum_{\substack{i+j=m \\ j\neq 0}} b_{i,n,0}\ b_{j,0,1} \ .$$

Since by Lemma 2.1 we know $b_{j,0,1} = \frac{1}{j!}$, we can rewrite the generating function as follows.

$$\begin{aligned} A(x,y,z) &= \left(\sum b_{m,n,0}\ x^m y^n\right)\left(1+\sum_{m=1}^{\infty} b_{m,0,1}\ x^m z\right) \\ &= \left(\sum b_{m,n,0}\ x^m y^n\right)\left(1+z\sum_{m=1}^{\infty} l_m\ x^m\right) \end{aligned}$$

We view the full dual bin arrangement A containing i_j elements in the ith bin as the sequence of full bins $A_{F_{i_1}}, A_{F_{i_2}}, \ldots, A_{F_{i_n}}$, corresponding to the polytopes $F_{i_1} \times [0,1)^{m-i_1}, [0,1)^{i_1} \times F_{i_2} \times [0,1)^{m-i_1-i_2}, \ldots, [0,1)^{m-i_n} \times F_{i_n}$. Then P_A is the intersection of the above polytopes, with volume $\prod_{j=1}^n f_{i_j}$. As any full arrangement of m elements into n bins is a composition of m into n parts, each part at least 2, we have

$$b_{m,n,0} = \sum_{\substack{i_1+\cdots+i_n=m \\ i_j\geq 2}} b_{i_1,1,0}\ldots b_{i_n,1,0}$$

and to calculate $\sum b_{m,n,0}$ we must sum over all such compositions. Thus

$$\sum_{m,n} b_{m,n,0}\ x^m y^n = 1 + \sum_m b_{m,1,0}\ x^m y + \left(\sum_m b_{m,1,0}\ x^m y\right)^2 + \cdots$$
$$= \frac{1}{1 - \sum\limits_m b_{m,1,0}\ x^m y}$$
$$= \frac{1}{1 - y \sum\limits_{m\geq 2} f_m x^m}$$

and after applying Lemma 2.1 the generating function becomes

$$A(x,y,z) = \frac{1 + z\sum_{m\geq 1} l_m\ x^m}{1 - y\sum_{m\geq 2} f_m\ x^m}$$
$$= \frac{1 + z\sum_{m\geq 1} \frac{x^m}{m!}}{1 - y\sum_{m\geq 2}\left(\frac{x^m}{(m-1)!} - \frac{x^m}{m!}\right)}$$
$$= \frac{1 + z(e^x - 1)}{1 - y[e^x(x-1)+1]}\ . \qquad \square$$

Theorem 2.4. *Given an integer $h \geq 2$, suppose the sizes of the objects in the dual bin packing problem are restricted to j/h for some j between 0 and $h-1$. Let $A^h(x,y,z) = \sum b^h_{m,n,p}\ x^m y^n z^p$, where $b^h_{m,n,p}$ is the number of m-tuples satisfying the above condition with the weight $x^m y^n z^p$. Then we have the following discrete dual bin threshold 1 generating function:*

$$A^h(x,y,z) = \frac{1 + z\left[\left(\frac{1}{1-x}\right)^h - 1\right]}{1 - y\left[\left(\frac{1}{1-x}\right)^h (hx-1) + 1\right]}\ .$$

Proof. The same argument holds as in Theorem 2.3, replacing $b_{m,n,p}$ by $b^h_{m,n,p}$. Lemma 2.2 gives $b^h_{m,0,1} = \left(\!\binom{h}{m}\!\right)$ and $b^h_{m,1,0} = h\ \left(\!\binom{h}{m-1}\!\right) - \left(\!\binom{h}{m}\!\right)$. Recalling that $\sum_{m=0}^{\infty} \left(\!\binom{h}{m}\!\right) x^m = \left(\frac{1}{1-x}\right)^h$, we get

$$A^h(x,y,z) = \frac{1 + z\sum_{m=1}^{\infty} x^m \left(\!\!\binom{h}{m}\!\!\right)}{1 - y\sum_{m=2}^{\infty} x^m \left[\left(\!\!\binom{h}{m-1}\!\!\right) h - \left(\!\!\binom{h}{m}\!\!\right)\right]}$$

$$= \frac{1 + z\left[\left(\frac{1}{1-x}\right)^h - 1\right]}{1 - yhx\left(\frac{1}{1-x}\right)^h + y\left(\left(\frac{1}{1-x}\right)^h - 1\right)}$$

$$= \frac{1 + z\left[\left(\frac{1}{1-x}\right)^h - 1\right]}{1 - y\left[\left(\frac{1}{1-x}\right)^h (hx-1) + 1\right]} . \qquad \square$$

To look at the proportion of the counted lattice points to all lattice points, we replace x^m with $(x/h)^m$. Taking the limit as $h \to \infty$ results in $A(x,y,z)$.

2.2. Expectation and Variance

We wish to know the expectation of the number of bins that a sequence of m objects fills, as well as its variance. We obtain these by examining the generating function $A(x,y,z)$. For the purposes of this argument, we modify the DUAL NEXT FIT algorithm slightly. Since we want the number of full bins, if the algorithm ends with a leftover bin, we consider its contents as being dumped back in the last full bin, following [**1**]. Thus we consider $A(x,y,1)$.

Recall the following:

Definition 2.3. Suppose Y is a random variable that takes on value n with probability p_n. We let $\mathrm{E(Y)} = \sum np_n$ be the expectation of Y, and write the variance of Y by $\mathrm{V(Y)} = \mathrm{E(E(Y) - Y)}^2$.

Remark 2.3. Let $Y(y) = \sum p_n y^n$ be the probability generating function of the random variable Y. Then it is immediate that E(Y) $= Y'(1)$. The variance can also be expressed in a similar form:

$$\begin{aligned}
\mathrm{V(Y)} &= \sum (\mathrm{E(Y)} - n)^2 p_n \\
&= \sum n^2 p_n - 2\,\mathrm{E(Y)} \sum n p_n + \mathrm{E(Y)}^2 \sum p_n \\
&= \sum n^2 p_n - \mathrm{E(Y)}^2 \\
&= Y''(1) + Y'(1) - \left(Y'(1)\right)^2 \ . \qquad (1)
\end{aligned}$$

Definition 2.4. Let $b_{m,n}$ be the probability that a random sequence of m objects will fill n bins under DUAL NEXT FIT. In the notation of the previous section, $b_{m,n} = b_{m,n,0} + b_{m,n,1}$. Let $\mathrm{B_m}$ be the random variable defined on sequences of m objects that takes on value n with probability $b_{m,n}$, and let $B_m(y) = \sum b_{m,n} y^n$ be its probability generating function.

Note that $\mathrm{E(B_m)}$ is the expected number of bins that m objects will fill.

Proposition 2.5.

$$\sum_m \mathrm{E(B_m)} x^m = \frac{e^{-x} + x - 1}{(1-x)^2} \qquad \textit{and} \qquad \mathrm{E(B_m)} = -1 + \sum_{i=0}^{m} (-1)^i \frac{m-i+1}{i!} \ .$$

Proof. Note that since $A(x,y,1) = \sum (b_{m,n,0} + b_{m,n,1}) x^m y^n = \sum B_m(y) x^m$, we can find $\sum_m \mathrm{E(B_m)} x^m = \sum_m B_m'(1) x^m$ by taking the first derivative of $A(x,y,1)$ with respect to y and evaluating at $y = 1$. Hence

$$\begin{aligned}
\sum_m \mathrm{E}(\mathrm{B_m})x^m &= \frac{\partial}{\partial y}\left(\frac{e^x}{1-y[e^x(x-1)+1]}\right)\Big|_{y=1} \\
&= \frac{e^{-x}+x-1}{(1-x)^2} \\
&= \left(\sum(-1)^m\frac{x^m}{m!}+x-1\right)\left(\sum(m+1)x^m\right) \\
&= \sum_{m=0}^{\infty} x^m\left[m-(m+1)+\sum_{i=0}^{m}\frac{(-1)^i}{i!}(m-i+1)\right] \\
&= \sum_{m=0}^{\infty} x^m\left[-1+\sum_{i=0}^{m}(-1)^i\frac{m-i+1}{i!}\right] . \qquad \square
\end{aligned}$$

For example, taking $m = 5$ we get $\mathrm{E}(\mathrm{B_5}) = -1+6-5+2-\frac{1}{2}+\frac{1}{12}-\frac{1}{120} = \frac{63}{40}$, so the expected number of bins needed for 5 objects is 63/40. (We could of course also have gotten this by expanding out the generating function $\sum \mathrm{E}(\mathrm{B_m})x^m$.)

Proposition 2.6. *With error term $|R| \leq 2/m!$, the expectation $\mathrm{E}(\mathrm{B_m})$ is asymptotically equal to $-1+(m+2)/e$.*

Proof. We examine the explicit formula for $\mathrm{E}(\mathrm{B_m})$.

$$\begin{aligned}
\mathrm{E}(\mathrm{B_m}) &= -1+\sum_{i=0}^{m}(-1)^i\frac{m-i+1}{i!} \\
&= -1+(m+1)\sum_{i=0}^{m}\frac{(-1)^i}{i!}-\sum_{i=1}^{m}\frac{(-1)^i}{(i-1)!} \\
&= -1+(m+1)\sum_{i=0}^{m}\frac{(-1)^i}{i!}+\sum_{i=0}^{m-1}\frac{(-1)^i}{i!} \\
&= -1+\frac{m+1}{e}+R_1+\frac{1}{e}+R_2 \\
&= -1+\frac{m+2}{e}+R ,
\end{aligned}$$

Where $R = R_1 + R_2$. Since $|R_1| \leq \frac{m+1}{(m+1)!} = \frac{1}{m!}$ and $|R_2| \leq \frac{1}{m!}$, the total error $|R|$ is at most $|R| \leq \frac{2}{m!}$. $\square$

We will use the following in the calculation of the variance.

Corollary 2.7. *With error* $|R| \leq \dfrac{4e^{-1} - 2}{m!} + \dfrac{2e^{-1}}{(m-1)!} + \dfrac{4}{(m!)^2}$,

$$\mathrm{E}(\mathrm{B_m})^2 = \frac{m^2}{e^2} + m\left(\frac{4}{e^2} - \frac{2}{e}\right) + \frac{4}{e^2} - \frac{4}{e} + 1 + R \ .$$

Notation. We rewrite equation (1) as $\mathrm{V}(\mathrm{B_m}) = B''_m(1) + B'_m(1) - (B'_m(1))^2$. Taking $V_{0,m} = B''_m(1) + B'_m(1)$ and $V_{1,m} = (B'_m(1))^2$, we have $\mathrm{V}(\mathrm{B_m}) = V_{0,m} - V_{1,m}$. We will denote $\sum_m V_{0,m} x^m$ by V_0.

Lemma 2.8. *For any* $\epsilon > 0$,

$$V_{0,m} = m^2 e^{-2} + m(7e^{-2} - 3e^{-1}) + 10e^{-2} - 6e^{-1} + 1 + O(\epsilon^m) \ .$$

Proof. As in the proof of Proposition 2.5, we can get $\sum_m B''_m(1) x^m$ by taking the second derivative of $A(x, y, 1)$ with respect to y and evaluating at $y = 1$. Hence

$$\begin{aligned} V_0 &= \frac{e^{-x} + x - 1}{(1-x)^2} + \frac{\partial^2}{\partial y^2}\left(\frac{e^x}{1 - y[e^x(x-1) + 1]}\right)\Bigg|_{y=1} \\ &= \frac{(1-x)^2 - 3e^{-x}(1-x) + 2e^{-2x}}{(1-x)^3} \ . \end{aligned}$$

To calculate the asymptotic behavior of V_0, we use the technique in [**15**]. Note that V_0 has as its only singularity a pole of order 3 at $x = 1$. The principal part of the Laurent series expansion around $x = 1$ is

$$L = \frac{-2e^{-2}}{(x-1)^3} + \frac{-3e^{-1} + 4e^{-2}}{(x-1)^2} + \frac{-1 + 3e^{-1} - 4e^{-2}}{(x-1)} \ .$$

If we take $V_0 - L$, we get an analytic function whose coefficients (expanded around $x = 0$) grow no faster than ϵ^m for any $\epsilon > 0$. Hence L is an asymptotic approximation of V_0.

Let

$$\begin{aligned} L &= \sum_{j=1}^{3} \frac{a_{-j}}{(x-1)^j} \\ &= \sum_{j=1}^{3} (-1)^j a_{-j} \left(\frac{1}{(1-x)^j} \right) \\ &= \sum_{j=1}^{3} (-1)^j a_{-j} \left(\sum_{m=0}^{\infty} \binom{m+j-1}{m} x^m \right) \\ &= \sum_{m=0}^{\infty} x^m \left(\sum_{j=1}^{3} (-1)^j a_{-j} \binom{m+j-1}{j-1} \right) . \end{aligned}$$

So

$$\begin{aligned} V_{0,m} &= \sum_{j=1}^{3} (-1)^j a_{-j} \binom{m+j-1}{j-1} + O(\epsilon^m) \\ &= -(-1 + 3/e - 4/e^2) + (m+1)(-3/e + 4/e^2) + \ (m+1)(m+2)/e^2 + O(\epsilon^m) \\ &= m^2/e^2 + m(7/e^2 - 3/e) + 10/e^2 - 6/e + 1 + O(\epsilon^m) \ . \end{aligned}$$

□

Proposition 2.9. *Given m and any $\epsilon > 0$, the variance* $\mathrm{V(B_m)}$ *is equal to* $(3e^{-2} - e^{-1})(m+2) + O(\epsilon^m) \approx .038126(m+2)$.

Proof. We have $\mathrm{V(B_m)} = V_{0,m} - V_{1,m}$. The former is calculated in the preceding lemma, the latter in Corollary 2.7. □

2.3. A Map to Hook Factorizations of Permutations

Consider the generating function $A(x, y, z)$ from before. If we take $y = 1$ and $z = 0$, we get

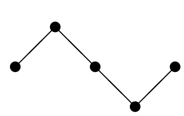
$$A(x, 1, 0) = \frac{1}{(1-x)e^x} = \frac{e^{-x}}{1-x} ,$$

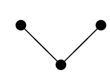
the generating function for the sequences that correspond to full dual bin arrangements. It is well known that this is the also the exponential generating function for the derangement numbers. Recall that a derangement is a permutation that has no fixed points, so if $\pi \in S_n$ is a derangement, then $\pi(i) \neq i$ for all $i \in [n]$.

In this section we introduce a transformation ϕ which relates dual bin arrangements and permutations. Among other things, it will explain the correspondence between derangement numbers and full dual bin arrangements.

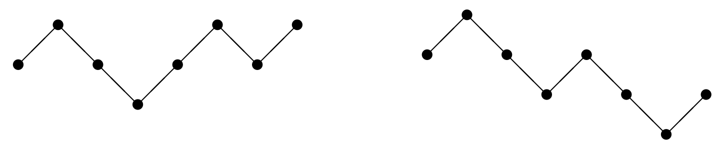
2.3.1. The Hook Factorization. Recall that if σ is a permutation of some set of integers, i is a descent of σ if $\sigma(i) > \sigma(i+1)$ and an ascent if $\sigma(i) < \sigma(i+1)$. For example, if $\sigma = 6\,3\,2\,1\,4\,5 \in S_6$, then σ has descent set $\{1, 2, 3\}$ and ascent set $\{4, 5\}$.

Definition 2.5. We define a *shape* to be a word in the letters a and d. We draw a shape by representing a as a northeast step and d as a southeast step. For instance, the shapes *adda* and *da* are drawn as follows.





We combine two shapes by writing either an a or a d between the two words. For instance, *adda* and *da* combine to form *addadda* and *addaada*.



Note that words do not combine uniquely.

We associate a shape to each permutation in the obvious way:

Definition 2.6. Define a function Sh which takes permutations of m integers to shapes of length $m-1$. If $\sigma \in S_m$ has a descent in the ith position, then the ith letter of $\mathrm{Sh}(\sigma)$ is d. If σ has an ascent in the ith position, then the ith letter of $\mathrm{Sh}(\sigma)$ is a.

For example, if σ_1 is $6\,3\,2\,1\,4\,5$ and σ_2 is $4\,2\,1\,3\,7\,5\,6$, then $\mathrm{Sh}(\sigma_1)$ and $\mathrm{Sh}(\sigma_2)$ are *dddaa* and *ddaada*:

The function Sh is certainly not injective: Sh also takes $6\,5\,4\,1\,2\,3$ to the first shape. However, it is surjective.

Definition 2.7. A *hook-shape* is a shape of the form $d^i a$, where i may be zero. We say this is a hook-shape of length $i+1$. A *hook* of length m is a permutation whose shape is a hook-shape of length $m-1$. A *tail-shape* is a shape of the form d^i, where we permit the empty word ($i=0$) to be considered a tail-shape. We say the tail-shape is of length i. A *tail* is a permutation whose shape is a tail-shape. We consider the empty permutation and the permutation of length 1 to be tails.

Proposition 2.10 (Gessel [**5**])**.** *Any permutation has a unique factorization into a (possibly empty) sequence of hooks, followed by a tail. We call this the* hook factorization*.*

Proof. If the permutation, say $\sigma \in S_m$, is actually $m\; m-1\; \ldots 3\,2\,1$ then it factors into an empty sequence of hooks followed by a tail of length m. Otherwise, say the first ascent occurs at $\sigma(i) < \sigma(i+1)$. Then $\sigma(1)\ldots\sigma(i)\,\sigma(i+1)$ forms the first hook. Any other initial sequence fails to be a hook. We repeat the analysis for the permutation $\sigma(i+2)\ldots\sigma(m)$. This gives a unique factorization of any permutation into a sequence of concatenated hooks α_i taken from left to right, with a tail β at the end. □

Our earlier example $\sigma_1 = 6\,3\,2\,1\,4\,5$ factors as $\sigma_1 = \alpha_1\beta$ with $\alpha_1 = 6\,3\,2\,1\,4$ and $\beta = 5$, while $\sigma_2 = 4\,2\,1\,3\,7\,5\,6$ factors as $\sigma_2 = \alpha_1\alpha_2$ with $\alpha_1 = 4\,2\,1\,3$ and $\alpha_2 = 7\,5\,6$.

We define the corresponding notion for shapes.

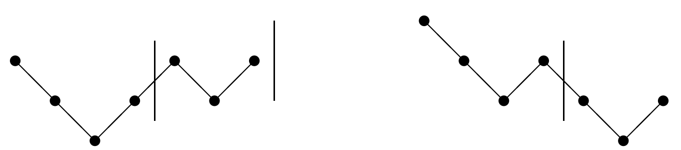
Definition 2.8. Given a (possibly empty) sequence of hook-shapes of lengths $(i_1, \ldots, i_l)$ followed by a tail-shape of length j, we define the *hook-shape decomposition set* indexed by $(i_1, \ldots, i_l; j)$ to be the set of all shapes formed by combining them. (If the sequence of hook-shapes is empty ($l = 0$), we say the decomposition set is indexed by (j).) Following this scheme, we define the sequence of hook-shapes of lengths $(i_1, \ldots, i_l)$ with no final tail-shape to be the hook-shape decomposition set indexed by $(i_1, \ldots, i_l; -1)$. The length of the shapes in the hook-shape decomposition set indexed by $(i_1, \ldots, i_l; j)$ is $i_1 + \cdots + i_l + l + j$.

Notation. We will denote the letters that combine the hook shapes (and tail shape) by $\not{a}$ and $\not{d}$. A final $\not{a}$ or $\not{d}$ will indicate a tail-shape of length zero. We will draw an additional vertical line after a final hook-shape to indicate a tail-shape of length -1.

For instance, the hook-shape of length 3 (dda) and the tail-shape of length 1 (d) combine to form the hook-shape decomposition set indexed by $(3; 1)$, which consists of shapes $dd\not{a}d$ and $dda\not{d}d$:

The hook-shape of length 4 ($ddda$) and the tail-shape of length zero combine to form the hook-shape decomposition set indexed by $(4; 0)$, which contains the shapes $ddd\not{a}$ and $ddda\not{d}$:

Finally, the hook-shape of length 3 (dda) and length 2 (da) combine to form the hook-shape decomposition set indexed by $(3, 2; -1)$. The relevant shapes are $dd\not{a}da|$ and $dda\not{d}da|$:



If $j \geq 0$, there are a total of 2^l shapes in the decomposition set indexed by $(i_1, \ldots, i_l; j)$. If $j = -1$, there are 2^{l-1} shapes in the decomposition set indexed by $(i_1, \ldots, i_l; -1)$.

The following is immediate.

Proposition 2.11. *If the permutation* σ *has hook factorization* $\alpha_1 \cdots \alpha_l \beta$ *with* α_k *of length* i_k *and* β *of length* j*, then* $\mathrm{Sh}(\sigma)$ *is in the hook-shape decomposition set indexed by* $(i_1 - 1, \ldots, i_l - 1; j - 1)$.

For example, taking $\sigma_1 = 6\,3\,2\,1\,4\,5$ and $\sigma_2 = 4\,2\,1\,3\,7\,5\,6$ as above, $\mathrm{Sh}(\sigma_1)$ is in the hook-shape decomposition set indexed by $(4; 0)$ and $\mathrm{Sh}(\sigma_2)$ is in the hook-shape decomposition set indexed by $(3, 2; -1)$. Notice that the hook-shape decomposition sets disjointly partition the set of shapes.

We associate polytopes with shapes and with hook-shape decomposition sets as follows.

Definition 2.9. Let S be a shape with $m - 1$ letters. We define the polytope $P_S \subset (0, 1]^m$ by the following inequalities. If the jth letter of S is d we have $y_{j-1} \geq y_j$; if it is a we have $y_{j-1} < y_j$. Finally, we take $0 < y_i \leq 1$ for all $i = 1, \ldots, m$.

Definition 2.10. Let H be a hook-shape decomposition set. We define the polytope P_H to be the union of the polytopes P_S for all shapes $S \in H$.

Notation. If H is indexed by $(i_1, \ldots, i_l; j)$, for each k between 1 and l we set $i'_k = i_1 + \cdots + i_k + k$ (letting i'_0 be 0). We say that the coordinates $y_{i'_{k-1}+1}$ through y_{i_k} come from the same hook-shape, and that $y_{i'_l+1}$ through y_m come from the tail-shape.

Note that in P_H the coordinates coming from the kth hook-shape satisfy the inequalities $y_{i'_{k-1}+1} \geq \cdots \geq y_{i'_k-1} < y_{i'_k}$ and the coordinates coming from the tail-shape satisfy $y_{i'_l+1} \geq \cdots \geq y_m$.

For example, for $H = (3, 2; 1)$, P_H is defined by $y_1 \geq y_2 \geq y_3 < y_4$, $y_5 \geq y_6 < y_7$ and $y_8 \geq y_9$. The coordinates y_1, y_2, y_3 and y_4 come from the same hook-shape, as do y_5, y_6, and y_7. Similarly, y_8 and y_9 come from the tail-shape.

2.3.2. The Functions ϕ and ϕ'.

Definition 2.11. Let A be a dual bin arrangement with m elements. We define the function ϕ_A from P_A to $[0, 1)^m$ with $\phi_A(x_1, \ldots, x_m) = (y_1, \ldots, y_m)$ as follows:

$$y_j = 1 - \sum_{i=b}^{j} x_i + \left\lfloor \sum_{i=b}^{j} x_i \right\rfloor$$

where b is the first element in the bin that contains j.

Proposition 2.12. *For each dual bin arrangement A, ϕ_A is an affine linear, volume preserving map.*

Proof. We will introduce dummy variables $x_0 = y_0 = 1$ and show that the action of ϕ_A can be written as an $m + 1$ by $m + 1$ invertible matrix M_A with determinant $(-1)^m$.

The first row of M_A is $[1, 0, \ldots, 0]$. Fix $j \geq 1$, and let b be the first element in the bin that contains j. By the definition of ϕ_A, since $y_j = 1 - \sum_{i=b}^{j} x_i + \lfloor \sum_{i=b}^{j} x_i \rfloor$, the $(j + 1)$st row of M_A will start with $\lfloor \sum_{i=b}^{j} x_i \rfloor + 1$, then will be followed by -1 in the $(b + 1)$st through $(j + 1)$st positions. The remaining $m - j$ entries of this $j + 1$st row are all zero. Hence M_A is a lower triangular matrix with 1 in the $(1,1)$ position and -1 elsewhere on the main diagonal. Thus M_A is invertible with determinant $(-1)^m$. □

This function is very similar to the one Stanley introduces in [**13**], which gives a combinatorial interpretation for slices of the unit hypercube perpendicular to $(1,\ldots,1)$.

Definition 2.12. Given $m > 0$, define a piecewise linear, volume preserving map ϕ from $[0,1)^m$ to $(0,1]^m$. Divide $[0,1)^m$ into polytopes P_A for all dual bin arrangements A of m objects. Then form ϕ by defining $\phi(x_1,\ldots,x_m) = \phi_A(x_1,\ldots,x_m)$ for $(x_1,\ldots,x_m) \in P_A$.

Definition 2.13. Given a hook factorization H, we define a function ϕ'_H by taking $\phi'_H(y_1,\ldots,y_m) = (x_1,\ldots,x_m)$ as follows:

$$x_j = \begin{cases} y_{j-1} - y_j, & \text{if } y_{j-1} \geq y_j \text{ and } y_{j-1} \text{ and } y_j \text{ come from the same hook-shape or tail-shape;} \\ 1 + y_{j-1} - y_j, & \text{if } y_{j-1} < y_j \text{ and } y_{j-1} \text{ and } y_j \text{ come from the same hook-shape;} \\ 1 - y_j, & \text{if } y_{j-1} \text{ and } y_j \text{ come from different hook-shapes.} \end{cases}$$

We consider y_0 to be 1. We then construct ϕ' on $(0,1]^m$ by piecing together the restrictions of each ϕ'_H to the polytope corresponding to H, just as we made ϕ from the ϕ_A's.

The following theorem provides an alternative combinatorial interpretation for dual bin arrangements.

Theorem 2.13. *Let A be the dual bin arrangement with n full bins and p left-over bins containing $m_1, \dots, m_n, m_0$ elements respectively. Let H be the hook-shape decomposition set indexed by $(m_1 - 1, \dots, m_n - 1; m_0 - 1)$. Then ϕ bijectively takes P_A to P_H, and $\phi_A^{-1} = \phi'_H$.*

We prove this using the following two lemmas.

Lemma 2.14. *Taking A and H as above, $\phi(P_A) \subseteq P_H$.*

Proof. For $1 \leq k \leq n$, set $m_1 + \cdots + m_k = m'_k$, so the kth bin contains the elements $m'_{k-1} + 1$ through m'_k. Consider x_i, for $m'_{k-1} + 1 \leq i < m'_k$. Since i is not the last element in a full bin, we have

$$\sum_{j=m'_{k-1}+1}^{i} x_j < 1, \quad \text{so} \quad \Big\lfloor \sum_{j=m'_{k-1}+1}^{i} x_j \Big\rfloor = 0 \, .$$

Hence

$$y_i = 1 - \sum_{j=m'_{k-1}+1}^{i} x_j = 1 - \sum_{j=m'_{k-1}+1}^{i-1} x_j - x_i = y_{i-1} - x_i$$

and $y_{i-1} \geq y_i$. Thus ϕ gives us the inequalities $y_{m'_{k-1}+1} \geq \cdots \geq y_{m'_k - 1}$.

Similarly, for the last element in the bin we have

$$\sum_{j=m'_{k-1}+1}^{m'_k} x_j \geq 1, \quad \text{and} \quad \Big\lfloor \sum_{j=m'_{k-1}+1}^{m'_k} x_j \Big\rfloor = 1$$

yielding

$$y_{m'_k} = 1 - \sum_{j=m'_{k-1}+1}^{m'_k} x_j + 1 = 1 - \sum_{j=m'_{k-1}+1}^{m'_k - 1} x_j + 1 - x_{m'_k} = y_{m'_k - 1} + 1 - x_{m'_k} \, .$$

Since $x_{m'_k}$ is strictly less than one, we get the inequality $y_{m'_k-1} < y_{m'_k}$. Combining these, we get that

$$y_{m'_{k-1}+1} \geq y_{m'_{k-1}+2} \geq \cdots \geq y_{m'_k-1} < y_{m'_k} \ .$$

But these are precisely the coordinates and inequalities that come from the kth hook-shape in H.

The same argument as above shows that, for the leftover bin, ϕ gives us the coordinates and inequalities that come from the tail-shape of H:

$$y_{m-m_0+1} \geq \cdots \geq y_m \ .$$

Hence, for $(x_1, \ldots, x_m) \in P_A$, $\phi(x_1, \ldots, x_m) = (y_1, \ldots, y_m)$ satisfies all the inequalities that define P_H and therefore $\phi(P_A) \subseteq P_H$. □

Lemma 2.15. *Taking A and H as before, $\phi'(P_H) \subseteq P_A$.*

Proof. Consider the action of ϕ' on the coordinates that come from the kth hook-shape. As in the previous lemma, we set $m_1 + \cdots + m_k = m'_k$. So the coordinates from this hook-shape are $y_{m'_{k-1}+1}$ through $y_{m'_k}$. Applying ϕ' gives

$$\begin{aligned} x_{m'_{k-1}+1} &= 1 - y_{m'_{k-1}+1} \\ x_{m'_{k-1}+2} &= y_{m'_{k-1}+1} - y_{m'_{k-1}+2} \\ &\vdots \\ x_{m'_k-1} &= y_{m'_k-2} - y_{m'_k-1} \ . \end{aligned}$$

If we then take $x_{m'_{k-1}+1} + \cdots + x_{m'_k-1}$, the sum telescopes to $1 - y_{m'_k-1}$, which, since $y_j > 0$, must be strictly less than 1. Hence

$$\sum_{j=m'_{k-1}+1}^{m'_k-1} x_j < 1 \ .$$

Since $y_{m'_k}$ is the last coordinate coming from the kth hook-shape, we have that $y_{m'_k-1} < y_{m'_k}$ and thus ϕ' gives $x_{m'_k} = 1 + y_{m'_k-1} - y_{m'_k}$. The sum $x_{m'_{k-1}+1} + \cdots + x_{m'_k}$ telescopes to $2 - y_{m'_k}$. Since $y_{m'_k} \leq 1$, this means

$$\sum_{j=m'_{k-1}+1}^{m'_k} x_j \geq 1 \ .$$

These two inequalities are precisely those that say that $m'_{k-1} + 1$ through m'_k fill a bin.

The same argument shows that ϕ' applied to the coordinates that come from the tail-shape yields the inequality $\sum_{j=m-m_0+1}^{m} x_j < 1$. This is the inequality which places $m - m_0 + 1$ through m in a leftover bin.

Thus, for $(y_1, \ldots, y_m) \in P_H$, $\phi'(y_1, \ldots, y_m) = (x_1, \ldots, x_m)$ satisfies all the inequalities that determine P_A. So $\phi'(P_H) \subseteq P_A$. □

Proof of Theorem 2.13. Take a point $(x_1, \ldots, x_m) \in P_A$. By Lemma 2.14 we know $\phi(x_1, \ldots, x_m) \in P_H$. By Lemma 2.15 we know $\phi'\phi(x_1, \ldots, x_m) \in P_A$. A simple check shows that $\phi'\phi(x_1, \ldots, x_m) = \phi'_H\phi_A(x_1, \ldots, x_m) = (x_1, \ldots, x_m)$. Similarly, for $(y_1, \ldots, y_m) \in P_H$, a simple check shows $\phi\phi'(y_1, \ldots, y_m) = \phi_A\phi'_H(y_1, \ldots, y_m) = (y_1, \ldots, y_m)$. □

2.3.3. Examples. We now return to our earlier observation that the special case of our dual bin generating function $A(x, 1, 0)$ is the exponential generating function for the derangement numbers. Note that by taking $A(x, 1, 0)$ we are determining the volumes of the polytopes of the full dual bin arrangements. Under the action of ϕ, this is the same as looking at the volume of the polytopes corresponding to hook-shape decomposition sets whose indices are of the form $(i_1, \ldots, i_l; -1)$.

Proposition 2.16 (Gessel [**5**])**.** *The derangement numbers are equinumerous with permutations that have an empty tail under the hook factorization.*

An equivalent formulation of this result appears in [**3**].

Since the volume of the polytopes corresponding to hook-shape decomposition sets whose indices are of the form $(i_1, \ldots, i_l; -1)$ is just the proportion of permutations with empty tail to all permutations, ϕ explains why the generating functions are the same.

We take as another example the dual bin arrangements with five elements, two full bins, and a leftover bin. Expanding the generating function $A(x, y, z)$ and looking at the coefficient of x^5y^2z gives $\frac{1}{4}$.

There is only one dual bin arrangement with 5 elements, 2 full bins, and 1 leftover bin: the one having two consecutive full bins with 2 elements, followed by a leftover bin with 1 element. Similarly, there is only one hook-shape decomposition set of length 4 whose index has the form $(i_1, i_2; j)$. Since $4 = i_1 + i_2 + 2 + j$, the hook-shape decomposition set must be indexed by $(1, 1; 0)$. Table 2.1 shows the four shapes in this decomposition set, as well as the permutations of 5 which have each shape.

There are 30 permutations whose shape is in the right hook-shape decomposition set. If we take $\frac{30}{5!} = \frac{30}{120}$ we get $\frac{1}{4}$ as needed.

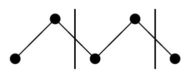
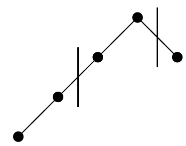
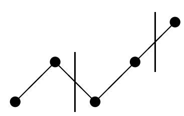
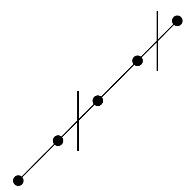
Sh(σ)	σ	Sh(σ)	σ
	14253, 14352 24153, 24351 34152, 34251 15243, 15342 25143, 25341 35142, 35241 45132, 45231 13254, 23154		12354, 12453 13452, 23451
	45123, 15234 25134, 35124 13245, 23145 14235, 24135 34125		12345

Table 2.1. Permutations whose shapes are in the hook-shape decomposition set indexed by $(1, 1; 0)$

Note that we could have made use of ϕ and calculated the generating function $A(x, y, z)$ by directly counting the number of permutations of m according to which hook-shape decomposition sets their shapes belong to.

CHAPTER 3

The Threshold k Case of the Dual Bin Problem

We now let the threshold be k for general positive integers k.

Note that the same sequence of objects will yield different dual bin arrangements depending on the actual value of k. For example, the sequence $v = (.1, .3, .7, .5, .1, .7)$ results in two full bins for $k = 1$, one full bin for $k = 2$, and one leftover bin for $k \geq 3$. The sequence $w = (.7, .5, .8, .8, .5)$ results in two full bins and one leftover bin for $k = 1$, one full and one leftover bin for $k = 2$, one full bin for $k = 3$, and one leftover bin for $k \geq 4$.

3.1. The Generating Function for a Continuous Distribution

Definition 3.1. We define $A_{m,k}$ to be the number of permutations of $[m]$ with $k-1$ descents, and we define $A_m(t)$ by setting $A_m(t) = \sum_k A_{m,k} t^k$. We call $A_{m,k}$ and $A_m(t)$ respectively Eulerian numbers and Eulerian polynomials.

Remark 3.1. The following generating function is well known.

$$\sum_{m,k} A_{m,k}\, t^k\, \frac{x^m}{m!} = \sum_m A_m(t)\, \frac{x^m}{m!} = \frac{1-t}{1-te^{x(1-t)}}\ . \tag{2}$$

Definition 3.2. Let F_m^k and L_m^k be the threshold k equivalents of the polytopes F_m and L_m in Definition 2.1, with f_m^k and l_m^k their respective volumes.

We will express these polytopes in terms of slices of the unit hypercube, and their volumes using Eulerian numbers.

Definition 3.3. Let P_m^k be the slice of the unit m-hypercube satisfying the inequality $k-1 \le \sum_{i=1}^m x_i < k$.

These are the slices that Stanley considers in [**13**]. It is well known that the volume of the slice P_m^k is $\frac{1}{m!}A_{m,k}$.

Lemma 3.1.

$$\textit{For } m \ge 1, \qquad l_m^k = \sum_{i=1}^k \frac{1}{m!}A_{m,i} \qquad \textit{and}$$

$$\textit{for } m \ge 2, \qquad f_m^k = \sum_{i=1}^k \frac{1}{(m-1)!}A_{m-1,i} - \sum_{i=1}^k \frac{1}{m!}A_{m,i} \ .$$

Proof. Since L_m^k is given by the inequality $0 \le \sum_{i=1}^m x_i < k$ and P_m^k is given by $k-1 \le \sum_{i=1}^m x_i < k$, we have that

$$L_m^k = \bigcup_{i=1}^k P_m^i \quad \text{and} \quad l_m^k = \sum_{i=1}^k \frac{1}{m!}A_{m,i} \ .$$

Similarly, F_m^k is given by both $0 \le \sum_{i=1}^{m-1} x_i < k$ and $\sum_{i=1}^m x_i \ge k$. We write this as

$$\Big\{(x_1, x_2 \dots, x_m) \mid x_1 + \dots + x_{m-1} < k \ \text{ and } \ 0 \le x_i < 1\Big\}$$
$$\Big\backslash \quad \Big\{(x_1, x_2 \dots, x_m) \mid x_1 + \dots + x_m \ge k \ \text{ and } \ 0 \le x_i < 1\Big\} \ .$$

The first set allows x_m to take on its full range of values of $[0,1)$, so the first set is $\bigcup_{i=1}^k [0,1) \times P_{m-1}^i$. Note that the volume of $[0,1) \times P_{m-1}^i$ is $\frac{1}{(m-1)!}A_{m-1,i}$, and that the polytopes don't intersect each other.

However, to form F_m^k we subtracted off the second set. Hence we need to subtract off the slices $P_m^1, \ldots, P_m^k$ from $\bigcup_{i=1}^k [0,1) \times P_{m-1}^i$. Since each slice P_m^i is contained completely in $[0,1) \times P_{m-1}^i$, we have that

$$f_m^k = \sum_{i=1}^k \frac{1}{(m-1)!} A_{m-1,i} - \sum_{i=1}^k \frac{1}{m!} A_{m,i} \ .$$

Finally, observe that while $A_{1,1}$ is defined to be 1, we wish f_1^k to be zero, since one element cannot possibly fill a bin. Thus we add the restriction that $m \geq 2$. □

We generalize Definition 2.2 to the threshold k case.

Definition 3.4. Given $k > 0$, we weight each point of $[0,1)^m$ by $x^m y^n z^p$, where n and p are the number of full and leftover bins in the dual bin arrangement obtained from applying the threshold k version of DUAL NEXT FIT to the m-tuple describing that point. Let $b_{m,n,p}^k$ be the volume of the region whose points are given the weight $x^m y^n z^p$. Set $A_k(x,y,z) = \sum b_{m,n,p}^k x^m y^n z^p$.

Remark 3.2. Note that $A_k(x,y,z)$ is the generating function for the continuous dual bin threshold k problem, and should not be confused with the similarly notated Eulerian polynomials. We also point out that $b_{m,1,0}^k = f_m^k$ and $b_{m,0,1}^k = l_m^k$.

Adapting the proof of Theorem 2.3, we have the following proposition.

Proposition 3.2.

$$A_k(x,y,z) = \frac{1 + z \sum_m l_m^k \ x^m}{1 - y \sum_m f_m^k \ x^m} \ .$$

Definition 3.5. Define the functions $\hat{A}(x,y,z,t) = \sum_{k=1}^{\infty} t^k(1 + z\sum l_m^k \; x^m)$ and $\check{A}(x,y,z,t) = \sum_{k=1}^{\infty} t^k(1 - y\sum f_m^k \; x^m)$. Note that

$$A_k(x,y,z) = \frac{[t^k] \; \hat{A}(x,y,z,t)}{[t^k] \; \check{A}(x,y,z,t)}$$

where $[t^k]f(t)$ denotes the coefficient of t^k in $f(t)$.

Lemma 3.3.

$$\sum_{k=1}^{\infty} t^k \left(\sum_{m=1}^{\infty} \frac{x^m}{m!} \sum_{i=1}^{k} A_{m,i} \right) = \frac{1}{1 - te^{x(1-t)}} - \frac{1}{1-t} \; .$$

Proof. We have

$$\begin{aligned}
\sum_{k=1}^{\infty} t^k \left(\sum_{m=1}^{\infty} \frac{x^m}{m!} \sum_{i=1}^{k} A_{m,i} \right) &= \sum_{m=1}^{\infty} \frac{x^m}{m!} \left(\sum_{k=1}^{\infty} t^k \sum_{i=1}^{k} A_{m,i} \right) \\
&= \sum_{m=1}^{\infty} \frac{x^m}{m!} \frac{A_m(t)}{1-t} \qquad (3) \\
&= \frac{1}{1-t} \left(-1 + \sum_{m=0}^{\infty} \frac{x^m}{m!} A_m(t) \right) \qquad (4) \\
&= \frac{1}{1-t} \left(\frac{1-t}{1-te^{x(1-t)}} - 1 \right) \qquad (5) \\
&= \frac{1}{1-te^{x(1-t)}} - \frac{1}{1-t} \; .
\end{aligned}$$

Equation (3) follows from the definition of the Eulerian polynomial: since we define $A_m(t)$ to be $\sum_k A_{m,k} t^k$, we rewrite $\sum_{k=1}^{\infty} t^k \left(\sum_{i=1}^{k} A_{m,i} \right)$ as $\frac{1}{1-t} A_m(t)$. Equation (4) uses the convention that $A_{0,0}$ is 1. Equation (5) uses the generating function for Eulerian polynomials stated in equation (2). ∎

Definition 3.6. We define the functions $g(x,t)$ and $g_k(x)$ as follows:

$$g(x,t) = \frac{1}{1-te^{x(1-t)}} = \sum_{k=0}^{\infty} g_k(x) \; t^k \; .$$

Expanding the first few terms of $g(x,t)$, we get

$$g(x,t) = 1 + e^x t + (e^x - x)e^x t^2 + \left((e^x - x)^2 - \frac{x^2}{2}\right) e^x t^3 + \cdots$$

Theorem 3.4. *The threshold k dual bin continuous case generating function is*

$$A_k(x,y,z) = \frac{1 + z(g_k(x) - 1)}{1 - y[(x-1)g_k(x) + 1]} \ .$$

Proof. First consider

$$\begin{aligned}
\hat{A}(x,y,z,t) &= \sum_{k=1}^{\infty} t^k \Big(1 + z \sum_{m=1}^{\infty} l_m^k \ x^m\Big) \\
&= \sum_{k=1}^{\infty} t^k \Big[1 + z \sum_{m=1}^{\infty} x^m \Big(\sum_{i=1}^{k} \frac{1}{m!} A_{m,i}\Big)\Big] && \text{(by Lemma 3.1)} \\
&= \sum_{k=1}^{\infty} t^k + z \sum_{k=1}^{\infty} t^k \Big(\sum_{m=1}^{\infty} \frac{x^m}{m!} \sum_{i=1}^{k} A_{m,i}\Big) \\
&= \frac{t}{1-t} + z \left(\frac{1}{1 - te^{x(1-t)}} - \frac{1}{1-t}\right) \ . && \text{(by Lemma 3.3)}
\end{aligned}$$

Taking the coefficient of t^k of both sides, we get $[t^k] \ \hat{A}(x,y,z,t) = 1 + z(g_k(x) - 1)$.

Similarly, consider

$$\begin{aligned}
\check{A}(x,y,z,t) &= \sum_{k=1}^{\infty} t^k \Big(1 - y \sum_{m=1}^{\infty} f_m^k \ x^m\Big) \\
&= \sum_{k=1}^{\infty} t^k \Big[1 - y \sum_{m=2}^{\infty} x^m \Big(\sum_{i=1}^{k} \frac{A_{m-1,i}}{(m-1)!} - \frac{A_{m,i}}{m!}\Big)\Big] && \text{(by Lemma 3.1)} \\
&= \sum_{k=1}^{\infty} t^k - yx \sum_{k=1}^{\infty} t^k \Big(\sum_{m=1}^{\infty} \frac{x^m}{m!} \Big(\sum_{i=1}^{k} A_{m,i}\Big)\Big) \\
&\quad + y \sum_{k=1}^{\infty} t^k \Big(-x + \sum_{m=1}^{\infty} \frac{x^m}{m!} \sum_{i=1}^{k} A_{m,i}\Big)
\end{aligned}$$

$$= \frac{t}{1-t} - yx\left(\frac{1}{1-te^{x(1-t)}} - \frac{1}{1-t}\right)$$

$$- \frac{yxt}{1-t} + y\left(\frac{1}{1-te^{x(1-t)}} - \frac{1}{1-t}\right) . \qquad \text{(by Lemma 3.3)}$$

Again taking the coefficient of t^k in both sides, we get

$$[t^k]\ \check{A}(x,y,z,t) = 1 - y[(x-1)g_k(x)+1] .$$

Hence

$$A_k(x,y,z) = \frac{1+z(g_k(x)-1)}{1-y[(x-1)g_k(x)+1]} . \qquad \square$$

3.2. The Generating Function for a Discrete Distribution

We now consider the case for the general threshold k where the object sizes are chosen uniformly at random from the set $\frac{0}{h}, \frac{1}{h}, \ldots, \frac{h-1}{h}$.

Definition 3.7. Let $b^{h,k}_{m,n,p}$ be the number of m-tuples which, under the threshold k version of DUAL NEXT FIT, yield a dual bin arrangement with n full bins and p leftover bins. Set $A^h_k(x,y,z) = \sum_{m,n,p} b^{h,k}_{m,n,p}\ x^m y^n z^p$. Then $A^h_k(x,y,z)$ is the generating function for the discrete dual bin threshold k problem.

We write the discrete analogue of Proposition 3.2, obtained using the same arguments.

Proposition 3.5.

$$A^h_k(x,y,z) = \frac{1+z\sum_m b^{h,k}_{m,0,1}\ x^m}{1-y\sum_m b^{h,k}_{m,1,0}\ x^m} .$$

Lemma 3.6.

$$b^{h,k}_{m,0,1} = [t^{hk}]\ \left(\frac{1-t^h}{1-t}\right)^{m+1} \frac{t(1-t^{hk})}{1-t^h} .$$

Proof. To calculate $b^{h,k}_{m,0,1}$, we need to count the number of integer solutions to the equation $\sum_{i=1}^{m} x_i < hk$, with the restriction that the integers are between 0 and $h-1$ inclusive. So we consider the number of compositions of $0, 1, \ldots, hk-1$ with m parts, each part of size $0, \ldots, h-1$. Since the number of compositions of j with m parts, each part in a set S is the coefficient of t^j in $(\sum_{i \in S} t^i)^m$, we have the following:

$$
\begin{aligned}
b^{h,k}_{m,0,1} &= \left(\sum_{i=0}^{h-1} t^i\right)^m + [t] \left(\sum_{i=0}^{h-1} t^i\right)^m + \cdots + [t^{hk-1}] \left(\sum_{i=0}^{h-1} t^i\right)^m \\
&= [t^{hk}] \quad (t^{hk} + t^{hk-1} + \cdots + t) \left(\frac{1-t^h}{1-t}\right)^m \\
&= [t^{hk}] \quad t \left(\frac{1-t^{hk}}{1-t}\right) \left(\frac{1-t^h}{1-t}\right)^m \\
&= [t^{hk}] \quad \frac{t}{1-t}(1-t^h)(1+t^h+\cdots+t^{h(k-1)}) \left(\frac{1-t^h}{1-t}\right)^m \\
&= [t^{kh}] \quad \left(\frac{1-t^h}{1-t}\right)^{m+1} \frac{t(1-t^{hk})}{1-t^h} .
\end{aligned}
$$

□

Lemma 3.7.

$$
b^{h,k}_{m,1,0} = [t^{hk}] \left(\frac{1-t^h}{1-t}\right)^{m-1} \left[\frac{ht}{1-t} - \frac{t}{1-t}\left(\frac{1-t^h}{1-t}\right)\right] .
$$

Proof. To calculate $b^{h,k}_{m,1,0}$, we need to count the number of integer solutions to the equations $\sum_{i=1}^{m-1} x_i < hk$ and $\sum_{i=1}^{m} x_i \geq hk$, with the integers between 0 and $h-1$ inclusive. We can consider the sum of x_1 through x_{m-1} to be $hk-j$ ($j = 1, \ldots, h-1$), forcing x_m to be one of $j, j+1, \ldots, h-1$.

In other words, we want the number of compositions of $hk-j$ with $m-1$ parts, each part of size $0, \ldots, h-1$. Each such composition yields $h-j$ sequences which exactly fill one bin, one for each of the possible values for the mth object. So given

m, we have

$$\begin{aligned}
b^{h,k}_{m,1,0} &= \sum_{j=1}^{h-1}\Big\{(h-j)\cdot[t^{hk-j}]\Big(\sum_{i=0}^{h-1} t^i\Big)^{m-1}\Big\} \\
&= \sum_{j=1}^{h-1}\Big\{(h-j)\cdot[t^{hk-j}]\left(\frac{1-t^h}{1-t}\right)^{m-1}\Big\} \\
&= [t^{hk}]\left(\sum_{j=1}^{h-1}(h-j)t^j\left(\frac{1-t^h}{1-t}\right)^{m-1}\right) \\
&= [t^{hk}]\left(\frac{1-t^h}{1-t}\right)^{m-1}\left(\sum_{j=1}^{h-1}(h-j)t^j\right) .
\end{aligned}$$

Since

$$\sum_{j=1}^{h-1} jt^j = t\cdot\frac{d}{dt}\left(\sum_{j=0}^{h-1} t^j\right) = t\cdot\left(\frac{1-t^h}{(1-t)^2} - \frac{ht^{h-1}}{1-t}\right) ,$$

after simple calculations, we get

$$b^{h,k}_{m,1,0} = [t^{hk}]\left(\frac{1-t^h}{1-t}\right)^{m-1}\left[\frac{ht}{1-t} - \frac{t}{1-t}\left(\frac{1-t^h}{1-t}\right)\right]$$

as needed. □

We will use the following in the determination of $A^h_k(x,y,z)$.

Definition 3.8. Set $g^h(x,t) = t[1-t-x(1-t^h)]^{-1}$ and $g^h_k(x) = [t^{hk}]\ g^h(x,t)$.

Proposition 3.8.

$$g^h_k(x) = \sum_{j=0}^{k}\frac{(-1)^j x^j}{(1-x)^{h(k-j)+j}}\left(\!\!\binom{h(k-j)}{j}\!\!\right) .$$

Proof. Divide the numerator and denominator of $g^h(x,t)$ by $1-x(1-t^h)$ to get

$$\begin{aligned}
g^h(x,t) &= \frac{t}{1-t-x(1-t^h)} \\
&= \frac{t}{1-x(1-t^h)}\cdot\frac{1}{1-\frac{t}{1-x(1-t^h)}}
\end{aligned}$$

$$= \frac{t}{1-x(1-t^h)} \cdot \sum_{i=0}^{\infty} \left(\frac{t}{1-x(1-t^h)}\right)^i$$

$$= \sum_{i=1}^{\infty} \left(\frac{t}{1-x(1-t^h)}\right)^i .$$

Since we are only interested in the powers of t^h, we need only look at the cases where i is a power of h. We thus can restrict ourselves to looking at $[t^{hk}]$ in

$$\sum_{j=0}^{\infty} \left(\frac{t}{1-x(1-t^h)}\right)^{jh} = \sum_{j=0}^{\infty} \frac{(t^h)^j}{(1-x(1-t^h))^{jh}} .$$

Now take $t^h = s$ and look at $[s^k]$ in the following:

$$\sum_{j=0}^{\infty} \left(\frac{s}{(1-x(1-s))^h}\right)^j = \left(1 - \frac{s}{(1-x(1-s))^h}\right)^{-1}$$

$$= \left(1 - s\left(1+\frac{xs}{1-x}\right)^{-h}(1-x)^{-h}\right)^{-1}$$

$$= \sum_{i=0}^{\infty} \left(s\left(1+\frac{xs}{1-x}\right)^{-h}(1-x)^{-h}\right)^i$$

$$= \sum_{i=0}^{\infty} \frac{s^i}{(1-x)^{hi}} \left(\sum_{j=0}^{\infty} (-1)^j \left(\frac{xs}{1-x}\right)^j \left(\!\!\binom{hi}{j}\!\!\right)\right) .$$

Since we need the coefficient of s^k, let $i = k - j$. Then $[s^k]$ in the above is

$$\sum_{j=0}^{k} \frac{(-1)^j x^j}{(1-x)^{h(k-j)+j}} \left(\!\!\binom{h(k-j)}{j}\!\!\right)$$

and we have our expression for $g_k^h(x)$. $\square$

Theorem 3.9.

$$A_k^h(x,y,z) = \frac{1 + z(g_k^h(x) - 1)}{1 - y[(hx-1)g_k^h(x) + 1]} .$$

Proof. First consider the numerator $1 + z \sum_m b_{m,0,1}^{h,k} x^m$. From Lemma 3.6, this becomes

$$1 + z \sum_{m=1}^{\infty} x^m \left\{ [t^{kh}] \quad \left(\frac{1-t^h}{1-t}\right)^{m+1} \frac{t(1-t^{hk})}{1-t^h} \right\} =$$

$$= 1 + z \cdot [t^{hk}] \left\{ \Big(\frac{1-t^h}{1-t}\Big) \frac{t(1-t^{hk})}{1-t^h} \sum_{m=1}^{\infty} x^m \Big(\frac{1-t^h}{1-t}\Big)^m \right\}$$

$$= 1 + z \cdot [t^{hk}] \quad \frac{t(1-t^{hk})}{1-t} \left\{ -1 + \sum_{m=0}^{\infty} x^m \Big(\frac{1-t^h}{1-t}\Big)^m \right\}$$

$$= 1 + z \cdot [t^{hk}] \quad \frac{t(1-t^{hk})}{1-t} \quad \left\{ -1 + \frac{1}{1 - x\left(\frac{1-t^h}{1-t}\right)} \right\}$$

$$= 1 + z \cdot [t^{hk}] \quad \left\{ \frac{t - t^{hk+1}}{1 - t - x(1-t^h)} - \frac{t(1-t^{hk})}{1-t} \right\} .$$

Note that, since we are interested only in $[t^{hk}]$, the part multiplied by t^{hk+1} contributes nothing. Using Definition 3.8, we can thus write the numerator of $A_k^h(x,y,z)$ as $1 + z(g_k^h(x) - 1)$.

From Lemma 3.7 the denominator $1 - y\sum_m b_{m,1,0}^{h,k}\ x^m$ becomes

$$1 - y \sum_{m=1}^{\infty} x^m \left\{ [t^{hk}] \quad \Big(\frac{1-t^h}{1-t}\Big)^{m-1} \Big[\frac{ht}{1-t} - \frac{t}{1-t}\Big(\frac{1-t^h}{1-t}\Big)\Big] \right\} =$$

$$= 1 - y \cdot [t^{hk}] \left\{ \frac{htx}{1-t} \sum_{m=1}^{\infty} x^{m-1} \Big(\frac{1-t^h}{1-t}\Big)^{m-1} \right\}$$

$$- y \cdot [t^{hk}] \left\{ -\frac{t}{1-t} \Big[-1 + \sum_{m=0}^{\infty} x^m \Big(\frac{1-t^h}{1-t}\Big)^m \Big] \right\}$$

$$= 1 - y \cdot [t^{hk}] \left\{ \frac{t(hx-1)}{1-t} \ \frac{1}{1 - x\left(\frac{1-t^h}{1-t}\right)} + \frac{t}{1-t} \right\}$$

$$= 1 - y \cdot [t^{hk}] \left\{ \frac{t(hx-1)}{1-t-x(1-t^h)} + \frac{t}{1-t} \right\} .$$

Using Definition 3.8 we rewrite this as $1 - y[(hx-1)g_k^h(x) + 1]$, yielding the formula

$$A_k^h(x,y,z) = \frac{1 + z(g_k^h(x) - 1)}{1 - y[(hx-1)g_k^h(x) + 1]} . \qquad \square$$

Using Proposition 3.8, we can also express this explicitly as

$$A_k^h(x,y,z) = \frac{1 - z + z\left[\sum_{j=0}^{k} \frac{(-1)^j x^j}{(1-x)^{h(k-j)+j}} \left(\!\!\binom{h(k-j)}{j}\!\!\right)\right]}{1 - y - y(hx-1)\left[\sum_{j=0}^{k} \frac{(-1)^j x^j}{(1-x)^{h(k-j)+j}} \left(\!\!\binom{h(k-j)}{j}\!\!\right)\right]} .$$

3.3. A Map to k-Hook-Shape Decomposition Sets

We now generalize the function ϕ of Section 2.3 to the threshold k case. Given a threshold k dual bin arrangement A, we look at the action of ϕ_A on P_A.

Notation. We let $T_j = \sum_{i=b}^{j} x_j$, where b is the first element in the bin containing the element j.

Using this notation, we recall from Definition 2.11 that $\phi(x_1, \ldots, x_m) = (y_1, \ldots, y_m)$ where y_j is defined to be $1 - T_j + \lfloor T_j \rfloor$.

As was discussed in Section 2.3, this results in a hook-shape decomposition set corresponding to each threshold 1 dual bin arrangement. Within any bin, we get descents and then an ascent when the running total T_j crosses over the integer 1 at the end, resulting in a hook-shape. In a leftover bin, since the running total never crosses over the threshold 1, we only get descents and hence the tail-shape of our decomposition set.

Now suppose we apply this same function in the threshold k case. Within a full bin, we once again generate all descents save for k ascents, one for each of the k instances when T_j crosses over an integer. Similarly, a leftover bin will correspond to a shape with less than k a's. This motivates the following definitions.

Definition 3.9. We define a k-hook-shape of length m to be a shape with m letters ending with a that includes precisely $k-1$ other a's amongst its other letters. A k-hook is a permutation whose shape is a k-hook-shape. Similarly, we define a k-tail-shape of length m to be a shape with m letters that has fewer than k of its letters be a's, and a k-tail to be a permutation whose shape is a k-tail-shape.

There was only one hook-shape and one tail-shape of length m. Note however that there are $\binom{m-1}{k-1}$ shapes of length m that are k-hook-shapes. For instance, the 3-hook-shapes of length 5 are *aadda*, *adada*, *addaa*, *daada*, *dadaa*, and *ddaaa*. There are $\sum_{i=0}^{k-1}\binom{m}{i}$ k-tail-shapes of length m.

Just as we could express any shape in terms of a sequence of hook-shapes, we can express any shape in terms of a sequence of k-hook-shapes.

Definition 3.10. Let $i_1, \ldots, i_n$ be positive integers and j be an integer greater than or equal to -1. We define the k-hook-shape decomposition set indexed by $(i_1, \ldots, i_n; j)$ to be the set of shapes of $i_1 + \cdots + i_n + j + n$ letters that can be expressed as a sequence of k-hook-shapes of lengths i_1 through i_n followed by a k-tail-shape of length j. Between any two k-hook-shapes, or between the last k-hook-shape and the k-tail-shape, there can be either a or d.

We frequently write the a or d that combines two k-hook-shapes as $\not{a}$ or $\not{d}$. We keep the same conventions about what a k-tail-shape of length -1 or 0 means as we had in Section 2.3.

Definition 3.11. Let H_k be a k-hook-shape decomposition set whose shapes have length m. Then we define a polytope $P_{H_k} \subset (0,1]^m$ by taking the union of the polytopes P_S, where the union is over all shapes S in the decomposition set H_k.

In order to examine the behavior of ϕ in the threshold k case, we need the following.

Definition 3.12. Let A be a dual bin arrangement of m elements, with a total of n full plus leftover bins. Consider a point $(x_1, \ldots, x_m) \in P_A$. Associated to this point we define a strictly increasing sequence

$$\bar{v} = (v_{1,1}, \ldots v_{1,k}, \ldots, v_{n-1,1}, \ldots, v_{n-1,k}, v_{n,1}, \ldots, v_{n,k'})$$

where $v_{l,i}$ is the minimum j in the lth bin for which $T_j \geq i$. If A was a full dual bin arrangement, then $k' = k$; if it was a leftover arrangement then $k' < k$. If in a leftover arrangement our chosen point has $T_m < 1$ resulting in $k' = 0$, then we consider the corresponding $\bar{v}$ as ending with the entry $v_{n-1,k}$. Note that $v_{l,k}$ is the last element in the lth bin. We define $P_{A_{\bar{v}}} \subset P_A$ to be the polytope consisting of all points with this same $\bar{v}$.

For example, let A be the following threshold 3 leftover dual bin arrangement.

1 2 3 4 5 6	7 8 9

Consider the point $(.3, .4, .3, .9, .5, .8, .1, .1, .3) \in P_A$. Since $T_1 = .3$, $T_2 = .7$, and $T_3 = 1.0$, we have $v_{1,1} = 3$. Similarly, $T_4 = 1.9$ and $T_5 = 2.4$, so $v_{1,2} = 5$. Since $T_6 = 3.2$, we have $v_{1,3} = 6$. Now $T_7 = .1$, $T_8 = .2$, and $T_9 = .5 < 1$, so we ended this $\bar{v}$ with $v_{1,3}$. So we can write $(.3, .4, .3, .9, .5, .8, .1, .1, .3) \in P_{A_{(3,5,6)}}$. On the other hand, the point $(.9, .8, .7, .1, .1, .7, .9, .9, .5) \in P_A$ has the associated $\bar{v} = (v_{1,1}, v_{1,2}, v_{1,3}, v_{2,1}, v_{2,2}) = (2, 3, 6, 8, 9)$ so $(.9, .8, .7, .1, .1, .7, .9, .9, .5) \in P_{A_{(2,3,6,8,9)}}$.

Proposition 3.10. *Given a threshold k dual bin arrangement A, ϕ_A is a piecewise affine linear, volume preserving map.*

Proof. If we subdivide P_A into the subpolytopes $P_{A_{\bar{v}}}$ of Definition 3.12, then ϕ is affine linear on each $P_{A_{\bar{v}}}$. Fix such a $\bar{v}$. By introducing the dummy variables $x_0 = y_0 = 1$, we can write the action of ϕ on $P_{A_{\bar{v}}}$ as an $m+1$ by $m+1$ matrix $M_{A_{\bar{v}}}$ with determinant $(-1)^m$, where m is the number of elements in A. In particular, suppose $v_{l,i-1} \leq j < v_{l,i}$. Then $\lfloor T_j \rfloor = i - 1$, so we know that $y_j = i - T_j$. Hence we can write the $j+1$st row of M explicitly as $[i, 0, \ldots, 0, -1, \ldots, -1, 0, \ldots, 0]$, where the -1's occur in the $v_{l-1,k}+2$nd through $j+1$st positions. The first row is $[1, 0, \ldots, 0]$. Hence $M_{A_{\bar{v}}}$ is invertible with determinant $(-1)^m$. □

As before, we define ϕ on all of $[0, 1)^m$ by taking $\phi(P_A) = \phi_A(P_A)$.

Theorem 3.11. *Let A be a threshold k dual bin arrangement with n full bins and p leftover bins of sizes $m_1, \ldots, m_n, m_0$ respectively. Let H_k be the k-hook-shape decomposition set indexed by $(m_1 - 1, \ldots, m_n - 1; m_0 - 1)$. Then ϕ bijectively takes P_A to P_{H_k}.*

The proof is straightforward and we omit it.

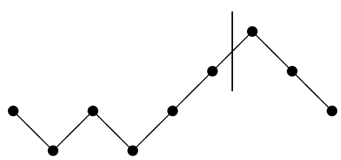
If we look at a particular $P_{A_{\bar{v}}}$, then we can say which particular shape each k-hook-shape in the decomposition set actually is. A shape in $\phi(P_{A_{\bar{v}}})$ will have the

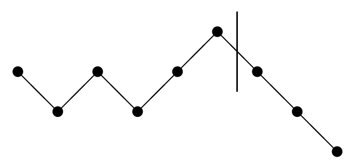
a's of the lth k-hook-shape as the $v_{l,1} - 1$st, $v_{l,2} - 1$st, … , and $v_{l,k} - 1$st letters of the shape. However, we still have full choice of $a\!|$'s and $d\!|$'s joining the k-hook-shapes and k-tail-shape.

For example, if we take the threshold 3 leftover dual bin arrangement

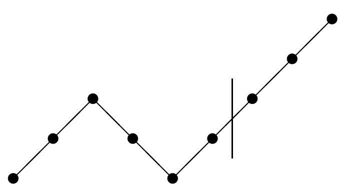
$$A \quad = \quad \boxed{1\;2\;3\;4\;5\;6} \quad \boxed{7\;8\;9}$$

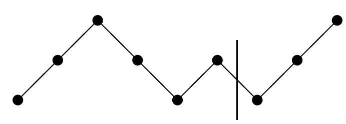
from our earlier example, and let $\phi(P_A) = P_{H_k}$, then the decomposition set H_k will include a total of 48 shapes. However, if we look at $P_{A_{(3,5,6)}}$, the corresponding shapes are $dadaa\,a\!|\,dd$ and $dadaa\,d\!|\,dd$.





For $P_{A_{(2,3,6,8,9)}}$, the corresponding shapes are $aadda\,a\!|\,aa$ and $aadda\,d\!|\,aa$.





However, aside from the threshold k dual bin problem and the discussion in the following section, it would be interesting to know whether k-hooks and k-hook shape decomposition sets have any other combinatorial interpretations.

3.4. Combinatorics of k-Hooks

Theorem 3.12. *The number of k-hooks of length m is $(m-k)A_{m-1,k}$.*

Proof. By definition, the number of k-hooks of length m is the number of permutations whose shape is a k-hook-shape of length $m-1$. By Theorem 3.11, this is $m!$ times the volume of the polytope for the threshold k dual bin arrangement consisting of one full bin with m elements. Lemma 3.1 calculates this volume to be $\sum_{i=1}^{k} \frac{1}{(m-1)!} A_{m-1,i} - \sum_{i=1}^{k} \frac{1}{m!} A_{m,i}$. So the number of k-hooks of length m is

$$\sum_{i=1}^{k} mA_{m-1,i} - \sum_{i=1}^{k} A_{m,i} \ .$$

Using the well known recursion for the Eulerian numbers

$$A_{m,k} = kA_{m-1,k} + (m-k+1)A_{m-1,k-1} \text{ for } m,k \geq 1 \ , \tag{6}$$

we get that this is

$$\begin{aligned}
&= \sum_{i=1}^{k} mA_{m-1,i} \ - \sum_{i=1}^{k} (iA_{m-1,i} + (m-i+1)A_{m-1,i-1}) \\
&= \sum_{i=1}^{k} (m-i)A_{m-1,i} \ - \sum_{i=1}^{k-1} (m-(i+1)+1)A_{m-1,i} \ - \ mA_{m-1,0} \\
&= (m-k)A_{m-1,k}
\end{aligned}$$

as needed. □

It is natural to consider a k-hook σ of length m as essentially a permutation of $m-1$ with $k-1$ descents; after all, if one ignores $\sigma(m)$ and hence the final descent, then the remaining $\sigma(1)$ through $\sigma(m-1)$ contains $k-1$ descents. We can make this more explicit.

Notation. If π is a permutation of $[m]$, we will call π' the permutation of $[m-1]$ obtained by removing $\pi(m)$ and relabeling appropriately. In other words, if $i < \pi(m)$ then $\pi'(\pi^{-1}(i)) = i$ and if $i > \pi(m)$ then $\pi'(\pi^{-1}(i)) = i - 1$.

Definition 3.13. Let $\pi \in S_m$. We define the augmented permutation $\pi_i \in S_{m+1}$ by taking $\pi_i(m+1) = i$, $\pi_i(\pi^{-1}(j)) = j$ for $j < i$, and $\pi_i(\pi^{-1}(j)) = j+1$ for $j \geq i$. If $i > \pi(m)$ then π_i ends in an ascent and we say it is an ascent-augmented permutation. If $i \leq \pi(m)$ then π_i is descent-augmented; if π had $k-1$ descents we say π_i is a descent-augmented (m,k)-permutation.

For example, if $\pi = 4\,3\,1\,2$ then $\pi' = 3\,2\,1$, $\pi'_1 = 4\,3\,2\,1$, $\pi'_2 = 4\,3\,1\,2$, $\pi'_3 = 4\,2\,1\,3$, and $\pi'_4 = 3\,2\,1\,4$. Only π'_1 is descent-augmented, and it is a descent-augmented $(3,3)$-permutation.

Proposition 3.13. *The set of k-hooks of length m and the set of descent-augmented $(m-1,k)$-permutations are the same. In particular, if σ is a k-hook, then $\sigma'_{\sigma(m)} = \sigma$.*

Proof. By construction, a descent-augmented $(m-1,k)$-permutation is a k-hook of length m. Now we let σ be a k-hook of length m. Since we form σ' by taking away $\sigma(m)$ and relabeling, descent-augmenting σ' by $\sigma(m)$ merely reverses the procedure. So $\sigma'_{\sigma(m)} = \sigma$. □

Definition 3.14. Let $A^i_{m-1,k}$ be the number of permutations $\pi \in S_{m-1}$ such that π has $k-1$ descents and $\pi(m-1) = i$.

Corollary 3.14. $(m-k)A_{m-1,k} = \sum_{i=1}^{m-1} i\, A^i_{m-1,k}$

Proof. Let π be a permutation of $m-1$ with $k-1$ descents, and set $i = \pi(m-1)$. Then π has i descent-augmentations $\pi_1, \ldots, \pi_i$, each of which is a k-hook of length

m. Since every k-hook can be gotten in such a manner, there are $\sum_{i=1}^{m-1} i\, A^i_{m-1,k}$ k-hooks. But from Theorem 3.12, this is $(m-k)A_{m-1,k}$. □

In [**4**], Ehrenborg *et al.* relate the mixed volumes of the adjacent slices of the unit hypercube given in Definition 3.3 to permutations ending in a particular element.

Theorem 3.15 (Ehrenborg-Readdy-Steingrímsson)**.** *The mixed volume* $V(P^k_m, m-i; P^{k+1}_m, i)$ *is equal to the number of permutations of* $[m{+}1]$ *with* k *descents and ending with the element* $i+1$*.*

Corollary 3.14 and the above give the following.

Corollary 3.16. $\sum_{i=1}^{m} i\, V(P^{k-1}_{m-1}, m-i\,;\, P^k_{m-1}, i-1) = (m-k+1)A_{m,k}$ *, which is the number of* k*-hooks of length* $m+1$*.*

So the weighted sum of the mixed volumes of two adjacent slices of the unit hypercube is counted by k-hooks.

While the preceeding discussion relates k-hooks to the permutations counted by $A_{m-1,k}$, it doesn't give any insight about why the factor of $(m-k)$ in particular appears in Theorem 3.12. A partial explanation follows.

Definition 3.15. Let $\mathcal{A}^+_{m,k+1}$ be the set of permutations of m with k descents whose shapes end with a, and let $|\mathcal{A}^+_{m,k+1}| = A^+_{m,k+1}$. Similarly, let $\mathcal{A}^-_{m,k+1}$ be the set of permutations of m with k descents whose shapes end with d, and let $|\mathcal{A}^-_{m,k+1}| = A^-_{m,k+1}$.

Since the number of k-hooks of length m is just $A^-_{m,k+1}$, we can write Theorem 3.12 as $(m-k)A_{m-1,k} = A^-_{m,k+1}$.

The recursion (6) for the Eulerian numbers can be derived by examining where m can be inserted in a permutation of $[m-1]$, and how this in turn affects the total number of descents. A similar analysis yields the following recursions.

$$A^+_{m,k+1} = (k+1)A^+_{m-1,k+1} + (m-k-1)A^+_{m-1,k} + A^-_{m-1,k+1} \tag{7}$$

$$A^-_{m,k+1} = kA^-_{m-1,k+1} + (m-k)A^-_{m-1,k} + A^+_{m-1,k} \tag{8}$$

Note that if we sum the middle term of (7) with the latter two terms of (8), we get

$$(m-k-1)A^+_{m-1,k} + (m-k)A^-_{m-1,k} + A^+_{m-1,k} = (m-k)(A^+_{m-1,k} + A^-_{m-1,k})$$

which is just $(m-k)A_{m-1,k}$.

We obtained the middle term of (7) by adding an m either to the start of or in any ascent (save the last one) of permutations $\sigma \in A^+_{m-1,k}$. This results in the set

$$B_1 = \Big\{\pi \in \mathcal{A}^+_{m,k+1} \ : \ \pi(1) = m \quad \text{or} \\ \Big(\pi(\pi^{-1}(m)+1) > \pi(\pi^{-1}(m)-1) \quad \text{and} \quad \pi(m-1) \neq m\Big)\Big\} \ .$$

We obtained the first term of (8) by adding an m in any descent of permutations $\sigma \in A^-_{m-1,k+1}$. This results in the set

$$B_2 = \Big\{\pi \in \mathcal{A}^-_{m,k+1} : \pi(\pi^{-1}(m)+1) < \pi(\pi^{-1}(m)-1)\Big\} \ .$$

It would be desirable to produce an explicit bijection between B_1 and B_2 and hence provide a more intuitive proof of Theorem 3.12. This remains an open problem.

CHAPTER 4

The Bin Packing Problem

We now turn our attention to the bin case. Recall that, unlike the dual bin problem, in the bin problem we wish to minimize the number of bins necessary to hold the objects. We let the object sizes be chosen uniformly at random from the interval $(0, 1]$, and require that if $i_1, \ldots, i_m$ are in a bin, then $\sum_{j=1}^{m} x_{i_j} \leq k$ where k is the capacity of the bin.

We will always assume our bin arrangements are obtained from applying the NEXT FIT algorithm to a sequence of objects. As in Proposition 1.1, we consider NEXT FIT as partitioning the unit m-hypercube (minus some boundary) of sequences of m objects into polytopes corresponding to the different possible bin arrangements.

4.1. A Map Between Bin Arrangements and Shapes

In the dual bin case, ϕ related dual bin arrangements and hook shape sequences. In the bin case, we will define a function ψ that will relate bin arrangements and shapes.

Definition 4.1. Let A be a bin arrangement with m elements. Define a function ψ_A from P_A to $[0, 1]^m$ with $\psi_A(x_1, \ldots, x_m) = (y_1, \ldots, y_m)$ as follows:

$$y_j = \begin{cases} 1 - \sum_{i=b}^{j} x_i, & \text{if } j \text{ is in an odd-numbered bin;} \\ \sum_{i=b}^{j} x_i, & \text{if } j \text{ is in an even-numbered bin.} \end{cases}$$

where b is the first element in the bin that contains j. For odd bins, y_i is just the remaining space left in the bin, and for even bins, it is just the total amount in the bin so far.

Proposition 4.1. *For each bin arrangement A, ψ_A is an affine linear, volume preserving map.*

Proof. Similar to the situation with ϕ_A, we will write the action of ψ_A as an invertible $m+1$ by $m+1$ matrix N_A with determinant ± 1. We introduce the dummy variables $x_0 = y_0 = 1$ write the first row of N_A as $[1, 0, \ldots, 0]$.

Now consider the $j+1$st row of N_A. Suppose b is the first element in the bin containing j. If this is an odd-numbered bin, then $y_j = 1 - \sum_{i=b}^{j} x_i$ and hence the $j+1$st row of N_A will start with 1, then contain -1 in the $(b+1)$st through $(j+1)$st positions, with zero everywhere else. If this is an even-numbered bin, then $y_j = \sum_{i=b}^{j} x_i$ and hence the $j+1$st row of N_A will contain 1's in the $(b+1)$st through $(j+1)$st positions, with zero everywhere else. So N_A is lower triangular with ± 1 on the main diagonal. □

For the odd numbered bins, this function is almost the same as ϕ_A.

Definition 4.2. Given $m > 0$, define a piecewise linear, volume preserving map ψ on $(0,1]^m$ into $[0,1]^m$. Divide $(0,1]^m$ into polytopes P_A for each bin arrangement A of m elements. Then form ψ by defining $\psi(x_1, \ldots, x_m) = \psi_A(x_1, \ldots, x_m)$ for $(x_1, \ldots, x_m) \in P_A$.

Proposition 4.2. *Given the bin arrangement A, the points in $\psi(P_A)$ satisfy the following inequalities.*

(1) $y_j > y_{j+1} \geq 0$ *if* $j+1$ *is in an odd-numbered bin*

(2) $y_j < y_{j+1} \leq 1$ *if* $j+1$ *is in an even bin*

(3) $0 \leq y_1 < 1$

Proof. Suppose $j+1$ is in an odd bin. Then y_{j+1} measures the space left in the bin, so $y_{j+1} \geq 0$. If j is in the same bin, then there is less space left after adding in the size of $j+1$, so $y_j > y_{j+1}$. If j was the last element in the previous (necessarily even-numbered) bin, y_j is the total amount in that bin and $1 - y_j$ the space left. Since $j+1$ is in the next bin, it clearly didn't fit. So $1 - y_j < x_{j+1}$, which rewrites to $y_j > 1 - x_{j+1} = y_{j+1}$.

Suppose $j+1$ is in an even bin. Then similar to before, y_{j+1} is the total in the bin after adding $j+1$, so $y_{j+1} \leq 1$. If j is in the same bin, there is of course more in the bin after adding $j+1$, so $y_j < y_{j+1}$. If j is in the previous bin, y_j measures the space left in that bin. But this is less than $x_{j+1} = y_{j+1}$, since $j+1$ didn't fit. Hence $y_j < y_{j+1}$.

Finally, $0 \leq y_1 < 1$ follows directly from the definition of $\psi_A(P_A)$ and the fact that $0 < x_1 \leq 1$. □

Recall from Definitions 2.5 and 2.6 that a shape is a word in the letters a and d, and that the function Sh takes a permutation to a shape by writing an a for an ascent and a d for a descent.

It will be convenient to write a shape in the form $d^{j_1}a^{j_2}d^{j_3}\cdots d^{j_n}$ (if n is odd) or $d^{j_1}a^{j_2}d^{j_3}\cdots d^{j_{n-1}}a^{j_n}$ (if n is even). We permit j_1 to be any non-negative integer, and require $j_2,\ldots,j_n$ to be positive integers. For example, $adda$ is $d^0a^1d^2a^1$. This sequence $(j_1,\ldots,j_n)$ uniquely determines the shape.

We will also sometimes mark a shape. For each link ad and da, we mark the latter letter with a dot: $a\dot{d}$ and $d\dot{a}$. If the first letter in the shape is an a, we also give it a dot. For example, we will write $d^0a^1d^1a^3d^2a^1d^2$ as $\dot{a}\dot{d}\dot{a}aa\dot{d}d\dot{a}\dot{d}d$ and $d^3a^4d^1a^2d^2a$ as $ddd\dot{a}aaa\dot{d}\dot{a}a\dot{d}d\dot{a}$. If S is a shape, we will denote its marked version by s. Note this marking is merely a bookkeeping device and adds no additional information to the shape.

Definition 4.3. Let s be a marked shape with $m-1$ letters. We define a polytope $P_s \subset [0,1]^m$ in the following way. If the ith letter of s is $\dot{d}$ or d, then we write the inequalities $y_i > y_{i+1}$ and $0 \leq y_{i+1} < 1$. If it is $\dot{a}$ or a we write $y_i < y_{i+1}$ and $0 < y_{i+1} \leq 1$. Finally, we have $0 < x_1 < 1$. These $2m-1$ inequalities determine P_s.

Remark 4.1. Notice that we define the polytope P_s slightly differently from the polytope P_S of Definition 2.9.

Remark 4.2. For any marked shape s, the coordinates of the vertices of the polytope P_s are all 0 or 1.

Remark 4.3. The disjoint union of the polytopes P_A is all of $(0,1]^m$. However, the disjoint union of the polytopes P_s is certainly not all of $[0,1]^m$. The hyperplanes of the form $y_i = y_{i+1}$ are missing, as is some of the boundary. Nonetheless, to calculate the volume of P_s for a particular marked shape s, it is irrelevant which inequalities forming the facets are strict and which are weak. We merely need to count the number of permutations $\sigma \in S_m$ such that $\mathrm{Sh}(\sigma) = S$, then divide by $m!$.

Definition 4.4. Let s be a (marked) shape containing $m-1$ letters. We define a function, $\psi'_s : P_s \to \mathbf{R}^m$ with $\psi'_s(y_1, \ldots, y_m) = (x_1, \ldots, x_n)$ as follows:

$$x_i = \begin{cases} 1 - y_i, & \text{if the } (i-1)\text{st letter in } s \text{ is } \dot{d}, \text{ or if } i = 1; \\ y_{i-1} - y_i, & \text{if the } (i-1)\text{st letter of } s \text{ is } d; \\ y_i, & \text{if the } (i-1)\text{st letter of } s \text{ is } \dot{a}; \\ y_i - y_{i-1}, & \text{if the } (i-1)\text{st letter of } s \text{ is } a. \end{cases}$$

The following provides an alternate combinatorial interpretation for bin arrangements.

Theorem 4.3. *Let A be the bin arrangement of m elements whose n bins contain $j_1, j_2, \ldots, j_n$ elements respectively. Let S be the shape $d^{j_1-1}a^{j_2} \cdots d^{j_n}$ if n is odd and the shape $d^{j_1-1}a^{j_2}d^{j_3} \cdots a^{j_n}$ if n is even. Then ψ bijectively takes P_A to P_s, and $\psi_A^{-1} = \psi'_s$.*

We prove this using the following two lemmas.

Lemma 4.4. *Taking A and S as above, $\psi(P_A) \subseteq P_s$.*

Proof. The inequalities described in Proposition 4.2 define P_s. □

Lemma 4.5. *Taking A and S as before, $\psi'_s(P_s) \subseteq P_A$.*

Proof. First, we write $j'_k = j_1 + j_2 + \cdots + j_k$. We take j'_0 to be 0.

Consider the sums

$$\sum_{i=j'_{k-1}+1}^{j'_k} x_i \qquad \text{and} \qquad \sum_{i=j'_{k-1}+1}^{j'_k+1} x_i \ .$$

For k odd (in other words, j_k is the exponent of a d), the first sum becomes

$$(1 - y_{j'_{k-1}+1}) + (y_{j'_{k-1}+1} - y_{j'_{k-1}+2}) \cdots + (y_{j'_k-1} - y_{j'_k}) = 1 - y_{j'_k} \ .$$

Since $0 \le y_{j'_k} < 1$, we get that $\sum_{i=j'_{k-1}+1}^{j'_k} x_i \le 1$.

Since k is odd, we know the j'_kth letter of s is $\dot{a}$. Hence we know that $x_{j'_k+1} = y_{j'_k+1}$, making the second sum $1 - y_{j'_k} + y_{j'_k+1}$. But k odd also means that $y_{j'_k} < y_{j'_k+1}$, so $\sum_{i=j'_{k-1}+1}^{j'_k+1} x_i > 1$

The calculations for the case where k is even (j_k is the exponent of an a) are similar. The first sum becomes

$$y_{j'_{k-1}+1} + (y_{j'_{k-1}+2} - y_{j'_{k-1}+1}) \cdots + (y_{j'_k} - y_{j'_k-1}) = y_{j'_k} \ .$$

Since $0 < y_{j'_k} \le 1$, we once again have $\sum_{i=j'_{k-1}+1}^{j'_k} x_i \le 1$.

We know if k is even that the j'_kth letter of s is $\dot{d}$. Then $x_{j'_k+1} = 1 - y_{j'_k+1}$ and the second sum becomes $y_{j'_k} + 1 - y_{j'_k+1}$. But it also means that $y_{j'_k} > y_{j'_k+1}$, so we once again have $\sum_{i=j'_{k-1}+1}^{j'_k+1} x_i > 1$.

Finally, a straightforward check shows that $\psi'_s(P_s)$ is contained in $(0,1]^m$.

Since we've derived all of the inequalities that determine P_A, the lemma is proved.

□

Proof of Theorem 4.3. Take a point $(x_1, \ldots, x_m) \in P_A$. By Lemma 4.4 we know $\psi(x_1, \ldots, x_m) \in P_s$. By Lemma 4.5 we know $\psi'_s\psi(x_1, \ldots, x_m) \in P_A$. Now a simple check shows that $\psi_s'\psi(x_1, \ldots, x_m) = (x_1, \ldots, x_m)$. Similarly, for $(y_1, \ldots, y_m) \in P_s$, a simple check shows $\psi\psi_s'(y_1, \ldots, y_m) = (y_1, \ldots, y_m)$. Hence $\psi_s' = \psi_A^{-1}$. □

Instead of finding the volume of a polytope P_A, Theorem 4.3 allows us to count permutations, a much easier problem.

There are several different ways we can determine which bin arrangement a permutation corresponds to. First, we can take its shape and write it in the form described on page 55. We then apply Theorem 4.3. For example, take $\sigma = 1\,3\,4\,2\,8\,7\,6\,9\,5$. We write $\mathrm{Sh}(\sigma) = aadaddad = d^0a^2d^1a^1d^2a^1d^1$. The corresponding bin arrangement is

$$\boxed{1}\ \ \boxed{2\ 3}\ \ \boxed{4}\ \ \boxed{5}\ \ \boxed{6\ 7}\ \ \boxed{8}\ \ \boxed{9}\,.$$

For the permutation $\tau = 6\,3\,2\,1\,4\,5$ we have $\mathrm{Sh}(\tau) = dddaa = d^3a^2$ which corresponds to the bin arrangement

$$\boxed{1\ 2\ 3\ 4}\ \ \boxed{5\ 6}\,.$$

Definition 4.5. Let σ be a permutation and suppose the ith through $i+j$th letters of $\mathrm{Sh}(\sigma)$ are all a's. Then we say $\sigma(i) < \cdots < \sigma(i+j+1)$ is an increasing run. If the $i-1$st and $i+j+1$st letters of $\mathrm{Sh}(\sigma)$ are both d's (or do not exist), then we say the above run is a maximal increasing run. We define decreasing and maximal decreasing runs similarly.

Corollary 4.6. *Given a permutation, the number of bins in the bin arrangement it represents is the number of its maximal increasing and decreasing runs, with the stipulation that we consider the first run to be necessarily decreasing.*

The number of items in the first bin is the length of the leading decreasing run. Subsequently, if the ith maximal run has length j, the ith bin contains $j-1$ items.

Proof. Because Theorem 4.3 assumes that a shape starts with d^{j_1-1}, the first run must be a decreasing one. The rest is immediate from the previous discussion and the theorem, keeping in mind that an increasing (decreasing) run of length j results in a^{j-1} (d^{j-1}) in the shape. □

For example, we split σ into maximal runs as follows: 1, 1 3 4, 4 2, 2 8, 8 7 6, 6 9, and 9 5. (Note that 1 is a decreasing run of length 1.) We of course get the same bin arrangement as before. We split τ into maximal runs 6 3 2 1 and 1 4 5, again yielding the desired bin arrangement.

Remark 4.4. If we consider all runs as necessarily increasing, we slightly modify the definition of 'run'. It should be clear from context which interpretation we assume. Let the $i-1$st through $i+j$th letters of a shape $\mathrm{Sh}(\sigma)$ be ad^ja, for $j \geq 2$. Instead of having a decreasing run, we view $\sigma(i)$ as part of the maximal increasing run ending with $\sigma(i-1) < \sigma(i)$ and we view $\sigma(i+j)$ as part of the maximal increasing run starting with $\sigma(i+j) < \sigma(i+j+1)$. We then have $j-1$ consecutive maximal increasing runs of length 1: $\sigma(i+1), \ldots,$ and $\sigma(i+j-1)$.

It thus suffices to consider only increasing runs.

Corollary 4.7. *If a permutation with i maximal increasing runs of length at least 2 ends in an ascent, then the number of bins it counts is $2i$. If it ends in a descent, then the number of bins is $2i+1$.*

Finally, note that in a shape, the links ad and da correspond to peaks and valleys, as shown below.

This motivates the following corollary.

Corollary 4.8. *Given a permutation* π*, define* p_π *to be the number of peaks,* v_π *to be the number of valleys, and* a_π *to be* 0 *if* $\pi(1) > \pi(2)$ *and* 1 *if* $\pi(1) < \pi(2)$*. Then the number of bins in the bin arrangement represented by* π *is* $1 + p_\pi + v_\pi + a_\pi$*.*

For example, σ starts with an ascent, has 3 peaks and 2 valleys for a total of $1 + p_\sigma + v_\sigma + a_\sigma = 1 + 3 + 2 + 1 = 7$ bins. The permutation τ starts with a descent, has no peaks and 1 valley, for a total of $1 + p_\tau + v_\tau + a_\tau = 1 + 0 + 1 + 0 = 2$ bins.

We can also rewrite Corollary 4.8 to read that the number of bins is one more than the sum of the number of $\dot{a}$'s and $\dot{d}$'s.

We will use the interpretation in Corollary 4.8 to calculate the expectation and variance of the number of bins.

4.2. Discrete Distribution of Object Sizes

Similar to the dual bin discrete case, we suppose that our object sizes are evenly distributed among $\frac{1}{h}, \ldots, \frac{h-1}{h}, \frac{h}{h}$ for some integer $h \geq 2$. We wish to count the number of sequences that result in a bin arrangement A; equivalently, we wish to count the integer lattice points in the polytope $h \cdot P_A$. Instead of the maps ψ_A and ψ'_s of Definitions 4.1 and 4.4, we consider the maps $h \cdot \psi_A : h \cdot P_A \to h \cdot [0, 1]$ and $h \cdot \psi'_s : h \cdot P_s \to h \cdot (0, 1]$ defined in the obvious way. Then the following is immediate.

Proposition 4.9. *The maps* $h \cdot \psi_A$ *and* $h \cdot \psi'_s$ *preserve integer lattice points.*

Corollary 4.10 (of Theorem 4.3)**.** *The number of integer lattice points in* $h \cdot P_A$ *is equal to the number of integer lattice points in* $h \cdot P_s$.

Corollary 4.11. *The Ehrhart polynomials of* P_A *and* P_s *are equal.*

Remark 4.5. Recall from Definition 1.6 that our polytopes are neither open nor closed, but instead the intersection of both open and closed half spaces. As a result, the Ehrhart polynomial above is actually the sum of the Ehrhart polynomials $\sum \bar{i}(P_f, h)$, where the summation is taken over each face P_f present in the polytope, and $\bar{i}(P, h)$ is as in [**12**, p. 235].

What does an integer lattice point in $h \cdot P_s$ look like? Since under the action of $h \cdot \psi_A$, $y_1 = h - hx_1$ and x_1 is one of $\frac{1}{h}, \frac{2}{h}, \ldots, \frac{h}{h}$, we have that y_1 is one of $0, \ldots, h-1$ but cannot be equal to h. Similarly, if the ith letter of s is not marked, then either $y_{i+1} = y_i + hx_{i+1}$ or $y_{i+1} = y_i - hx_{i+1}$. Since we do not permit x_i to be equal to $\frac{0}{h}$, we cannot have y_{i+1} equal to y_i. Finally, suppose the ith letter of s is marked. By Theorem 4.3, this is equivalent to saying $i+1$ starts a new bin in A. So $\sum_{j=b}^{i+1} x_i > 1$ and $\sum_{j=b}^{i} x_i \leq 1$, where b is the first element in the bin containing i. So either $y_i = h(1 - \sum_{j=b}^{i} x_i) < hx_{i+1} = y_{i+1}$ or $y_i = h(\sum_{j=b}^{i} x_i) > h(1 - x_{i+1}) = y_{i+1}$, and once again the inequalities are strict.

In other words, an integer lattice point in $h \cdot P_s \subset [0,1]^m$ is a sequence of m numbers $(y_1, \ldots, y_m)$ chosen with repetition from the set $\{0, 1, \ldots, h\}$, where $y_1 \neq h$, $y_i > y_{i+1}$ if the ith letter of S is d, and $y_i < y_{i+1}$ is the ith letter of S is a.

It thus suffices to count sequences of numbers in $[h]$ according to their numbers of maximal runs (or peaks and valleys), with the restrictions that no two adjacent numbers are equal, and that the first number in the sequence cannot be h. This enumeration remains an open problem.

4.3. Expectation and Variance

To calculate the expectation and variance for the bin problem, we make use of Corollary 4.8.

Definition 4.6. We define the following random variables. Suppose m is given. Let X_i be the random variable defined on permutations of m, $1 \leq i \leq m-1$, where X_i takes on the value 1 if $\pi(i) > \pi(i+1)$ (the ith letter of $\mathrm{Sh}(\pi)$ is d) and the value 0 if the ith letter of $\mathrm{Sh}(\pi)$ is a. Let Z be a random variable on permutations so that $Z(\pi)$ is the number of bins in the bin arrangement A, where $\psi(P_A) = P_{\mathrm{Sh}(\pi)}$.

First we express Z in terms of $X_1, \ldots, X_{m-1}$.

Proposition 4.12. *For $m \geq 3$,*

$$Z = 1 + (1 - X_1) + \sum_{|i-j|=1} X_i(1 - X_j) \ .$$

Proof. We restate Corollary 4.8 in terms of the random variables X_i.

The random variable $1 - X_1$ corresponds to a_π; it takes on value 1 if the first letter of $\mathrm{Sh}(\pi)$ is a and 0 if it is d. Corresponding to v_π we have

$$X_i(1 - X_{i+1})(\pi) = \begin{cases} 1, & \text{if } \pi(i-1) > \pi(i) < \pi(i+1) \\ 0, & \text{otherwise} \end{cases}$$

and to p_π we have

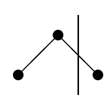
$$(1 - X_i)X_{i+1}(\pi) = \begin{cases} 1, & \text{if } \pi(i-1) < \pi(i) > \pi(i+1) \\ 0, & \text{otherwise} \end{cases} .$$

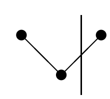
Summing over all the locations where a valley or peak can occur, we get

$$\sum_{i=1}^{m-2} X_i(1 - X_{i+1}) + \sum_{i=1}^{m-2} (1 - X_i)X_{i+1} = \sum_{|i-j|=1} X_i(1 - X_j) ,$$

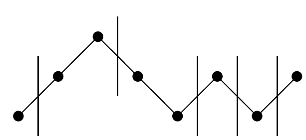
the final summand of Z as needed. □

We draw lines showing where a new bin starts after the crest of a peak and the trough of a valley, as shown below.





For example, consider the (marked) permutation shape $\dot{a}a\dot{d}d\dot{a}\dot{d}\dot{a}$. This shape counts 6 bins: the automatic one to start with, one due to the initial ascent, two peaks, and two valleys.



To calculate the expected number of bins under NEXT FIT we need to find $E(Z)$, the expectation of Z.

Theorem 4.13. *Given $m \geq 2$ objects, the expected number of bins they use under* NEXT FIT *is $\frac{2m}{3} + \frac{1}{6}$. (If $m = 1$, the expected number is 1.)*

Proof. As noted in Section 2.2, we find the expectation of a random variable by multiplying a value it can take on by the probability it does so, and summing over all values it can take on. Each summand of Z can only take on the values 0 and 1, so we need only figure out the probability that each summand takes on the value 1.

If $m = 2$, then the permutation 1 2 gives 2 bins and 2 1 gives 1 bin, for an expected value of $\frac{3}{2}$. Clearly if $m = 1$ the expected value is 1.

Now assume $m \geq 3$. We had $Z = 1 + (1 - X_1) + \sum_{|i-j|=1} X_i(1 - X_j)$. The probability that 1 takes on the value 1 is 1, and the probability that $1 - X_1$ takes on the value 1 is $\frac{1}{2}$ (half the permutations start in an ascent). So these two summands contribute $\frac{3}{2}$ to $E(Z)$. Now consider

$$\sum_{|i-j|=1} X_i(1 - X_j) = \sum_{|i-j|=1} X_i - \sum_{|i-j|=1} X_i X_j \ .$$

The probability that X_i is 1 is $\frac{1}{2}$ (denote by $P(X_i = 1) = \frac{1}{2}$). If $|i - j| = 1$, $P(X_i X_j = 1) = \frac{1}{6}$. We can see this by rewriting the requirement as follows. Assume for example that $i = j + 1$. Then $X_i X_j(\pi) = 1$ means that $\pi(i-1) > \pi(i) > \pi(i+1)$, which occurs with probability $\frac{1}{6}$. There are $2(m - 2)$ pairs (i, j) with $|i - j| = 1$, so $\sum_{|i-j|=1} X_i$ contributes $\frac{2(m-2)}{2}$ to $E(Z)$ and $-\sum_{|i-j|=1} X_i X_j$ contributes $-\frac{2(m-2)}{6}$. Summing all the contributions, we get $E(Z) = \frac{3}{2} + m - 2 - \frac{m-2}{3} = \frac{2m}{3} + \frac{1}{6}$. □

Next we consider the variance of Z. As discussed in Section 2.2, the variance of Z is equal to $E(Z^2) - E(Z)^2$. By using Proposition 4.12, we get an expression for Z^2 in terms of the X_i:

$$Z^2 = 2Z - 2X_1 + X_1^2 - 2X_1 \sum_{|i-j|=1} X_i(1 - X_j) + 2 \sum_{|i-j|=1} X_i(1 - X_j) + \sum_{\substack{|i-j|=1 \\ |k-l|=1}} X_i(1 - X_j) X_k(1 - X_l) \ . \quad (9)$$

We use Equation (9) to calculate $E(Z^2)$. We assume $m \geq 4$. Note that in the process of calculating $E(Z)$ in Theorem 4.13, we already calculated $E(\sum_{|i-j|=1} X_i(1-X_j))$. Hence only the expectations of the first and third summands of Equation (9) will require additional calculation.

Lemma 4.14. *The expectation* $E(X_1 \sum_{|i-j|=1} X_i(1-X_j))$ *is equal to* $(m-2)/3$.

Proof. We analyze the probability that $X_1 \sum_{|i-j|=1} X_i(1-X_j)$ takes on the value 1 for various values of i and j. Note that the presence of X_1, X_i, and $1-X_j$ means that there are descents in the first and ith positions (the first and ith letters of a shape must be d), and an ascent in the jth position (the jth letter of the shape must be a). We calculate the probability that a permutation will have these shapes, as follows:

$$P\{X_1X_i(1-X_j)=1\} = \begin{cases} 1/3, & \text{if } i=1, j=2 \text{ (Sh}(\pi)\text{ starts with } da); \\ 0, & \text{if } i=2, j=1; \\ 1/8, & \text{if } i=2, j=3 \text{ (Sh}(\pi)\text{ starts with } dda); \\ 5/24, & \text{if } i=3, j=2 \text{ (Sh}(\pi)\text{ starts with } dad); \\ 1/6, & \text{otherwise (Sh}(\pi)\text{ starts with } d\cdots ad \text{ or } d\cdots da); \end{cases}$$

For example, in the case where $i=2, j=3$, we are concerned with shapes of length $m-1$ whose first three letters are dda. Since subsequent letters do not affect the result, we need only count the number of permutations of 4 with shape dda. There are three such permutations: 4 3 1 2, 4 2 1 3, and 3 2 1 4, so the probability that a permutation's shape will start with dda is $\frac{3}{24} = \frac{1}{8}$.

In the final case, the first shape $d \cdots ad$ results when $i = j+1$ and $j \geq 3$. The peak in the ith position occurs with probability 1/3, and the initial descent with probability 1/2. Since they are sufficiently separated, these are independent of each other, for a total probability of 1/6. Since j can be any of $3, \ldots, m-2$, there are $m-4$ such choices of i and j, so such shapes contribute a total of $(m-4)/6$ to the expectation. Similarly, $d \cdots da$ occurs when $j = i+1$ and $i \geq 3$. Once again, there are $m-4$ such choices of i and j, each with probability 1/6, for a total contribution of $(m-4)/6$.

Note that there is only one choice of i and j that yields each shape in the first three (non-trivial) cases, so they contribute 1/3, 1/8, and 5/24 respectively. Adding this all up gives the desired result. □

Lemma 4.15. *The expectation of the third summand of Equation (9) is as follows:*

$$E\Big(\sum_{\substack{|i-j|=1\\|k-l|=1}} X_i(1-X_j)X_k(1-X_l)\Big) = \frac{4m^2}{9} - \frac{8m}{5} + \frac{131}{90}\ .$$

Proof. Table 4.1 lists the possibilities for what $X_i(1-X_j)X_k(1-X_l)$ can look like (depending on the arrangement of i, j, k and l), together with the corresponding shape of the permutation in those places, the probability a permutation has that shape (denoted in the table by P), and the number of such i, j, k, l so arranged. For example, the first shape occurs when $a = i = k = j-1 = l-1$, where i can be any of $1, \ldots, m-2$. The third, fourth, and seventh listings contain $X_{a+1}(1-X_{a+1})$, hence can never occur as permutations (you cannot have both a descent and an

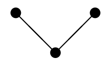
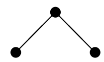
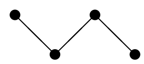
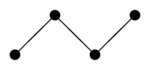
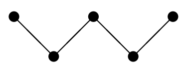
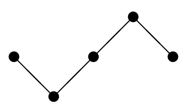
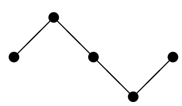
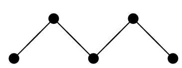
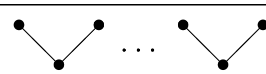
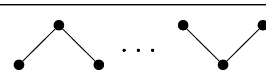
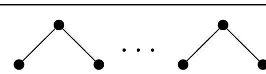
summand of Z^2	picture of shape (if applicable)	P	# of such summands in Z^2
$X_a(1-X_{a+1})X_a(1-X_{a+1})$		$1/3$	$m-2$
$(1-X_a)X_{a+1}(1-X_a)X_{a+1}$		$1/3$	$m-2$
$X_a(1-X_{a+1})(1-X_a)X_{a+1}$		0	$2(m-2)$
$X_a(1-X_{a+1})X_{a+1}(1-X_{a+2})$		0	$2(m-3)$
$X_a(1-X_{a+1})(1-X_{a+1})X_{a+2}$		$5/24$	$2(m-3)$
$(1-X_a)X_{a+1}X_{a+1}(1-X_{a+2})$		$5/24$	$2(m-3)$
$(1-X_a)X_{a+1}(1-X_{a+1})X_{a+2}$		0	$2(m-3)$
$X_a(1-X_{a+1})X_{a+2}(1-X_{a+3})$		$16/120$	$2(m-4)$
$X_a(1-X_{a+1})(1-X_{a+2})X_{a+3}$		$11/120$	$2(m-4)$
$(1-X_a)X_{a+1}X_{a+2}(1-X_{a+3})$		$11/120$	$2(m-4)$
$(1-X_a)X_{a+1}(1-X_{a+2})X_{a+3}$		$16/120$	$2(m-4)$
$X_a(1-X_{a+1})X_b(1-X_{b+1})$		$1/9$	$m^2-9m+20$
$X_a(1-X_{a+1})(1-X_b)X_{b+1}$		$1/9$	$m^2-9m+20$
$(1-X_a)X_{a+1}X_b(1-X_{b+1})$		$1/9$	$m^2-9m+20$
$(1-X_a)X_{a+1}(1-X_b)X_{b+1}$		$1/9$	$m^2-9m+20$

TABLE 4.1. Calculations for determining the expectation of the summand $\sum_{\substack{|i-j|=1\\|k-l|=1}} X_i(1-X_j)X_k(1-X_l)$ in Z^2

ascent in the $a+1$st place!) and have no corresponding shape. The third shape occurs either when $l=j=i+1=k-1$ and $i=a$ is any of $1,\ldots,m-3$, or when $l=j=i-1=k+1$ and $i=a+2$ is any of $3,\ldots,m-1$. (For simplicity's sake, we relabel in the table so that a is the minimum of the four subscripts.)

In the last four entries, we assume that $a-b\geq 3$. This occurs when i,j,k,l are not all in sequence: there is a gap between the two pairs. For example, the second to last picture occurs either when $i=j+1$, $k+1=l$, and $k-i\geq 2$ (so $a=j$ and $b=k$) or when $k=l+1$, $i+1=j$, and $i-k\geq 2$ (so $a=l$ and $b=i$). We can choose a and b from $1,\ldots,m-2$, for a total of $\binom{m-2}{2}$. However, we have the condition that $a-b\geq 3$, so we must subtract off the $m-3$ possibilities for $a-b=1$ and the $m-4$ possibilities for $a-b=2$. Combining all of this, we get $\frac{1}{2}(m^2-9m+20)$. Since there are two possible assignments of a and b in terms of the subscripts i,j,k,l we get a total of $m^2-9m+20$ summands of that shape. The other three shapes are similar.

As in the proof of Lemma 4.14, given the shape of a permutation one merely counts the number of permutations with the given shape to determine the probability. Multiplying the probability by the number of summands of each shape and adding gives the desired result. □

We collect these calculations together in the following.

Lemma 4.16. *For $m\geq 4$,*

$$E(Z^2)=\frac{4m^2}{9}+\frac{2m}{5}-\frac{2}{45}\ .$$

Proof. We examine the expectation of each term of Equation (9). Theorem 4.13 gives that the expectation of the first term $E(2Z)$ is $\frac{4m}{3} + \frac{1}{3}$. Clearly the expectation $E(-2X_1)$ is -1 and the expectation $E(X_1^2)$ is $\frac{1}{2}$. Lemma 4.14 finds the expectation of the following term. The proof of Theorem 4.13 shows the expectation of the next term is $\frac{4(m-2)}{3}$. Finally, Lemma 4.15 gives the expectation of the last term. Adding these all together gives the result. □

Theorem 4.17. *Given m objects, the variance of the number of bins they use under* NEXT FIT *is*

$$V(Z) = E(Z^2) - E(Z)^2 = \begin{cases} 8m/45 - 13/180, & \text{if } m \geq 4; \\ 17/36, & \text{if } m = 3; \\ 1/4, & \text{if } m = 2; \\ 0, & \text{if } m = 1. \end{cases}$$

Proof. For $m \geq 4$, we use Theorem 4.13 and Lemma 4.16:

$$\begin{aligned} E(Z^2) - E(Z)^2 &= \frac{4m^2}{9} + \frac{2m}{5} - \frac{2}{45} - \left(\frac{2m}{3} + \frac{1}{6}\right)^2 \\ &= \frac{8m}{45} - \frac{13}{180} . \end{aligned}$$

For $m = 3$, we look at how $E(Z^2)$ changes. The expectations of the first three terms in Equation (9) are unaffected. The first summand becomes $-2[X_1^2(1 - X_2) + X_1X_2(1 - X_1)]$, for a contribution to the expectation of $\frac{-2}{3}$. The second summand becomes $2[X_1(1 - X_2) + X_2(1 - X_1)]$, thus adding $\frac{4}{3}$ to the expectation. For the third summand, if we examine table 4.1, only the first two lines are consistent with $m = 3$, so the contribution is $\frac{2}{3}$. Thus $E(Z^2) = \frac{31}{6}$ and $V(Z) = \frac{31}{6} - (\frac{13}{6})^2 = \frac{17}{36}$.

For $m = 2$, the first, second, and third summands of Equation (9) contribute nothing, yielding $V(Z) = 1/4$. Finally, for $m = 1$, the only permutation of 1 gives one bin, which is the expectation. Hence the variance is 0. □

Hofri [**9, 10**] obtains the same expectation and variance using much different methods. Note that we did not need the generating function in order to calculate these, just the transformation ϕ.

4.4. The Generating Function

We use Corollary 4.7 to calculate the generating function for the bin capacity 1 continuous case.

In [**8**, pp. 231–239], Goulden and Jackson provide a method of counting sequences based on their numbers of "maximal strings". In particular, we will consider just those sequences that are permutations, and take the strings to be increasing runs. We will state and apply Goulden and Jackson's result in this special case.

Definition 4.7. Let f_i be the weight of a maximal (non-final) increasing run of length i, and g_i the weight of a final maximal increasing run of length i. We write $F(x) = 1 + f_1 x + f_2 x^2 + \cdots$ and $G(x) = g_1 x + g_2 x^2 + \cdots$.

Theorem 4.18 (Goulden and Jackson). *Let* $\sum_n u_n x^n = F(x)^{-1}$ *and* $\sum_n v_n x^n = G(x)F(x)^{-1}$. *Then the number of permutations of* m *with* i_k *maximal increasing runs of length* k *is*

$$\left[\frac{z^m}{m!} f_1^{i_1} f_2^{i_2} \cdots\right] \left(\sum_n u_n \frac{z^n}{n!}\right)^{-1}$$

and the number with i_k non-final maximal increasing runs of length k and a final maximal increasing run of length j is

$$\left[\frac{z^m}{m!} g_j f_1^{i_1} f_2^{i_2} \cdots\right] \left(\sum_n u_n \frac{z^n}{n!}\right)^{-1} \left(\sum_n v_n \frac{z^n}{n!}\right)$$

where $[y^i]\ H(y)$ denotes the coefficient of y^i in $H(y)$.

Corollary 4.19. *There are*

$$\left[\frac{z^m}{m!} f^i\right] \quad \frac{\dfrac{\cosh(z\sqrt{1-f}) - f}{1-f} - \dfrac{\sinh(z\sqrt{1-f})}{\sqrt{1-f}}}{\cosh(z\sqrt{1-f}) - \dfrac{\sinh(z\sqrt{1-f})}{\sqrt{1-f}}}$$

permutations of $[m]$ that count $2i$ bins and

$$\left[\frac{z^m}{m!} f^i\right] \quad \frac{\dfrac{\sinh(z\sqrt{1-f})}{\sqrt{1-f}} - \dfrac{\cosh(z\sqrt{1-f}) - 1}{1-f}}{\cosh(z\sqrt{1-f}) - \dfrac{\sinh(z\sqrt{1-f})}{\sqrt{1-f}}}$$

permutations of $[m]$ that count $2i+1$ bins.

Proof of Corollary 4.19. Following Corollary 4.7, we consider the cases of permutations ending in ascents and descents separately.

First note that we only want to count increasing runs of length at least 2, but that we don't need to keep track of the exact length. Hence in Definition 4.7 we can take $f_1 = 1$ and $f_2 = f_3 = \cdots = f$, so

$$F(x) = 1 + x + \frac{fx^2}{1-x} = \frac{1 - x^2(1-f)}{1-x}$$

and

$$F^{-1}(x) = \frac{1-x}{1-x^2(1-f)}$$

$$= (1-x)\sum_{i=0}^{\infty} x^{2i}(1-f)^i$$
$$= \sum_{i\geq 0} x^{2i}\ (1-f)^i - \sum_{i\geq 0} x^{2i+1}\ (1-f)^i$$
$$= \sum_n u_n x^n\ .$$

So

$$\left(\sum_n u_n \frac{z^n}{n!}\right)^{-1} = \left(\sum_{i\geq 0} \frac{z^{2i}}{(2i)!}\ (1-f)^i - \sum_{i\geq 0} \frac{z^{2i+1}}{(2i+1)!}\ (1-f)^i\right)^{-1}$$
$$= \left(\cosh(z\sqrt{1-f}) - \frac{\sinh(z\sqrt{1-f})}{\sqrt{1-f}}\right)^{-1}\ . \qquad (10)$$

Now consider only those permutations which end in a descent. Then the length of the final increasing run is 1 (a descent is just an increasing run of length 1). We take $g_1 = g$ and $0 = g_2 = g_3 = \cdots$ in Definition 4.7, so $G(x) = gx$. Then

$$G(x)F^{-1}(x) = g\left(\sum_{i\geq 0} x^{2i+1}\ (1-f)^i - \sum_{i\geq 0} x^{2i+2}\ (1-f)^i\right)$$
$$= \sum_n v_n x^n$$

and

$$\sum_n v_n \frac{z^n}{n!} = g\sum_{i\geq 0} \frac{z^{2i+1}}{(2i+1)!}\ (1-f)^i - g\sum_{i\geq 0} \frac{z^{2i+2}}{(2i+2)!}\ (1-f)^i$$
$$= g\frac{\sinh(z\sqrt{1-f})}{\sqrt{1-f}} - g\frac{\cosh(z\sqrt{1-f})-1}{1-f}\ .$$

Since the coefficient of gy^i in $gH(y)$ is the same as the coefficient of y^i in $H(y)$, the number of permutations of m counting $2i+1$ bins is

$$\left[\frac{z^m}{m!}f^i\right] \quad \frac{\dfrac{\sinh(z\sqrt{1-f})}{\sqrt{1-f}} - \dfrac{\cosh(z\sqrt{1-f})-1}{1-f}}{\cosh(z\sqrt{1-f}) - \dfrac{\sinh(z\sqrt{1-f})}{\sqrt{1-f}}}. \tag{11}$$

Realizing that the number of permutations ending in an ascent is the total number minus those ending in a descent, the number of permutations of $[m]$ counting $2i$ bins is the coefficient of $\frac{z^m}{m!}f^i$ in equation (10) minus equation (11). This is just

$$\left[\frac{z^m}{m!}f^i\right] \quad \frac{\dfrac{\cosh(z\sqrt{1-f})-f}{1-f} - \dfrac{\sinh(z\sqrt{1-f})}{\sqrt{1-f}}}{\cosh(z\sqrt{1-f}) - \dfrac{\sinh(z\sqrt{1-f})}{\sqrt{1-f}}}$$

as needed. □

Theorem 4.20. *The generating function for the ordinary bin capacity 1 case is*

$$\frac{1-y^2+y\sqrt{1-y^2}\,\sinh(x\sqrt{1-y^2})}{(1+y)(1-y\cosh(x\sqrt{1-y^2}))}$$

where $[x^m y^n]$ *is the probability that a sequence of* m *objects can fit in* n *bins.*

Remark 4.6. This differs from the generating function that Hofri derives in [**10**] by $1-y$. In his base case, an empty sequence needs a bin, while we consider the empty sequence not to need one.

Proof. We merely combine the two parts of Corollary 4.19 appropriately. In the first expression, we replace f with y^2. In the second, we again replace f with y^2, and then multiply the whole expression by y. We then just add the expressions together and simplify. In keeping with the notation for the dual bin case, we replace z with x so that in order to get the volume we consider $[x^m y^n]$. □

4.5. Barred Permutations and Another Derivation of the Generating Function

In this section, we present a more combinatorial derivation of the bin generating function. The calculation will require the following decomposition of a permutation into disjoint alternating decreasing and increasing runs.

Proposition 4.21. *Any permutation of* $[2m]$ *can be split uniquely into a sequence of disjoint alternating decreasing and increasing runs, each of maximal even (possibly zero) length. Any permutation of* $[2m+1]$ *can be similarly split into alternating increasing and decreasing runs of maximal even length following an initial decreasing run of maximal odd length.*

For example, consider $\sigma_1 = 4\,3\,2\,1\,7\,6\,5\,8\,9\,10$. This splits into the decreasing run $4\,3\,2\,1$, an increasing run of length zero, the decreasing run $7\,6$, and the increasing run $5\,8\,9\,10$. If we take $\sigma_2 = 4\,3\,2\,1\,7\,6\,5\,8\,9$, we split it into runs $4\,3\,2$, $1\,7$, $6\,5$ and $8\,9$. We split $\sigma_3 = 4\,5\,6\,7\,3\,2\,1$ into 4 (a decreasing run of length 1), $5\,6$ and $7\,3\,2\,1$.

Proof. Let σ be a permutation of $[2m]$. We will mark $\mathrm{Sh}(\sigma)$ by drawing for each i a line through the $2i$th letter of $\mathrm{Sh}(\sigma)$ unless the $2i-1$st, $2i$th, and $2i+1$st letters are all the same.

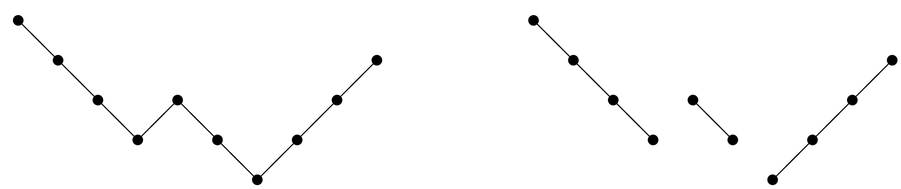
This results in dividing $\mathrm{Sh}(\sigma)$ into subwords of the form a^j and d^j (where j is odd), separated by $\not a$'s and $\not d$'s. Under this division, a subword a^j consisting of the $k+1$st through $k+j$th letters of $\mathrm{Sh}(\sigma)$ corresponds to the increasing run $\sigma(k+1) < \sigma(k+2) < \cdots < \sigma(k+j+1)$. Similarly, a subword d^j consisting of the $k+1$st through $k+j$th letters of $\mathrm{Sh}(\sigma)$ corresponds to the decreasing run $\sigma(k+1) > \sigma(k+2) > \cdots > \sigma(k+j+1)$.

If two consecutive separated subwords are both of the form a^j, we say there is a decreasing sequence of length zero between them; if they are both of the form d^j, we say there is an intervening increasing sequence of length zero. If $\mathrm{Sh}(\sigma)$ starts with an a, we say there is an initial decreasing run of length zero.

The construction for permutations of odd length is similar. Let τ be a permutation of $[2m+1]$. We mark $\mathrm{Sh}(\tau)$ by drawing a line through the $2i+1$th letter of $\mathrm{Sh}(\tau)$ (for $i \geq 1$) unless the $2i$th, $2i+1$st and $2i+2$nd letters are all the same. We also draw a line through the first letter, unless both it and the second letter are d's. Similar to before, this results in subwords of the form d^j and a^j separated by $\not{d}$'s and $\not{a}$'s. The first subword has an even exponent, the rest odd exponents. If the first letter of the marked $\mathrm{Sh}(\tau)$ is $\not{a}$ or $\not{d}$, the exponent of the first subword is zero and the initial decreasing run of $\tau(1)$ has length 1. Otherwise, we form the increasing and decreasing sequences as we did before. □

Remark 4.7. Note that the 'maximality' does not refer to the runs being as long as possible, merely to being as long as possible under the condition that all runs are even length. Take for example a permutation σ whose shape is *daaad*. It certainly has an increasing run of length 4: $\sigma(2) < \sigma(3) < \sigma(4) < \sigma(5)$. However, in order to insure all runs are of even length, the algorithm above results in both $\sigma(2)$ and $\sigma(5)$ being part of decreasing runs.

To illustrate this construction, consider $\sigma_1 = 4\,3\,2\,1\,7\,6\,5\,8\,9\,10$ from above. We mark $\mathrm{Sh}(\sigma_1) = dddaddaaa$ as $ddd\not{a}d\not{d}aaa$. The first subword d^3 means that the first decreasing run is $\sigma_1(1)\,\sigma_1(2)\,\sigma_1(3)\,\sigma_1(4) = 4\,3\,2\,1$. Since the next subword is d, there is an intervening empty increasing run followed by the decreasing run $\sigma_1(5)\,\sigma_1(6) = 7\,6$. Finally the last subword a^3 results in the increasing run $5\,8\,9\,10$. We illustrate this by leaving blank the northeast step for the $\not{a}$ and the southeast step for the $\not{d}$ as follows.



For $\sigma_2 = 4\,3\,2\,1\,7\,6\,5\,8\,9$, we write $\mathrm{Sh}(\sigma_2) = dddaddaa$ as $dd\not{d}a\not{d}d\not{a}a$.

For $\sigma_3 = 4\,5\,6\,7\,3\,2\,1$ we have $aaaddd$ and hence $\not{a}a\not{a}ddd$.

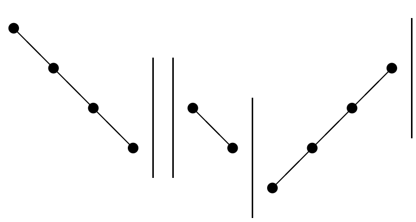
Definition 4.8. A barred permutation of m letters is a permutation which can have lines (or bars) drawn in the $m+1$ spaces between, before, and after the letters.

For example, $||3\,4|2\,5|||1$ and $3\,4\,2||||5\,1||$ are both barred permutations of the set $[5]$ with 6 bars. For more on barred permutations, see [**11**] and [**7**].

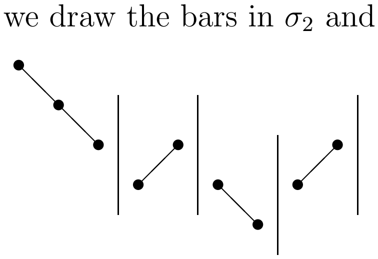
We will be counting barred permutations that have additional restrictions on where the bars may appear.

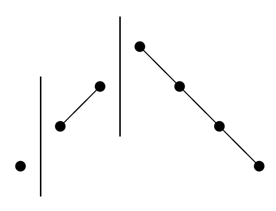
Definition 4.9. Given a permutation of $[m]$, we decompose it into alternating decreasing and increasing runs as in Proposition 4.21. We then place a bar after each run, in the space described by the $a\!\!|$ or $d\!\!|$ that marks the end of that run's corresponding subword. (An increasing run of length zero has its bar in the $a\!\!|$ that splits the surrounding d^i subwords, and an empty decreasing run's bar is in the $d\!\!|$ splitting the surrounding a^i subwords. The bar for the last run is in the mth space.) We define $P_{m,n}$ to be the number of permutations of $[m]$ that have n such bars, and we write $P_m(y) = \sum_n P_{m,n} y^n$.

In our earlier examples, σ_1 would have two bars in the fourth place (one for the initial decreasing run described by d^3, and one for the empty increasing run), a bar in the sixth place, and a bar at the end in the tenth place. We write it $4\,3\,2\,1||7\,6|5\,8\,9\,10|$ and draw it as shown below.

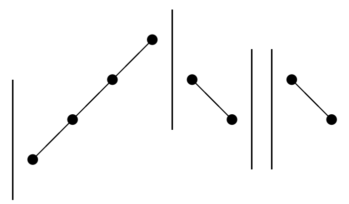


Similarly, we draw the bars in σ_2 and σ_3 as follows.





If a permutation of $[2m]$ starts with an ascent, there will be a bar in the 0th position, corresponding to the initial empty decreasing run. We show the placement of the bars for $\sigma_4 = 3\,4\,5\,8\,7\,1\,6\,2$.



Remark 4.8. We note that for m even, the bars are in the even spaces, and that for m odd they are in the odd spaces.

Proposition 4.22. *The number of permutations of* $[m]$ *whose shape corresponds to a bin arrangement with* n *bins is* $P_{m,n}$.

Proof. Each maximal increasing or decreasing run has exactly one bar associated with it. If the run is of length at least 3, then at least part of it will form one of the runs described in Proposition 4.21, and hence is guaranteed a bar. If the run has length 2 (in other words, it is described by a single a or d in the shape) then either it forms a run in Proposition 4.21, or it becomes an $a\!\!\mid$ surrounded by d's or a $d\!\!\mid$ surrounded by a's. In either case, the construction gives it a bar. Finally, an initial ascent (the first decreasing run is of length 1) gets a bar as well: in the 0th place for m even and in the first place for m odd.

From Corollary 4.6, since we assume the first run is a decreasing one, this is the number of bins. □

Proposition 4.23. *We have*

$$(1+y)\sum_{m=0}^{\infty}\frac{P_{2m}(y)}{(1-y^2)^{m+1}}\ \frac{x^{2m}}{(2m)!}=\frac{1}{1-y\cosh x} \tag{12}$$

and

$$(1+y)\sum_{m=0}^{\infty}\frac{P_{2m+1}(y)}{(1-y^2)^{m+1}}\ \frac{x^{2m+1}}{(2m+1)!}=\frac{y\sinh x}{1-y\cosh x}\ . \tag{13}$$

Proof. For each equation, the left and right sides describe different ways of constructing certain barred permutations. We will let a barred permutation of $[m]$ with n bars have the weight $y^n\frac{x^m}{m!}$.

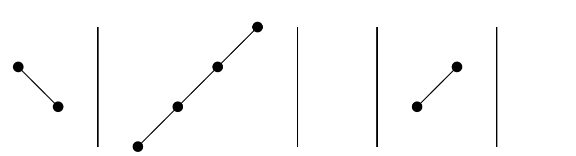
The right side of equation (12) counts alternating even-length labeled decreasing and increasing sequences, with a bar after each one. We label as follows. Given sequences of length $i_1,\dots,i_n$ (for non-negative even integers i_j), a labeling is a permutation $\sigma\in S_{i_1+\cdots+i_n}$ such that for even j

$$\sigma(i_1+\cdots+i_{j-1}+1)<\sigma(i_1+\cdots+i_{j-1}+2)<\cdots<\sigma(i_1+\cdots+i_{j-1}+i_j)$$

and for odd j

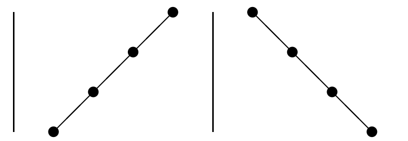
$$\sigma(i_1+\cdots+i_{j-1}+1)>\sigma(i_1+\cdots+i_{j-1}+2)>\cdots>\sigma(i_1+\cdots+i_{j-1}+i_j)\ .$$

We will illustrate these alternating barred sequences as shapes of the form d^i and a^i. For example, we may have sequences of lengths 2, 4, 0, 2, and 0, as in the picture below.



Two sample labelings are the permutations 3 2 1 4 5 7 6 8 and 2 1 3 4 5 6 7 8.

Sequences of lengths 0, 4, and 4 would result in the following picture.



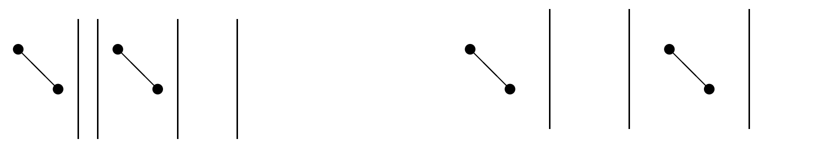
(Whether there is an ascent or descent between any two sequences depends on the precise choice of labeling. No assumption is made on this in the above pictures.) All sequences are of even length, each marked by a bar. If the sum of the lengths of the sequences is m, the labels will be drawn without repetition from $[m]$. This results in a factor of $(1 - y\cosh x)^{-1}$.

For the right side of equation (13), we start by drawing an odd-length decreasing sequence followed by a bar, and then continue alternating even-length increasing and decreasing sequences, once again placing a bar after each sequence. We label similarly to before, which gives us $(y \sinh z)(1 - y\cosh x)^{-1}$.

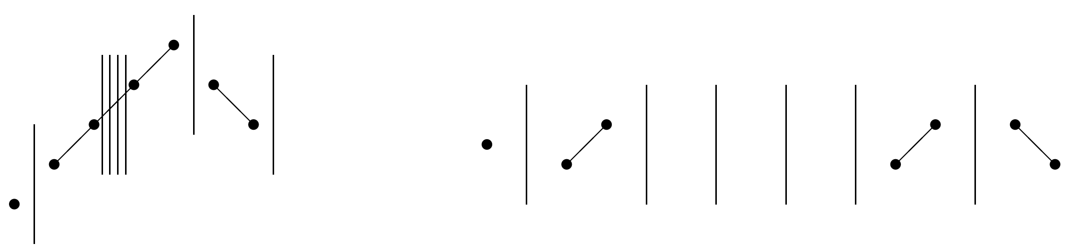
On the left side of both equation (12) and (13), the barred permutations discussed in Definition 4.9 give the factors $P_{2m}(y)$ and $P_{2m+1}(y)$ respectively. We then permit any number of additional optional pairs of bars in each of the $m+1$ even spaces for equation (12), and in each of the $m+1$ odd spaces for equation (13), for a factor of $(1-y^2)^{m+1}$. Finally, in each expression we wish to allow any number of bars, not just an odd number, at the end of a permutation. Hence we multiply each summation by a factor of $1 + y$. □

This analysis is similar to one in [**6**, ch. 7].

For example, consider the barred permutation $2\,1||4\,3||$. Since Definition 4.9 places bars in $2\,1\,4\,3$ to form $2\,1||4\,3|$, we obtain $2\,1||4\,3||$ on the left side of (12) by inserting an additional bar at the end. On the right, we get it from choosing the labels appropriately for the sequences of lengths 2, 0, 2, and 0.



Similarly, Definition 4.9 places bars in $1\,2\,3\,4\,7\,6\,5$ to form $1|2\,3\,4\,7|6\,5|$. We get $1|2\,3||||4\,7|6\,5|$ from the left side of (13) by inserting an extra two pairs of bars in the 3rd space. On the right side we take sequences of lengths 1, 2, 0, 0, 0, 2, and 2 with the appropriate labeling.



Theorem 4.24. *We have*

$$\sum P_{m,n} y^n \frac{x^m}{m!} = \frac{1 - y^2 + \sqrt{1-y^2}\; y \sinh(x\sqrt{1-y^2})}{(1+y)(1 - y\cosh(x\sqrt{1-y^2}))} \;.$$

Proof. We replace x with $x\sqrt{1-y^2}$, multiply equation (12) by $(1-y^2)(1+y)^{-1}$ and (13) by $\sqrt{1-y^2}(1+y)^{-1}$, and add. □

Since Proposition 4.22 tells us that $P_{m,n}$ "counts" bin arrangements with m elements and n bins, this provides another proof of Theorem 4.20.

We can use a similar method to more combinatorially derive the generating functions of Corollary 4.19 as well. We sketch the procedure below.

Definition 4.10. Given a permutation of $[m]$, we decompose it into alternating decreasing and increasing runs as in Proposition 4.21. We then place a bar only after each increasing run, in the $\not{a}$ or $\not{d}$ that marks the end of its subword. (If the increasing run has length zero, we place the bar in the $\not{a}$ that splits the surrounding d^i subwords.) If $\mathrm{Sh}(\sigma)$ ends in an a, we place the bar in the mth space, the one at the very end of the word. We define $R_{m,i}$ be the number of permutations of $[m]$ with i such bars. We take $R^a_{m,i}$ to be the number of those that end in an ascent, and $R^d_{m,i}$ to be the number that end in a descent. We write $R_m = \sum_i R_{m,i} f^i$, $R^a_m = \sum_i R^a_{m,i} f^i$, and $R^d_m = \sum_i R^d_{m,i} f^i$.

In other words, we form our barred permutations as we did before except we omit half the bars, starting with the first and alternating.

Proposition 4.25. *The number of permutations of $[m]$ with i maximal increasing runs of length at least 2 is $R_{m,i}$.*

Hence in order to calculate the generating functions in Corollary 4.19 we need to find $\sum R^d_m(f) z^m/m!$ and $\sum R^a_m(f) z^m/m!$. As we did in Proposition 4.23 we split each into even and odd parts. It is useful to first do so for $\sum R_m(f) z^m/m!$.

Proposition 4.26 (Gessel [**6**])**.** *We have*

$$\sum_{m=0}^{\infty} \frac{R_{2m}(f)}{(1-f)^{m+1}} \frac{z^{2m}}{(2m)!} = \frac{\cosh z}{1 - f\cosh^2 z} \tag{14}$$

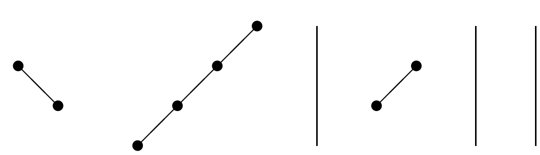
and

$$\sum_{m=0}^{\infty} \frac{R_{2m+1}(f)}{(1-f)^{m+1}} \frac{z^{2m+1}}{(2m+1)!} = \frac{\sinh z}{1 - f\cosh^2 z} \ . \tag{15}$$

Proof. For each equation, the left and right sides describe different ways of constructing the same set.

Similar to before, on the left side of equations (14) and (15), the barred permutations of Definition 4.10 give the factors of $R_{2m}(f)$ and $R_{2m+1}(f)$. We then permit any number of additional optional bars in the $m+1$ even spaces of a permutation of $[2m]$ and $m+1$ odd spaces of a permutation of $[2m+1]$, for a factor of $(1-f)^{-(m+1)}$ in each case.

For the right side of equation (14), we start by writing pairs of even-length (labeled) decreasing and increasing sequences, with a bar after the increasing one. For example, one pair may be of lengths 2 and 4, the next of lengths 0 and 2, and the last of lengths 0 and 0, forming the picture below.



(Whether there is an ascent or descent between any two sequences depends on the precise choice of labeling. No assumption is made on this in the picture above.) All sequences are of even length, but only one of each pair has a bar. This results in a factor of $(1 - f\cosh^2 z)^{-1}$. Finally, we permit a decreasing even-length sequence (with no bar) at the end, for the factor of $\cosh z$.

For the right side of equation (15), we start by drawing an odd-length decreasing sequence for a factor of $\sinh z$. We then continue with pairs of even-length increasing and decreasing sequences, once again placing a bar after the increasing sequence. This time however we write the increasing sequence first in the pair. Once again this gives the factor of $(1 - f\cosh^2 z)^{-1}$.

Each case is just another way to count these particular barred permutations according to their length and number of bars. □

Proposition 4.27. *We have the following.*

$$\sum_{m=0}^{\infty} \frac{R^d_{2m}(f)}{(1-f)^m} \frac{z^{2m}}{(2m)!} = \frac{\cosh z - 1}{1 - f\cosh^2 z} \tag{16}$$

$$\sum_{m=0}^{\infty} \frac{R^d_{2m+1}(f)}{(1-f)^m} \frac{z^{2m+1}}{(2m+1)!} = \sinh z + \frac{\sinh z(\cosh z - 1) f \cosh z}{1 - f\cosh^2 z} \tag{17}$$

$$\sum_{m=0}^{\infty} \frac{R^a_{2m}(f)}{(1-f)^m} \frac{z^{2m}}{(2m)!} = 1 + \frac{(\cosh z - 1) f \cosh z}{1 - f\cosh^2 z} \tag{18}$$

$$\sum_{m=0}^{\infty} \frac{R^a_{2m+1}(f)}{(1-f)^m} \frac{z^{2m+1}}{(2m+1)!} = \frac{f \sinh z(\cosh z - 1)}{1 - f\cosh^2 z} \tag{19}$$

Proof. As the arguments are all similar to those in the proofs of Propositions 4.23 and 4.26, we will only give the proof of equation (14).

On the left, the situation is the same as in equation (14), save that we no longer permit any additional bars to be placed in the last spot. Since the permutations we are concerned with here all end with an ascent, this means that each permutation ends in exactly one bar.

On the right, the 1 takes care of the base case (we say the empty permutation, counted by $R_{0,0}$ is counted by $R^a_{0,0}$ and not $R^d_{0,0}$). Otherwise, we construct our even-length alternating decreasing and barred increasing sequences, as in Proposition 4.26. We end by requiring the final pair to have a non-empty increasing sequence (with a bar at the end), for an additional factor of $f(\cosh z - 1)\cosh z$. □

Corollary 4.28. *We have the following.*

$$\sum_{m=0}^{\infty} R^d_{2m}(f)\ \frac{z^{2m}}{(2m)!} = \frac{\cosh(z\sqrt{1-f}) - 1}{1 - f\cosh^2(z\sqrt{1-f})} \tag{20}$$

$$\sum_{m=0}^{\infty} R^d_{2m+1}(f)\ \frac{z^{2m+1}}{(2m+1)!} = \frac{\sinh(z\sqrt{1-f})\left[1 - f\cosh(z\sqrt{1-f})\right]}{\sqrt{1-f}\left(1 - f\cosh^2(z\sqrt{1-f})\right)} \tag{21}$$

$$\sum_{m=0}^{\infty} R^a_{2m}(f)\ \frac{z^{2m}}{(2m)!} = \frac{1 - f\cosh(z\sqrt{1-f})}{1 - f\cosh^2(z\sqrt{1-f})} \tag{22}$$

$$\sum_{m=0}^{\infty} R^a_{2m+1}(f)\ \frac{z^{2m+1}}{(2m+1)!} = \frac{f\sinh(z\sqrt{1-f})\left[\cosh(z\sqrt{1-f}) - 1\right]}{\sqrt{1-f}\left(1 - f\cosh^2(z\sqrt{1-f})\right)} \tag{23}$$

Proof. To get from (16) to (20) we merely replace z with $z\sqrt{1-f}$. To get from (17) to (21) we use the identity

$$1 + \frac{(\cosh z - 1)f\cosh z}{1 - f\cosh^2 z} = \frac{1 - f\cosh z}{1 - f\cosh^2 z} \tag{24}$$

then replace z with $z\sqrt{1-f}$ and divide by $\sqrt{1-f}$. To get from (18) to (22), apply (24) and replace z with $z\sqrt{1-f}$. Finally, to get from (19) to (23) replace z with $z\sqrt{1-f}$ and divide by $\sqrt{1-f}$. □

If we add (22) and (23) and simplify, we get the first part of Corollary 4.19 . If we add (20) and (21) and simplify, we get its second part.

A similar analysis provides another method of deriving the generating function for the Eulerian numbers.

4.6. Transfer Matrices and the Inversion Theorem Applied to the Generating Function

In this section we make use of techniques discussed in Gessel's thesis [**6**] to provide yet another derivation of the bin capacity 1 generating function.

Definition 4.11. Let D be a digraph with weights on its edges and with vertices $v_1, \ldots, v_n$. Let $w_{i,j}$ be the weight of the edge going from v_i to v_j (if there is no such edge, we consider $w_{i,j}$ to be zero). We then define a matrix M by taking $M_{i,j}$ to be $w_{i,j}$. We call M the adjacency matrix of D.

The powers of M encode the paths in D, as Stanley discusses in [**12**, pp. 241-243].

Theorem 4.29. *Let M be the adjacency matrix of a digraph D. Then $M^k_{i,j}$ is the sum of the weights of all paths of length k in D that start at v_i and end at v_j.*

The following corollary is immediate.

Corollary 4.30. *The sum of the weights of all paths in D starting at v_i and ending at v_j is $(I - M)^{-1}_{i,j}$.*

In our case, we will construct a digraph whose edges record the weights of increasing runs of different lengths. Paths in the digraph then correspond to sequences of increasing runs. For example, suppose an increasing run r_1 of length l_1 is recorded

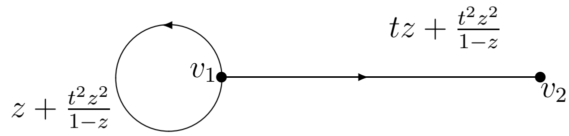


FIGURE 4-1. The digraph D for the bin capacity 1 case

by the edge v_iv_j, where it is given weight w_{i,j,l_1}. Suppose an increasing run r_2 of length l_2 is recorded by the edge v_jv_k, where it is given weight w_{j,k,l_2}. Then this sequence of runs r_1r_2 is recorded by the path $v_iv_jv_k$ with weight $w_{i,j,l_1}w_{j,k,l_2}$.

In particular, we recall Corollary 4.7, in which each maximal increasing run of length ≥ 2 counts two bins, and a terminal maximal increasing run of length one (i.e., the permutation ends in a descent) counts one bin. If we weight a run of length k that counts i bins by t^iz^k, we can codify this information by constructing a digraph D with two vertices. The edge v_1v_1 (in which we wish to record non-terminal runs) has weight $z + \sum_{i=2}^{\infty} t^2z^i$. The edge v_1v_2 (in which we wish to record terminal runs) has weight $zt + \sum_{i=2}^{\infty} t^2z^i$. We are only concerned with paths in D that start at v_1 and end at v_2. We draw D in Figure 4-1.

See Section 5.2 for less trivial examples of such digraphs.

By counting paths in D (or any other such digraph) and recording their weights, we can count sequences of increasing runs. For example, using doubly indexed elements x_i^j, let $r_1 = x_1^1 < x_2^1 < x_3^1$, $r_2 = x_1^2$, $r_3 = x_1^3 < x_2^3$, and $r_4 = x_1^4$ be increasing runs of lengths 3, 1, 2, and 1 respectively. Then the sequence of runs $r_1r_2r_3r_4$ corresponds in D to the path $v_1v_1v_1v_1v_1$ with weight t^4z^7 and the path $v_1v_1v_1v_1v_2$ with weight t^5z^7. Since we are only concerned with paths that start at v_1 and end at v_2, only the latter will count $r_1r_2r_3r_4$.

However, there is an added complication. We have not specified any relation between the last element in one run and the first in the next. In our example we could have had $x_1^1 < x_2^1 < x_3^1 \geq x_1^2 < x_1^3 < x_2^3 < x_1^4$ or $x_1^1 < x_2^1 < x_3^1 < x_1^2 \geq x_1^3 < x_2^3 \geq x_1^4$, to name just two possibilities. But because Corollary 4.7 required that the increasing runs be maximal, we only want to count those sequences of runs where the last element of one run is greater than or equal to the first element of the next.

For example, we wish the permutation 1 2 3 4 to correspond to a single run of length four for a weight of t^2z^4 instead of, say, a run of length two (1 2) followed by a terminal run of length two (3 4) (weight $t^2z^2 \cdot t^2z^2 = t^4z^4$) or a run of length three (1 2 3) followed by a terminal run of length one (4) (weight $t^2z^3 \cdot tz = t^3z^4$). By requiring that each run be maximal, we guarantee that each permutation gets broken down into a unique sequence of runs and thus a unique path in D.

We use the techniques of [**2**] and [**6**] to deal with this issue as follows.

Theorem 4.31. *Let A be an alphabet and A^* the free monoid on A. Let $L \subseteq A \times A$ and let $A^{(L)} \subset A^*$ be those words for which every consecutive subsequence of length 2 is in L. Let $\bar{L} = A \times A \setminus L$ and let $A^{(\bar{L})} \subset A^*$ be those words for which every consecutive subsequence of length 2 is in $\bar{L}$. For $w \in A^*$, let $l(w)$ be its length. Then*

$$\sum_{w \in A^{(L)}} w = \Big(\sum_{w \in A^{(\bar{L})}} (-1)^{l(w)} w \Big)^{-1} .$$

Theorem 4.32. *Let D be a digraph encoding a certain pattern of weighted run lengths, and M its adjacency matrix. Then the sum of the weights of the permutations of $[m]$ whose maximal increasing run sequence corresponds to a path from i to j in D is*

$$[\frac{z^m}{m!}]\ \left[\Phi((I+M)^{-1})\right]^{-1}_{i,j}$$

where Φ is the linear operator which takes $\sum a_i z^i$ to $\sum a_i z^i/i!$.

Proof. In the notation of Theorem 4.31, let the alphabet A consist of increasing runs, and let L be those pairs of consecutive runs for which the final element of the first run is greater than or equal to the initial element of the second run. Then words in $A^{(L)}$ are those sequences of runs such that every run is of maximal length. We wish to count the number of permutations that can be written as words in $A^{(L)}$, weighted according to their corresponding paths in D.

By Theorem 4.31, $\sum_{w\in A^{(L)}} w = \left(\sum_{w\in A^{(\bar{L})}}(-1)^{l(w)}w\right)^{-1}$. We first consider what form words in $A^{(\bar{L})}$ take. They are those sequences of runs in which the last element of one run is always strictly less than the first element of the next run. Combining a sequence of runs in this fashion results in a single increasing run. Since in our context this increasing run is actually an increasing permutation, we apply the operator Φ.

We must weight each word $w \in A^{(\bar{L})}$ by $(-1)^{l(w)}$, as well as by the path in D it corresponds to. Recall however that $l(w)$ is not the number of elements in the single increasing run, but rather the number of runs that combined to form it. In other words, if we formed w by taking a sequence of runs of lengths $i_1, \ldots, i_k$ corresponding to a path of length k in D, we need to weight it by $(-1)^k$. Since Theorem 4.29 gives the weights of the paths of length k by M^k, we need to replace this by $(-1)^k M^k$. Summing over k, we get $\sum_k (-1)^k M^k = (I+M)^{-1}$.

So the number of permutations corresponding to words in $A^{(\bar{L})}$, weighted according to paths in D and multiplied by (-1) to the path length, is $\Phi((I+M)^{-1})$. By Theorem 4.31, the number of permutations corresponding to words in $A^{(L)}$, weighted according to path in D, is $\left[\Phi((I+M)^{-1})\right]^{-1}$ as needed. □

This provides us with another method of proving Theorem 4.20.

Proof. We will apply Theorem 4.32 to the digraph in Figure 4-1. We wish to find the $1,2$ entry of $\left[\Phi((I+M)^{-1})\right]^{-1}$.

First, we write down the matrix $I+M$

$$I + M = \begin{pmatrix} 1 + z + \dfrac{t^2 z^2}{1-z} & tz + \dfrac{t^2 z^2}{1-z} \\ \\ 0 & 1 \end{pmatrix}$$

and calculate its inverse:

$$(I+M)^{-1} = \begin{pmatrix} \dfrac{1-z}{1-z^2(1-t^2)} & \dfrac{-zt(1-z(1-t))}{1-z^2(1-t^2)} \\ \\ 0 & 1 \end{pmatrix}.$$

Since

$$(I+M)^{-1}_{1,1} = \frac{1-z}{1-z^2(1-t^2)}$$

$$= (1-z)\sum_{i=0}^{\infty} z^{2i}(1-t^2)^i$$

$$= \sum_{i=0}^{\infty}(z\sqrt{1-t^2})^{2i} - \frac{1}{\sqrt{1-t^2}}\sum_{i=0}^{\infty}(z\sqrt{1-t^2})^{2i+1}$$

we have

$$\Phi((I+M)^{-1}_{1,1}) = \sum_{i=0}^{\infty}\frac{(z\sqrt{1-t^2})^{2i}}{(2i)!} - \frac{1}{\sqrt{1-t^2}}\sum_{i=0}^{\infty}\frac{(z\sqrt{1-t^2})^{2i+1}}{(2i+1)!}$$

$$= \cosh(z\sqrt{1-t^2}) - \frac{\sinh(z\sqrt{1-t^2})}{\sqrt{1-t^2}} \ .$$

Similarly,

$$(I+M)^{-1}_{1,2} = \frac{-zt(1-z(1-t))}{1-z^2(1-t^2)}$$

$$= [-zt(1-z(1-t))]\sum_{i=0}^{\infty} z^{2i}(1-t^2)^i$$

$$= \frac{-t}{\sqrt{1-t^2}}\sum_{i=0}^{\infty}(z\sqrt{1-t^2})^{2i+1} + \frac{t(1-t)}{1-t^2}\sum_{i=0}^{\infty}(z\sqrt{1-t^2})^{2i+2}$$

so

$$\Phi((I+M)^{-1}_{1,2}) = \frac{-t}{\sqrt{1-t^2}}\sum_{i=0}^{\infty}\frac{(z\sqrt{1-t^2})^{2i+1}}{(2i+1)!} + \frac{t}{1+t}\sum_{i=1}^{\infty}\frac{(z\sqrt{1-t^2})^{2i}}{(2i)!}$$

$$= \frac{-t}{\sqrt{1-t^2}}\sinh(z\sqrt{1-t^2}) + \frac{t}{1+t}(\cosh(z\sqrt{1-t^2}) - 1) \ .$$

Further simple calculations yield

$$\left[\Phi((I+M)^{-1})\right]^{-1}_{1,2} =$$

$$= \frac{t}{1+t}\ \frac{(1+t)\sinh(z\sqrt{1-t^2}) - \sqrt{1-t^2}\cosh(z\sqrt{1-t^2}) + \sqrt{1-t^2}}{\sqrt{1-t^2}\cosh(z\sqrt{1-t^2}) - \sinh(z\sqrt{1-t^2})} \ .$$

Finally, the transfer matrix method defines M^0 to be the identity matrix I. Since we consider all our permutations to start at v_1 and end at v_2, we need to modify the function above by adding a 1 for the base case. But after suitable calculations, this is the same generating function as before. □

We can use the techniques in this section to prove the special case of the theorem of Goulden and Jackson.

Proof of Theorem 4.18. As in Definition 4.7, we let f_i be the weight of a (non-final) maximal increasing run of length i, and we let g_i be the weight of a final maximal increasing run of length i. We construct the same digraph as in Figure 4-1, and weight its edges as indicated by the following adjacency matrix J.

$$J = \begin{pmatrix} \sum_{i=1}^{\infty} f_i x^i & \sum_{i=1}^{\infty} g_i x^i \\ 0 & 0 \end{pmatrix}$$

Following Theorem 4.32 and using the notation from Theorem 4.18, we take

$$I + J = \begin{pmatrix} F(x) & G(x) \\ 0 & 1 \end{pmatrix}$$

and

$$(I+J)^{-1} = \begin{pmatrix} F(x)^{-1} & -F(x)^{-1}G(x) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sum_n u_n z^n & -\sum_n v_n z^n \\ 0 & 1 \end{pmatrix}$$

Applying the operator Φ yields

$$\Phi((I+J)^{-1}) = \begin{pmatrix} \sum_n u_n \frac{z^n}{n!} & -\sum_n v_n \frac{z^n}{n!} \\ 0 & 1 \end{pmatrix}$$

and inverting gives

$$(\Phi((I+J)^{-1}))^{-1} = \begin{pmatrix} \left(\sum_n u_n \frac{z^n}{n!}\right)^{-1} & \left(\sum_n u_n \frac{z^n}{n!}\right)^{-1}\left(\sum_n v_n \frac{z^n}{n!}\right) \\ 0 & 1 \end{pmatrix} .$$

If we do not choose to distinguish the terminal run, then the $(1, 1)$ entry gives the first part of Theorem 4.18. The $(1, 2)$ entry gives the second part of Theorem 4.18, in which we distinguish the final run. $\square$

CHAPTER 5

The Capacity k Case of the Bin Problem

5.1. Capacity k Bin Arrangements and k-Marked Shapes

We now consider bins of capacity k. As before, we let the object sizes be chosen uniformly at random from the interval $(0, 1]$, and require that if $i_1, \ldots, i_m$ are in a bin, then $\sum_{j=1}^{m} x_{i_j} \leq k$ where $k \in \mathbb{N}^+$ is the capacity of the bin. We assume all bin arrangements are obtained from applying the NEXT FIT algorithm to a sequence of objects, and we consider NEXT FIT to partition $(0, 1]^m$ into polytopes corresponding to the different possible bin arrangements.

As we did in the case where $k = 1$, we will define a function that will transform polytopes corresponding to bin arrangements into polytopes corresponding to permutation shapes. However, as in the dual bin case, a single bin arrangement will turn out to correspond to a set of shapes.

Definition 5.1. Suppose $(x_1, \ldots, x_m) \in P_A$, where A is a capacity k bin arrangement with n bins. As in the dual bin case, we let $T_j = \sum_{i=b}^{j} x_i$, where b is the first element in the bin containing j. We define a strictly increasing sequence

$$\bar{u} = (u_{1,0}, \ldots, u_{1,k-1}, \ldots, u_{n-1,0}, \ldots, u_{n-1,k-1}, u_{n,0}, \ldots, u_{n,k'})$$

where $u_{l,i}$ is the minimum j in the lth bin for which $T_j > i$, and $k' = \lceil T_m \rceil - 1$. Note that $u_{l,0}$ is just the first item in the lth bin. Define $P_{A_{\bar{u}}} \subset P_A$ to be the polytope consisting of all points with this same $\bar{u}$.

Definition 5.2. For each $\bar{u}$, we associate the related sequence

$$\bar{u}' = (u'_{1,1}, \ldots, u'_{1,k}, \ldots, u'_{n-1,1}, \ldots, u'_{n-1,k}, u'_{n,1}, \ldots, u'_{n,k'}, u'_{n,k'+1})$$

where

$$u'_{l,i} = \begin{cases} m - u_{n,k'}, & \text{if } l = n \text{ and } i = k'+1; \\ u_{l+1,0} - u_{l,k-1} - 1, & \text{if } i = k; \\ u_{l,i} - u_{l,i-1} - 1, & \text{otherwise.} \end{cases}$$

The sequence $\bar{u}$ lists those objects for which the running total in the appropriate bin crosses over an integer. The elements of $\bar{u}'$ are one less than the differences between successive elements of $\bar{u}$, with the last being the difference between m and $u_{n,k'}$.

For example, let A be the following capacity 3 bin arrangement.

1 2 3 4 5 6 7	8 9 10

Consider the point $(.2, .2, .3, .4, .9, .1, .8, .8, .3, .8) \in P_A$. We automatically have that $u_{1,0} = 1$. Since $T_1 = .2$, $T_2 = .4$, $T_3 = .7$, and $T_4 = 1.1$, we have $u_{1,1} = 4$. Similarly, $T_5 = 2.0$ and $T_6 = 2.1$, so $u_{1,2} = 6$. Now $T_7 = 2.9$ and $T_7 + x_8 = 3.7$, so 8 starts a new bin, $u_{2,0} = 8$, and $T_8 = x_8 = .8$. Finally, $T_9 = 1.1$ and $T_{10} = 1.9$ so $u_{n,k'} = u_{2,1} = 9$. Hence $\bar{u} = (u_{1,0}, u_{1,1}, u_{1,2}, u_{2,0}, u_{2,1}) = (1, 4, 6, 8, 9)$ and the associated $\bar{u}'$ is $(u'_{1,1}, u'_{1,2}, u'_{1,3}, u'_{2,1}, u'_{2,2}) = (2, 1, 1, 0, 1)$.

Definition 5.3. Let A be a bin arrangement with m elements. We define $\bar{\psi}_A$ with $\bar{\psi}_A(x_1,\ldots,x_m)=(y_1,\ldots,y_m)$ as follows:

$$y_j=\begin{cases}\lceil T_j\rceil - T_j, & \text{if } j \text{ is in an odd-numbered bin;}\\ T_j-\lceil T_j\rceil+1, & \text{if } j \text{ is in an even-numbered bin.}\end{cases}$$

where T_j is as in Definition 5.1, and $\lceil x\rceil$ is the least integer greater than x.

Note that for $k=1$, $\bar{\psi}_A$ reduces to ψ_A from before. Similarly, if we take $(x_1,\ldots,x_m)\in P_A$, then $\bar{\psi}_A(x_1,\ldots,x_m)\in[0,1]^m$.

Proposition 5.1. *Given the bin arrangement A, $\bar{\psi}_A$ is a piecewise affine linear, volume preserving map.*

Proof. The situation is the same as in Proposition 3.10. If we subdivide P_A into the subpolytopes $P_{A_{\bar{u}}}$ of Definition 5.1, then $\bar{\psi}$ is linear on each $P_{A_{\bar{u}}}$. In particular, if $u_{l,i}\le j<u_{l,i+1}$ then $i<T_j\le i+1$ and hence $i=\lceil T_j\rceil-1$. (If we take $i=k-1$ then $u_{l,k-1}\le j<u_{l+1,0}$ and $k-1<T_j\le k$, so $k-1=\lceil T_j\rceil-1$.) As a result, we have the formulas $y_j=i+1-T_j$ if l is odd and $y_j=T_j-i$ if l is even. Then, as in Proposition 4.1, we can easily write the action of $\bar{\psi}$ on $P_{A_{\bar{u}}}$ as an explicit matrix with determinant ±1. □

Definition 5.4. We define $\bar{\psi}$ on $(0,1)^m$ by setting $\bar{\psi}(x_1,\ldots,x_m)=\bar{\psi}_A(x_1,\ldots,x_m)$ for $(x_1,\ldots,x_m)\in P_A$.

The following proposition is straightforward.

Proposition 5.2. *We take a bin arrangement A and fix a sequence $\bar{u}$, and consider the inequalities satisfied by the points in $\bar{\psi}_{A_{\bar{u}}}$. For notational purposes, we will consider $u_{l,k}$ to be equal to $u_{l+1,0}$. Suppose j is in the lth bin. Then the following hold.*

(1) *If l is odd, then*

  (a) $y_j < y_{j+1}$ *if* $j+1 = u_{l,i}$ *for some* $i = 1, \ldots, k$

  (b) $y_j > y_{j+1}$ *otherwise*

  (c) $0 \leq y_j < 1$.

(2) *If l is even, then*

  (a) $y_j > y_{j+1}$ *if* $j+1 = u_{l,i}$ *for some* $i = 1, \ldots, k$

  (b) $y_j < y_{j+1}$ *otherwise*

  (c) $0 < y_j \leq 1$.

(3) $0 < y_1 < 1$.

In other words, if the bin is an odd one, the corresponding y_j's form a sequence of descents, with an ascent every time the running total crosses over an integer. After the kth such cross over, the corresponding ascent marks the start of a new bin. Similarly, even bins yield descents when the running total reaches each integer, and ascents elsewhere.

As a result, we introduce another way to mark shapes.

Definition 5.5. Fix k, and let S be a shape. We define the k-marked shape s_k as follows. Starting from the left, we mark the kth a as $\dot{a}$, and the kth d after that as $\dot{d}$. We continue in this fashion, alternating whether we are counting the a's or the d's.

Note that this is consistent with the method of marking in Section 2.3; we just took $k = 1$ as we would expect in the capacity 1 case.

We thus write each marked word s_k in the form

$$\left((d^*a)^{k-1}d^*\dot{a}(a^*d)^{k-1}a^*\dot{d}\right)^* \tag{25}$$

followed by some truncation of $(d^*a)^{k-1}d^*\dot{a}(a^*d)^{k-1}a^*\dot{d}$. As usual, if w is a word in a free monoid, then w^* indicates that w appears to some non-negative power.

Definition 5.6. Let S_k^m be the set of k-marked shapes of length m. We define $S^m_{k,(i_1,\ldots,i_n)} \subset S_k^m$ by $s_k \in S^m_{k,(i_1,\ldots,i_n)}$ if the first marked letter (an $\dot{a}$) in s_k is the i_1st letter of s_k, the second marked letter (a $\dot{d}$) is the i_2nd letter after that, the third (an $\dot{a}$) is the i_3rd letter after that, and so on.

For example, take $S = dddaadaddaddaa$ and $S' = daddaaddadddad$. Then

$$s_2 = ddda\dot{a}dad\dot{d}add\dot{a}a \qquad (= d^3ad^0\dot{a}a^0da^1\dot{d}d^1ad^2\dot{a}a^1)$$

$$s_2' = dadd\dot{a}add\dot{a}ddd\dot{a}d \qquad (= d^1ad^2\dot{a}a^1da^0\dot{d}d^0ad^3\dot{a}a^0d)$$

$$s_3 = dddaad\dot{a}dda\dot{d}daa \qquad (= d^3ad^0ad^1\dot{a}a^0da^0da^1\dot{d}d^1ad^0a)$$

and $$s_3' = dadda\dot{a}dda\dot{d}ddad \qquad (= d^1ad^2ad^0\dot{a}a^0da^0da^1\dot{d}d^2ad^1)\ .$$

Both s_2 and s_2' are in $S^{14}_{2,(5,3,5)}$. However, $s_3 \in S^{14}_{3,(7,4)}$ and $s_3' \in S^{14}_{3,(6,4)}$.

Definition 5.7. Let s_k be the k-marking of the shape S. We define the polytope P_{s_k} to be equal to the polytope P_s of Definition 4.3. We define $P_{S^m_{k,(i_1,\dots,i_n)}}$ to be the union of the polytopes P_{s_k} for all $s_k \in S^m_{k,(i_1,\dots,i_n)}$.

Definition 5.8. Suppose we have sequences $\bar{u}$ and $\bar{u}'$ as in Definitions 5.1 and 5.2. For $l \neq n$ let

$$i_l = u'_{l,1} + \cdots + u'_{l,k} + k \ .$$

Then we define a shape $s_{\bar{u}} \in S^{m-1}_{k,(i_1,\dots,i_{n-1})}$ in terms of the formulation of (25) as follows. We define the $u_{l,0}$th through $u_{l+1,0} - 1$st letters of $s_{\bar{u}}$ to be

$$d^{u'_{l,1}} a d^{u'_{l,2}} a \dots d^{u'_{l,k}} \dot{a} \qquad \text{for } l \text{ odd, } l \neq n;$$

$$a^{u'_{l,1}} d a^{u'_{l,2}} d \dots a^{u'_{l,k}} \dot{d} \qquad \text{for } l \text{ even, } l \neq n.$$

Then, for the last truncated copy of (25), we have one of the following:

$$d^{u'_{n,1}} a \dots d^{u'_{n,k'}} a d^{u'_{n,k'+1}} \qquad \text{for } n \text{ odd;}$$

$$a^{u'_{n,1}} d \dots a^{u'_{n,k'}} d a^{u'_{n,k'+1}} \qquad \text{for } n \text{ even.}$$

Theorem 5.3. *Let A be the capacity k bin arrangement of m elements that contains i_l elements in the lth bin for $l \in 1, \dots, n$. Then $\bar{\psi}$ bijectively takes P_A to $P_{S^{m-1}_{k,(i_1,\dots,i_{n-1})}}$. In particular, $\bar{\psi}$ takes each $P_{A_{\bar{u}}}$ bijectively to the corresponding $P_{s_{\bar{u}}}$.*

Proof. This follows from the definitions and Proposition 5.2. We omit the tedious details. □

We note that, as in the capacity 1 case, each $\dot{a}$ and $\dot{d}$ marks the start of a new bin; if the ith letter of a shape is marked, then $i + 1$ starts a new bin. Furthermore, if we write a k-marked shape in the form given in (25), we can immediately read off $\bar{u}'$. Similarly, given $\bar{u}'$, we can immediately write the corresponding k-marked shape. For instance, recall the first example of this section, where we had $\bar{u}' = (u'_{1,1}, u'_{1,2}, u'_{1,3}, u'_{2,1}, u'_{2,2}) = (2,1,1,0,1)$. Then the corresponding shape $s_{\bar{u}}$ will be $d^2ad^1ad^1\dot{a}a^0da^1 = ddadad\dot{a}da$.

In the capacity 1 case, one could look locally at the shape of a permutation and determine information about its corresponding bin arrangement. Each run or, alternatively, each peak or valley corresponded to a bin. For $k > 1$, this is no longer the case. In order to determine whether an a or d in a shape is actually marked as $\dot{a}$ or $\dot{d}$, representing the start of a new bin, one needs to know information about all the preceeding letters in that shape.

The methods of Section 4.6 allow us to keep track of such information.

5.2. Capacity 2 and 3

We will use the methods of Section 4.6 to explicitly calculate the generating function for the capacity 2 bin problem. We will set up, but not solve, the same for the capacity 3 case.

First, we need to write down a digraph that encodes which increasing runs start new bins in the capacity 2 case. We start by looking at the 2-marked shapes in terms of the form of (25), and then rewrite in terms of the lengths of the maximal increasing runs. All 2-marked shapes can be written in the form

$$(d^*ad^*\dot{a}a^*da^*\dot{d})^* \qquad (26)$$

followed by a truncation of $d^*ad^*\dot{a}a^*da^*\dot{d}$.

Assume for the moment that the second d^* is non-empty (the second d^* is of the form d^{i_2} for some $i_2 > 0$). Then a shape starting with $d^{i_1}ad^{i_2}\dot{a}a^{i_3}da^{i_4}\dot{d}$ will start with i_1 maximal increasing runs of length 1, followed by a run of length 2, and then i_2 runs of length 1. The d at the end of the combination $\dot{a}a^{i_3}d$ ends a maximal increasing run of length $i_3 + 2$, and the $\dot{d}$ at the end of the combination $a^{i_4}\dot{d}$ ends a run of length $i_4 + 1$.

We will record this pattern of maximal increasing run lengths by

$$1^* \ 2 \ 1^* \geq \mathbf{2} \ \ \geq \mathbf{1}$$

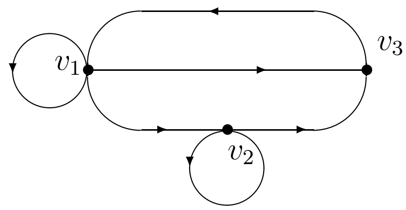
where the last two runs are in bold to denote that they each include a marked letter (and hence correspond to starting a new bin).

Now suppose the second d^* was actually d^0. Then we can write such a shape as starting with $d^{i_1}ad^0\dot{a}a^{i_3}da^{i_4}\dot{d} = d^{i_1}a\dot{a}a^{i_3}da^{i_4}\dot{d}$. Once again, it starts with i_1 maximal increasing runs of length 1. Now, however, the d at the end of the combination $a\dot{a}a^{i_3}d$ ends an increasing run of length $i_3 + 3$, and once again the $\dot{d}$ at the end of $a^{i_4}\dot{d}$ forms the end of a run of length $i_4 + 1$. The run pattern then looks like

$$1^* \ \ \geq \mathbf{3} \ \ \geq \mathbf{1}$$

with bold face numbers indicating runs including marked letters as before. We combine these two cases in the following diagram.

$$1^* \left\{ \begin{array}{lll} 2 & 1^* & \geq \mathbf{2} \\ \geq \mathbf{3} & & \end{array} \right\} \geq \mathbf{1} \qquad (27)$$



$$\begin{pmatrix} z & z^2 & \dfrac{tz^3}{1-z} \\ 0 & z & \dfrac{tz^2}{1-z} \\ \dfrac{tz}{1-z} & 0 & 0 \end{pmatrix}$$

FIGURE 5-1. Preliminary digraph and adjacency matrix for the bin capacity 2 case

Starting the pattern (26) again merely restarts the pattern of maximal run lengths given by (27). This gives us a preliminary version of the digraph, as shown in Figure 5-1. An edge corresponding to a run of length k will have weight z^k; it has weight tz^k if it marks the start of a new bin. We are interested in paths that start at the vertex v_1.

Now we consider the truncation of $d^*ad^*\dot{a}a^*da^*\dot{d}$ at the end of a 2-marked shape. In general, truncating (26) will merely truncate the pattern of run lengths given by (27). However, the final (bold face) run of length $\geq \mathbf{1}$ depends on the $\dot{d}$ ending the combination $a^*\dot{d}$. If the shape ends with $d^*ad^*\dot{a}a^*da^*$, then while the final maximal increasing run will indeed be of length ≥ 1, it will not be ended by $\dot{d}$ and hence will not correspond to starting a new bin.

To take this into account, we modify the digraph in Figure 5-1 by adding vertex v_4 and edge v_3v_4, which we give weight $z(1-z)^{-1}$. This corresponds to a terminal (non-boldface!) run of length ≥ 1. Note that if the truncation is trivial (i.e., the shape ends with a full copy of $d^*ad^*\dot{a}a^*da^*\dot{d}$, say $d^{i_1}ad^{i_2}\dot{a}a^{i_3}da^{i_4}\dot{d}$), then the penultimate run is a marked one of length $i_4 + 1$ and the final run is of length 1. We can thus introduce vertex v_5 and edge v_1v_5 with weight z, and specify that we will only count

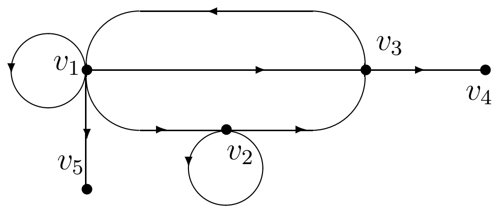


FIGURE 5-2. The digraph D_2 for the bin capacity 2 case

the paths that start at v_1 and end elsewhere. (This forces us to count the paths that end in v_3v_4 or v_1v_5 instead of v_3v_1 or v_1v_1 respectively.)

Our final version of the digraph D_2 for the capacity 2 bin case appears in Figure 5-2. Note that each shape, when split into maximal increasing runs, corresponds to exactly one path in D_2, starting at v_1 and ending at some other vertex.

We write the adjacency matrix N of the digraph D_2 as follows.

$$N = \begin{pmatrix} z & z^2 & \dfrac{tz^3}{1-z} & 0 & z \\ 0 & z & \dfrac{tz^2}{1-z} & 0 & 0 \\ \dfrac{tz}{1-z} & 0 & 0 & \dfrac{z}{1-z} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Notation. We use the abbreviations $C^+ = \cosh(z\sqrt{1+t})$, $S^+ = \sinh(z\sqrt{1+t})$, $C^- = \cosh(z\sqrt{1-t})$, and $S^- = \sinh(z\sqrt{1-t})$. In terms of exponentials, we introduce the abbreviations $p = e^{z\sqrt{1+t}}$ and $n = e^{z\sqrt{1-t}}$.

Theorem 5.4. *Taking*

$$f = \sqrt{1-t^2}\ S^+ \left[(t^2-3t)C^- + t^2 - t + 2\right] + 2(1-t)^{5/2}\left(1 + C^- - C^+ - C^+C^-\right)$$
$$+ S^- \left[C^+(t^3-t) - 2 - t^3 + 2t^2 + t\right] + 2S^+S^-\sqrt{1+t}\ (1-t)$$

and

$$g = 2S^-(1-t)\left[C^+ + t\right] - \sqrt{1-t}\ (1-t^2)\left[C^+C^- + 1\right]$$
$$+ 2\sqrt{1-t^2}\ S^+\left[C^- - t\right] - 3\sqrt{1+t}\ (1-t)S^+S^-\ ,$$

the generating function for the ordinary bin capacity 2 case is $1+f/g$. *The coefficient of* $t^n z^m/m!$ *in* $1+f/g$ *counts the number of permutations of* m *corresponding to* n *capacity 2 bins.*

Proof. We apply Theorem 4.32 to the digraph D_2 of Figure 5-2 and its adjacency matrix N.

First, it is necessary to calculate the matrix $(I+N)^{-1}$, and apply Φ to its entries. (Recall that Φ is the linear operator that takes $\sum a_i z^i$ to $\sum a_i z^i/i!$.) In order to calculate $\Phi((I+N)^{-1}_{i,j})$, we write $(I+N)^{-1}_{i,j}$ in terms of its partial fraction expansion. Straightforward manipulations then yield expressions in C^+, S^+, C^- and S^-. See Table 5.1 for the actual expressions. (The fourth and fifth rows of $(I+N)^{-1}$ are just $[0,0,0,1,0]$ and $[0,0,0,0,1]$ respectively.) Inverting $\Phi((I+N)^{-1})$ and taking the sum

$$\left[\Phi((I+N)^{-1})\right]^{-1}_{1,2} + \left[\Phi((I+N)^{-1})\right]^{-1}_{1,3} + \left[\Phi((I+N)^{-1})\right]^{-1}_{1,4} + \left[\Phi((I+N)^{-1})\right]^{-1}_{1,5}$$

i, j	$(I+N)^{-1}_{i,j}$	partial fraction expansion of $(I+N)^{-1}_{i,j}$	$\Phi((I+N)^{-1}_{i,j})$
$1,1$	$\frac{-(1-z)(1-z)^2}{t^2z^4-(z^2-1)^2}$	$\frac{1-z}{2(1-z^2(1+t))} + \frac{1-z}{2(1-z^2(1-t))}$	$\frac{C^+}{2} - \frac{S^+}{2\sqrt{1+t}} + \frac{C^-}{2} - \dots$
$1,2$	$\frac{z^2(1-z)^2}{t^2z^4-(z^2-1)^2}$	$\frac{-1}{1-t^2} + \frac{1}{2t(1+t)} \cdot \frac{-2-t+2z(1+t)}{1-z^2(1+t)} + \frac{1}{2t(1-t)} \cdot \frac{2-t-2z(1-t)}{1-z^2(1-t)}$	$\frac{-1}{1-t^2} - \frac{(2+t)C^+}{2t(1+t)} + \frac{S^+}{t\sqrt{1+t}} + \dots$
$1,3$	$\frac{tz^3(1-z)}{t^2z^4-(z^2-1)^2}$	$\frac{t}{1-t^2} + \frac{1}{2(t+1)} \cdot \frac{1-z(1+t)}{1-z^2(1+t)} - \frac{1}{2(1-t)} \cdot \frac{1-z(1-t)}{1-z^2(1-t)}$	$\frac{t}{1-t^2} + \frac{C^+}{2(1+t)} - \frac{S^+}{2\sqrt{1+t}} - \dots$
$1,4$	$\frac{-tz^4}{t^2z^4-(z^2-1)^2}$	$\frac{t}{1-t^2} + \frac{1}{2(1+t)} \cdot \frac{1}{1-z^2(1+t)} - \frac{1}{2(1-t)} \cdot \frac{1}{1-z^2(1-t)}$	$\frac{t}{1-t^2} + \frac{C^+}{2(1+t)} - \dots$
$1,5$	$\frac{z(1+z)(1-z)^2}{t^2z^4-(z^2-1)^2}$	$\frac{-1}{1-t^2} + \frac{1}{2(1+t)} \cdot \frac{1-z(1+t)}{1-z^2(1+t)} + \frac{1}{2(1-t)} \cdot \frac{1-z(1-t)}{1-z^2(1-t)}$	$\frac{-1}{1-t^2} + \frac{C^+}{2(1+t)} - \frac{S^+}{2\sqrt{1+t}} + \dots$
$2,1$	$\frac{-t^2z^3}{t^2z^4-(z^2-1)^2}$	$\frac{tz}{2(1-z^2(1+t))} - \frac{tz}{2(1-z^2(1-t))}$	$\frac{tS^+}{2\sqrt{1+t}} - \frac{tS^-}{2\sqrt{1-t}}$
$2,2$	$\frac{-1+z+z^2-z^3+t^2z^4}{t^2z^4-(z^2-1)^2}$	$\frac{-t^2}{1-t^2} + \frac{1}{2(1+t)} \cdot \frac{1-z(1+t)}{1-z^2(1+t)} + \frac{1}{2(1-t)} \cdot \frac{1-z(1-t)}{1-z^2(1-t)}$	$\frac{-t^2}{1-t^2} + \frac{C^+}{2(1+t)} - \frac{S^+}{2\sqrt{1+t}} + \dots$
$2,3$	$\frac{tz^2(1-z^2)}{t^2z^4-(z^2-1)^2}$	$\frac{t}{1-t^2} - \frac{1}{2(1+t)} \cdot \frac{t}{1-z^2(1+t)} - \frac{1}{2(1-t)} \cdot \frac{t}{1-z^2(1-t)}$	$\frac{t}{1-t^2} - \frac{tC^+}{2(1+t)} - \dots$
$2,4$	$\frac{-tz^3(1+z)}{t^2z^4-(z^2-1)^2}$	$\frac{t}{1-t^2} + \frac{1}{2(1+t)} \cdot \frac{1+z(1+t)}{1-z^2(1+t)} - \frac{1}{2(1-t)} \cdot \frac{1+z(1-t)}{1-z^2(1-t)}$	$\frac{t}{1-t^2} + \frac{C^+}{2(1+t)} + \frac{S^+}{2\sqrt{1+t}} - \dots$
$2,5$	$\frac{t^2z^4}{t^2z^4-(z^2-1)^2}$	$\frac{-t^2}{1-t^2} - \frac{1}{2(1+t)} \cdot \frac{t}{1-z^2(1+t)} + \frac{1}{2(1-t)} \cdot \frac{t}{1-z^2(1-t)}$	$\frac{-t^2}{1-t^2} - \frac{tC^+}{2(1+t)} + \dots$
$3,1$	$\frac{tz(1-z^2)}{t^2z^4-(z^2-1)^2}$	$\frac{-tz}{2(1-z^2(1+t))} - \frac{tz}{2(1-z^2(1-t))}$	$\frac{-tS^+}{2\sqrt{1+t}} - \frac{tS^-}{2\sqrt{1-t}}$
$3,2$	$\frac{-tz^3(1-z)}{t^2z^4-(z^2-1)^2}$	$\frac{-t}{1-t^2} - \frac{1}{2(1+t)} \cdot \frac{1-z(1+t)}{1-z^2(1+t)} + \frac{1}{2(1-t)} \cdot \frac{1-z(1-t)}{1-z^2(1-t)}$	$\frac{-t}{1-t^2} - \frac{C^+}{2(1+t)} + \frac{S^+}{2\sqrt{1+t}} + \dots$
$3,3$	$\frac{-(1+z)^2(1-z)^2}{t^2z^4-(z^2-1)^2}$	$\frac{1}{1-t^2} + \frac{1}{2(1+t)} \cdot \frac{t}{1-z^2(1+t)} - \frac{1}{2(1-t)} \cdot \frac{t}{1-z^2(1-t)}$	$\frac{1}{1-t^2} + \frac{tC^+}{2(1+t)} - \dots$
$3,4$	$\frac{z(1-z)(1+z)^2}{t^2z^4-(z^2-1)^2}$	$\frac{1}{1-t^2} - \frac{1}{2(1+t)} \cdot \frac{1+z(1+t)}{1-z^2(1+t)} - \frac{1}{2(1-t)} \cdot \frac{1+z(1-t)}{1-z^2(1-t)}$	$\frac{1}{1-t^2} - \frac{C^+}{2(1+t)} - \frac{S^+}{2\sqrt{1+t}} - \dots$
$3,5$	$\frac{-tz^2(1-z^2)}{t^2z^4-(z^2-1)^2}$	$\frac{-t}{1-t^2} + \frac{1}{2(1+t)} \cdot \frac{t}{1-z^2(1+t)} + \frac{1}{2(1-t)} \cdot \frac{t}{1-z^2(1-t)}$	$\frac{-t}{1-t^2} + \frac{tC^+}{2(1+t)} + \dots$

TABLE 5.1. Calculations for determining $\Phi((I+N)^{-1})$ in the capacity 2 bin case

yields the desired result, after more calculations. (In particular, we had to multiply the expression by t, since as stated the first bin never got its weight. As in the capacity 1 case, we must add 1 to conform to our base case convention.) □

In terms of our (abbreviations for) exponentials, the generating function (minus its base case) has numerator

$$2(1-t)^2(p-1)^2(1+n)^2+2\sqrt{1-t^2}(p^2-1)(1-n^2)+t(1-t)\sqrt{1+t}(n+1)^2(p^2-1) \\ +\sqrt{1+t}(p^2-1)\left[2t(n^2+1)-4n\right]+(1+t)\sqrt{1-t}(1-n^2)(-t(p-1)^2-4p)$$

and denominator

$$(1-t^2)[(1+pn)^2+(p+n)^2]+3\sqrt{1-t^2}(1-n^2)(1-p^2)+ \\ 2\sqrt{1-t}\left[(1-n^2)(1+p^2)+2tp(1-n^2)\right]+2\sqrt{1+t}[(1-p^2)(1+n^2)-2tn(1-p^2)]\,.$$

Expanding, the first few terms are

$$1+tz+tz^2+\left(\frac{5}{6}t+\frac{1}{6}t^2\right)z^3+\left(\frac{12}{24}t+\frac{12}{24}t^2\right)z^4 \\ +\left(\frac{27}{120}t+\frac{87}{120}t^2+\frac{6}{120}t^3\right)z^5+\left(\frac{58}{720}t+\frac{520}{720}t^2+\frac{142}{720}t^3\right)z^6 \\ +\left(\frac{121}{5040}t+\frac{2803}{5040}t^2+\frac{2045}{5040}t^3+\frac{71}{5040}t^4\right)z^7+\cdots$$

For the capacity 3 case, we produce the digraph and its adjacency matrix, but do not explicitly present the generating function.

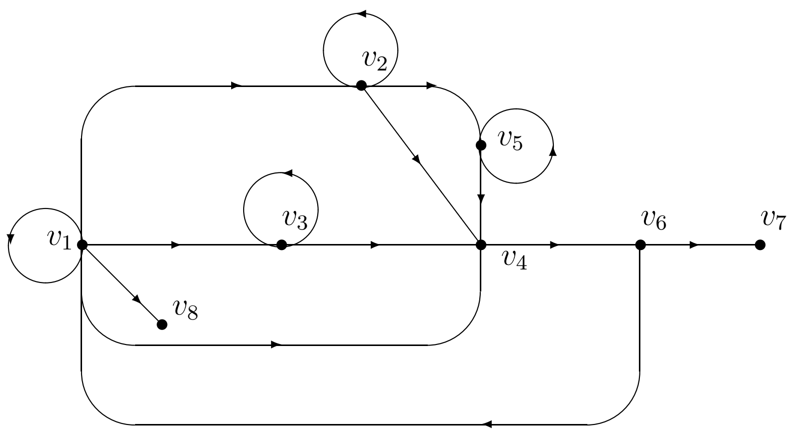
We write each 3-marked shape in the form given by (25):

$$\left((d^*a)^2d^*\dot{a}(a^*d)^2a^*\dot{d}\right)^*=\left(d^*ad^*ad^*\dot{a}a^*da^*da^*\dot{d}\right)^*$$

As we did for capacity 2, we rewrite this according to the lengths of the maximal increasing runs. We omit the details of the analysis, and present the following pattern of run lengths. The bold face numbers represent runs that include $\dot{a}$ or are ended by $\dot{d}$, and hence result in the start of a new bin.

$$1^* \left\{ \begin{array}{lll} 2 & 1^* & \left\{ \begin{array}{lll} 2 & 1^* & \geq \mathbf{2} \\ \geq \mathbf{3} & & \end{array} \right. \\ 3 & 1^* & \geq \mathbf{2} \\ \geq \mathbf{4} & & \end{array} \right\} \geq 1 \quad \geq \mathbf{1}$$

We can convert this information into a digraph. As before, we need to consider how the final truncation of (25) changes which runs represent a new bin, and modify the digraph accordingly. This results in the digraph and adjacency matrix pictured in Figure 5-3. Each sequence of maximal increasing run lengths, weighted according to run length and where the bin lines are drawn, corresponds to a unique path in this digraph, starting at v_1 and ending at any other vertex.



$$\begin{pmatrix} z & z^2 & z^3 & \frac{tz^4}{1-z} & 0 & 0 & 0 & z \\ 0 & z & 0 & \frac{tz^3}{1-z} & z^2 & 0 & 0 & 0 \\ 0 & 0 & z & \frac{tz^2}{1-z} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{z}{1-z} & 0 & 0 \\ 0 & 0 & 0 & \frac{tz^2}{1-z} & z & 0 & 0 & 0 \\ \frac{tz}{1-z} & 0 & 0 & 0 & 0 & 0 & \frac{z}{1-z} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

FIGURE 5-3. The capacity 3 digraph and its adjacency matrix

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