Applications of Computer Algebra to the Theory of Hypergeometric Series

A Dissertation

Presented to

The Faculty of the Graduate School of Arts and Sciences

Brandeis University

Department of Mathematics

Professor Ira Gessel, Advisor

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

by

Ping Zhou

May, 2003

This dissertation, directed and approved by **Ping Zhou**'s Committee, has been accepted and approved by the Faculty of Brandeis University in partial fulfillment of the requirements for the degree of :

DOCTOR OF PHILOSOPHY

____

Dean of Arts and Sciences

Dissertation Committee:

____

Ira Gessel, Mathematics

____

Susan Parker, Mathematics

____

Martin Cohn, Computer Science, Brandeis University

Acknowledgements

I would like to express my deepest gratitude to Ira Gessel, my advisor, for his mathematical insight and his guidance, and for the tremendous patience he has shown me and the enormous amount of time he has shared with me so generously. Without his help, I do not think I could even come close to finishing this work.

I thank my family for always being there for me and would like to dedicate this thesis to my parents.

ABSTRACT

Applications of Computer Algebra to the Theory of Hypergeometric Series

A dissertation presented to the faculty of
the Graduate School of Arts and Sciences of
Brandeis University, Waltham, Massachusetts

by Ping Zhou

This thesis consists of two parts which both deal with the application of computers to the theory of hypergeometric series. In the first part of this thesis we study how symbolic computational software, like Maple, can be used to generate hypergeometric transformations systematically. Based on the observation that $\sum_i\Big(\sum_j a_{ij}\Big) = \sum_j\Big(\sum_i a_{ij}\Big)$, we generate double sums which both inner sums can be evaluated by known hypergeometric summation theorems. In a similar way, we generate transformations for two-variable hypergeometric series. In the second part we focus on the WZ(Wilf-Zeilberger) method. We use a WZ pair to assign weight to a step and derive the path independence theorem which states that sum of the weights along paths depend only on the endpoints. We derive the change of variable theorem. Then we give some applications of path independence theorem. We also extend the WZ method to Euler's and Pfaff's transformations. We generalized the application of the path independence theorem from hypergeometric functions to symmetric functions in the end.

Contents

Chapter 0. Introduction

In chapter 1 we look at double sums of the form

$$\sum_{i,j} \frac{(a_1)_{p_1 i+q_1 j} \cdots (a_n)_{p_n i+q_n j}}{i!j!} {c_1}^i {c_2}^j.$$

We associate such a double sum with a multiset $\{(p_1, q_1), \cdots (p_n, q_n)\}$, which we call the partition associated with the sum. We study the use of Maple to generate all the partitions whose corresponding double sums can be evaluated when first summed on i and when first summed on j. This gives us transformations for hypergeometric series. In chapter 2, we start with a hypergeometric series with two variables, say $f(x, y) = \sum a_{ij} x^i y^j$, then we do a simple transformation on the variables, like $(1-x)^p(1-y)^q f(\frac{x}{1-x}, \frac{y}{1-y})$ and express the coefficient of $x^i y^j$ as hypergeometric series. We then evaluate the hypergeometric series by known hypergeometric summation theorems and obtain a hypergeometric transformations in two variables.

In chapter 3, we study WZ forms for hypergeometric series. In particular, we derive the path independence theorem for WZ forms and give some applications of the path independence theorem. We prove the change of variables theorem for WZ forms which is a straightforward consequence of the path independence theorem. We show how the change of variables theorem gives the connection between various forms of a hypergeometric evaluation and its WZ forms. We also extend the WZ method to Euler's and Pfaff's transformations. In the end of chapter 3, we generalize the application of the path independence theorem from hypergeometric functions to symmetric functions.

Chapter 1. Hypergeometric Transformations from Double Summation

§1.1 **Introduction**

A hypergeometric series $\sum_{i\geq 0} a_i$ is a series in which the ratio of every two consecutive terms, $\frac{a_{i+1}}{a_i}$, is a rational function of the summation index i. If we define a rising factorial in a to be

$$(a)_n = \begin{cases} a(a+1)\cdots(a+n-1), & \text{if } n > 0; \\ \dfrac{(-1)^n}{(1-a)_{-n}}, & \text{if } n < 0; \\ 1, & \text{if } n = 0 \end{cases}$$

then obviously, the following is a hypergeometric series

$$\sum_{i=0}^{\infty} \frac{(a_1)_i \cdots (a_n)_i}{i!(b_1)_i \cdots (b_m)_i} v^i, \tag{1.1.1}$$

and we denote it by

$${}_nF_m \left(\begin{matrix} a_1, \cdots, a_n \\ b_1, \cdots, b_m \end{matrix} \middle| v \right). \tag{1.1.2}$$

We call $a_1, \cdots, a_n$ its numerator parameters, $b_1, \cdots, b_m$ its denominator parameters, and z its argument. Note that

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$$

as long as a is not a negative integer. There are some well-known hypergeometric identities such as the binomial theorem

$${}_1F_0 \left(\begin{matrix} a \\ - \end{matrix} \middle| x \right) = (1-x)^{-a},$$

Gauss's theorem

$${}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| 1 \right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

where $R(c-a-b) > 0$, or a or b is a nonpositive integer and Saalschütz's theorem

$${}_3F_2 \left(\begin{matrix} a, b, -n \\ c, d \end{matrix} \middle| 1 \right) = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n},$$

where $a+b-n+1 = c+d$, and n is a nonnegative integer. In Saalschütz's theorem the hypergeometric series is said to be balanced since its parameters satisfy the relation $a+b-n+1 = c+d$ [3, (2.5)]. An identity which transforms one hypergeometric series to another one is called a hypergeometric transformation. There are many ways to get hypergeometric transformations. One of the most powerful techniques is based on the observation that

$$\sum_i \Big(\sum_j a_{ij} \Big) = \sum_j \Big(\sum_i a_{ij} \Big).$$

This technique has been used widely in the theory of hypergeometric series and various authors have studied this method somewhat systematically [3, 4], but have restricted their attention to what appeared to be the most promising cases, because looking at all possible cases would be too tedious. However, the computer, more specifically the mathematical symbolic computation software such as Maple, makes it possible for us to consider all possibilities. It can help generate all possible double sums, examine each one of them and evaluate it if possible. For example, we can come up with a Maple program to systematically apply the method of double summation in the cases in which one side is evaluated by Gauss's, the binomial or Saalschütz's theorem and the other side is evaluated by one of these theorems as well.

§1.2 Deriving hypergeometric series by double summation

We start with a double sum on i and j of the form

$$\sum_{i,j} \frac{(a_1)_{p_1 i+q_1 j} \cdots (a_n)_{p_n i+q_n j}}{i!j!} {c_1}^i {c_2}^j \tag{1.2.1}$$

where the p_l and q_l are integers (positive, negative, or zero) with p_l and q_l not both zero. Such a sum is determined (up to a change of the names of the parameters) by the multiset $\{(p_1, q_1), \ldots, (p_n, q_n)\}$, which we call the *partition* associated to the sum and each (p_i, q_j) is called a *part* belonging to the partition. We use $P(s_1, s_2; t_1, t_2)$ to represent the set of partitions, which satisfy the following conditions:

(1) the sum of the positive p_l is s_1,

(2) the sum of the negative p_l is $-s_2$,

(3) the sum of the positive q_l is t_1,

(4) the sum of the negative q_l is $-t_2$.

For such a partition we defined its *cutoff* to be $[s_1, s_2, t_1, t_2]$. Now we take an arbitrary double sum of the form (1.1.1) which corresponds to a partition belonging to $P(s_1, s_2; t_1, t_2)$. For such a double sum if we first sum on i we get a hypergeometric series in i

$$\sum_j \frac{(a_1)_{q_1 j} \cdots (a_n)_{q_n j} {c_2}^j}{j!} \left(\sum_i \frac{(a_1 + q_1 j)_{p_1 i} \cdots (a_n + q_n j)_{p_n i} {c_1}^i}{i!} \right). \tag{1.2.2}$$

Notice that we have

$$(a)_{pi} = p^{pi} (\frac{a}{p})_i \cdots (\frac{a+p-1}{p})_i,$$

when p is a positive integer and

$$(a)_{pi} = \frac{p^{pi}}{(\frac{1-a}{-p})_i \cdots (\frac{-p-a}{-p})_i}$$

when p is a negative integer. So the number of numerator parameters of (1.2.2) is s_1 and the number of denominator parameters is s_2. If we want v to be the argument of this hypergeometric series we need to have $c_1 = {p_1}^{-p_1} \cdots p_n^{-p_n} v$. In particular if we want this summation to be a ${}_1F_0(v)$, which can be evaluated by the binomial theorem, we need to have $s_1 = 1, s_2 = 0$ and $c_1 = {p_1}^{-p_1} \cdots p_n^{-p_n} v$. On the other hand if we first sum on j and want to have ${}_2F_1$ which can be evaluated by Gauss's theorem we need to have $t_1 = 2, t_2 = 1$ and $c_2 = {q_1}^{-q_1} \cdots q_n^{-q_n}$. We summarize what we have here in the following claim.

Claim 1. *The double sum* (1.2.1) *has the property that summing first on the variable i results in a ${}_1F_0(v)$, while summing on the variable j results in a ${}_2F_1(1)$ if and only if the corresponding partition is in $P(1,0;2,1)$ and $c_1 = p_1^{-p_1} \cdots p_n^{-p_n} v$ and $c_2 = q_1^{-q_1} \cdots q_n^{-q_n}$.*

With this observation we can generate hypergeometric transformations systematically with the binomial theorem and Gauss's theorem. The procedure is straightforward. We first generate the set $P(1,0;2,0)$. Then for each partition we form a double sum of the form (1.2.1) and with $c_1 = p_1^{-p_1} \cdots p_n^{-p_n} v$ and $c_2 = q_1^{-q_1} \cdots q_n^{-q_n}$. Finally, we sum the double sum first on i and evaluate the summation by the binomial theorem to get the left-hand side of the transformation, then sum the double sum first on j and evaluate the summation by Gauss's theorem to get the right-hand side of the transformation. Here to avoid any convergence problem, we assume that we are dealing with formal power series, and when applying Gauss's theorem, we need one of the numerator parameters to be non-positive for the series to terminate. For example, we take the partition $\{(1,-1),(0,1),(0,1)\}$ and form the double sum

$$\sum_{i,j} (a)_{i-j} (b)_j (c)_j \frac{(-1)^j \, v^i}{i! \, j!}. \tag{1.2.3}$$

The sum on i of the summand of (1.2.3) is

$$\frac{(a)_{-j}(-1)^j (b)_j (c)_j}{j!} {}_1F_0 \left(\begin{matrix} a-j \\ - \end{matrix} \middle| v \right),$$

which by the binomial theorem may be written as

$$(a)_{-j}(-1)^j (b)_j (c)_j (1-v)^{-a+j}.$$

Thus (1.2.3) is equal to

$$(1-v)^{-a} {}_2F_1 \left(\begin{matrix} b, c \\ 1-a \end{matrix} \middle| 1-v \right)$$

If we sum (1.2.3) first on j we get

$$\frac{v^i (a)_i}{i!} {}_2F_1 \left(\begin{matrix} b, c \\ 1-a-i \end{matrix} \middle| 1 \right). \tag{1.2.4}$$

This ${}_2F_1$ converges for all i if and only if b or c is a nonpositive integer. In this case let's set $b = -n$, where n is a nonnegative integer. By Gauss's theorem (1.2.4) may be written as

$$v^i (a)_i \frac{\Gamma(1-a-i)\Gamma(1-a-i-c+n)}{\Gamma(1-a-i+n)\Gamma(1-a-i-c)}.$$

Thus (1.2.3) is also equal to

$$\frac{\Gamma(1-a)\Gamma(1-a-b+n)}{\Gamma(1-a+n)\Gamma(1-a-c)} {}_2F_1 \left(\begin{matrix} a-n, a+c \\ a+c-n \end{matrix} \middle| v \right),$$

and we get a well known linear transformation

$$
\begin{aligned}
(1-v)^{-a}{}_2F_1\left(\begin{matrix}-n, c\\ 1-a\end{matrix}\middle|\, 1-v\right) &= \frac{\Gamma(1-a)\Gamma(1-a-c+n)}{\Gamma(1-a+n)\Gamma(1-a-c)}{}_2F_1\left(\begin{matrix}a-n, a+c\\ a+c-n\end{matrix}\middle|\, v\right)\\
&= \frac{(1-a-c)_n}{(1-a)_n}{}_2F_1\left(\begin{matrix}a-n, a+c\\ a+c-n\end{matrix}\middle|\, v\right).
\end{aligned}
$$

§1.3 Generation of hypergeometric transformations via Maple

As we mentioned before, mathematical symbolic computation software will be very helpful to get all the transformations which can be derived by the method we described in the previous sections. Now we are going to see how we can use Maple to carry out the procedures to generate the hypergeometric transformations. We will always start with a cutoff and a set of parts, and generate the set of partitions from them. The set of parts should include all possible parts that could appear in partitions with the given cutoff. One observation is that the double sum formed by a partition with a part such as $(2,0)$ is a special case of the double sum formed by the same partition with $(2,0)$ replaced by $(1,0),(1,0)$. Therefore in the parts set there is no need to include parts $(p,0)$, where the absolute value of p is not 1. A similar argument results in excluding $(0,q)$ from the parts set if the absolute value of q is not 1. Considering that we are only using the binomial theorem, Gauss's theorem, and Saalschütz's theorem we come up with the following parts set:

$$\begin{aligned}&\{(-2,-2),(-2,-1),(-2,1),(-2,2),(-2,3),(-1,-2),\\&(-1,-1),(-1,0),(-1,1),(-1,2),(-1,3),(0,-1),(0,1),(1,-2),\\&(1,-1),(1,0),(1,1),(1,2),(1,3),(2,-2),(2,-1),(2,1),(2,2),\\&(2,3),(3,-2),(3,-1),(3,1),(3,2),(3,3)\}.\end{aligned}$$

With this parts set, for a given cutoff $[s_1, s_2, t_1, t_2]$, we call a procedure `createpartitions` to generate the set $P(s_1, s_2; t_1, t_2)$. The returned partition set is stored in a global variable `fullpartitions` which is a Maple list structure. Notice that a partition produces a trivial identity if every part of it has zero as one of the entries, i.e., it is of the form either $(p,0)$ or $(0,q)$. Such partitions are eliminated by `createpartitions`. There is usually some more work needed to refine the partition set `fullpartitions.` Such work depends on the theorems we apply later. For example, if we want to apply Gauss's theorem twice, the cutoff would be $[2,1,2,1]$. Here, every partition is symmetric to another partition (could be itself) in the sense that if we switch the two entries in all the parts of the first partition we get the second partition. Two partitions which are symmetric to each other produce the same transformations, so we need to get rid of one of the two. We call a procedure `remove_symmetry` for this purpose. On the other hand, if we want to apply Saalschütz's theorem once and then Gauss's theorem once, then we do not have the symmetry problem since the cutoff is $[3,2,2,1]$. Yet, we run into another problem due to the fact that the parameters of the hypergeometric series that we want to evaluate by Saalschütz's theorem have to be balanced, which we mentioned in section one. We can express one of the numerator or denominator parameters in terms of the other parameters, but some partitions generate an expression which contains the other summation index. For example, partition $((-1,2),(1,-1),(1,0),(1,0),(-1,0))$ has cutoff $[3,2,2,1]$. We form a double sum of the form (1.2.1) from this partition and first sum on i we get

$$\frac{(a)_{-i+2\,j}(b)_{i-j}(c)_i(d)_i(e)_{-i}(-1)^{2i+j}(2)^{-2j}}{i!\,j!}.$$

If we first sum this sum on i we get

$$\sum_j \frac{(a)_{2\,j}\left(\frac{1}{4}\right)^j(b)_{-j}\,(-1)^j}{j!}\,{}_3F_2\left(\begin{matrix} b-j,c,d \\ 1-a-2\,j,1-e\end{matrix}\middle|1\right).$$

In order to use Saalschütz's theorem the parameters of ${}_3F_2$ needs to satisfy the condition

$$b-j+c+d+1=1-a-2j+1-e.$$

Since condition is dependent on j it cannot produce a valid variable substitution expression. If the coefficient of j on the left hand side of the equation is equal to the coeffient of j on the right hand side, then we can get a variable substition expression independent of j. In fact the coefficient on the left hand side is the sum of the second tuple for all parts whose first tuple is positive, while the coefficient on the right hand side is the negative of the sum of the second tuple for all parts whose first tuple is negative. In general, a partition P from $P(3,2;2,1)$ produces a valid substitution expression if and only if the terms containing the summation index cancel each other, i.e.:

$$\sum_{(p,q)\in P, p>0} q = \sum_{(p,q)\in P, p<0} (-q).$$

We call a procedure `select_balanced` to just include such good partitions in the `fullpartitions` list. The following is the Maple output when calling some of the procedures we just described.

```
> cutoff := [2, 1, 2, 1]

> createpartitions(cutoff);

> nops(fullpartitions);

24

> remove_symmetry();
```

[[[−1, 2], [0, −1], [1, 0], [1, 0]], [[−1, −1], [0, 1], [0, 1], [1, 0], [1, 0]],

[[−1, 1], [0, −1], [0, 1], [1, 0], [1, 0]], [[−1, 2], [1, −1], [1, 0]], [[−1, 1], [0, 1], [1, −1], [1, 0]],

[[−1, −1], [1, 1], [1, 1]], [[−1, 0], [0, −1], [1, 1], [1, 1]], [[−1, 1], [1, −1], [1, 1]],

[[−1, 0], [0, 1], [1, −1], [1, 1]], [[−1, −1], [0, 1], [1, 0], [1, 1]],

[[−1, 0], [0, −1], [0, 1], [1, 0], [1, 1]], [[−1, 0], [1, −1], [1, 2]], [[−1, −1], [1, 0], [1, 2]],

[[−1, 0], [0, −1], [1, 0], [1, 2]], [[−1, 2], [2, −1]], [[−1, −1], [2, 2]], [[−1, 0], [0, −1], [2, 2]]]

```
>nops(fullpartitions);
17
```

For every partition in `fullpartitions`, we call a procedure, whose definition depends on the theorems we want to apply, to generate the transformations. We have written `bg_get_iden, bs_get_iden, gg_get_iden` and `sg_get_iden.` The first two letters in these procedures provide the information about the theorems used to evaluate the two hypergeometric series. Obviously, `b` is for the binomial theorem, `g` is for Gauss's theorem and `s` is for Saalschütz's theorem. These procedures are quite similar to one another since they all follow the procedures described in the previous section to derive the transformation. We now pick one procedure, `gg_get_iden`, to illustrate how the transformation is derived by these procedures. Given a partition, the first thing `gg_get_iden` does is to pick parts which correspond to parameters making the double sum terminating. These parameters have to be negative integers and are denoted by $-n$ or $-m$. The procedure to do this is called `lookup_part`. In the case that the hypergeometric series obtained by summing on i is evaluated by the binomial theorem while the other one obtained by summing on j is evaluated by Gauss's or Saalschütz's theorem, since we assume the series is a formal power series, we only need to pick a terminating parameter for the second series. In the case that only Gauss's or Saalschütz's theorem will be applied we need to pick terminating parameters to make both series terminate. Moreover, we also want the original double sum terminating so that we can switch the order of the summation. The observation is that if we pick a terminating parameter whose part has both entry positive then the original double sum terminating. This is the most simple case. Now we look at other choices we may have. A parameter which corresponds to a part, say (p, q) can make the series which is obtained by summing on i first terminating, as far as $p > 0$ and q can be any nonpositive integer. But to make the series which is obtained by summing on j first terminating we need to pick another part say (p_1, q_1) where p_1 is any nonpositive integer and $q_1 > 0$. Moreover to make the whole double sum terminating we need $|pq_1| > |p_1 q|$. With `lookup_part` we can specify, via passing an argument *opt*, if we want the picked parts to make the first series terminate ($opt = 1$), or the second series terminate ($opt = 2$), or the double sum terminate ($opt = 0$). In `gg_get_iden` we want `lookup_part` to return the index of parts which make the double sum terminate. Here is a Maple output from `lookup_part`.

```
> p:=fullpartitions[3];
p := [[1, -2], [-2, 1]]
> lookup_part(p,0);
```

[]

In this case, we get the following double sum corresponding to p

$$\sum_{i,j} \frac{(a)_{i-2j}(b)_{-2i+j} 2^{2i}\, 2^{2j}}{i!\, j!}.$$

Since we want to get a terminating hypergeometric series summing first on the first index we need to set a a negative integer, and we also need to set b a negative integer to make the series summing first on the second index terminating. So we get something as below

$$\sum_{i,j} \frac{(-n)_{i-2j}(-m)_{-2i+j} 2^{2i}\, 2^{2j}}{i!\, j!}.$$

Yet, since these two terminating parts donot satisfy the condition we mentioned earlier there are infinitely many of i, j which makes the summand nonezero, so the double sum itself is non-terminating. Therefore there is no way to get a non-terminating series out of this partition.

```
> p:=fullpartitions[4];
```

p := $[[-1,-1],[0,1],[0,1],[1,0],[1,0]]$

```
> lookup_part(p,0);
```

```
[[4, 2]]
```

This means the there is only one way to make the double sum formed from this partition terminating. That is setting both the parameter corresponding to the 4th part and that corresponding to the second part to negative integers.

```
> p:=fullpartitions[12];
```

p := $[[-1,0],[0,1],[1,-1],[1,1]]$

```
> lookup_part(p,0);
```

```
[[4], [2, 3]]
```

This means that there are two ways for the double sum formed from this partition to terminate. One way is to set the parameter corresponding to the 4th part a negative integer. The other way is to set both the parameter corresponding to the second part and that corresponding to the 3rd part negative integers.

For each choice of terminating parameters we call `doublesum(L)` to form a double sum of the form (1.1.1) with the terminating parameters represented by $-n$ or $-m$. In this procedure if the first entry of a part is negative then we put the rising factorial for that part on the denominator.

We then call `toF(summand, i)` to convert hypergeometric series obtained by i into the form defined by (1.1.2) and procedure `gauss` evaluates this hypergeometric series by Gauss's theorem. We use `hypercombine` to convert the summand to rising factorials in j and combine factorials in fractional j, if such factorials exist, to one factorial in j if possible. In the end `tor` is called to convert the `summand` to be a product of rising factorials in n or m. In this way, we obtain one side of the transformation and the other side is obtained in a similar manner.

The following shows how all the procedures can be used to generate transformations from a partition. Note that " in Maple denotes the result of the last calculation.

```
> var:=[a,b,c,d,e,f,g,h];
```

> part:= $[[-1,0],[0,1],[1,-1],[1,1]]$:

```
> lookup_part(",0);
[[4], [2, 3]]
> doublesum(part);
```

$$\frac{(-1)^j\,(b)_j\,(c)_{i-j}\,(d)_{i+j}}{i!\,j!\,(a)_i}$$

```
> d_sum:=subs(var[4]=-n, ");
```

$$d\_sum := \frac{(-1)^j\,(b)_j\,(c)_{i-j}\,(-n)_{i+j}}{i!\,j!\,(a)_i}$$

```
> L0:=toF(d_sum, j);
```

$$L0 := \frac{(c)_i(-n)_i F([b,-n+i],[1-c-i],1)}{i!(a)_i}$$

```
> L1:=gauss(L0);
```

$$L1 := \frac{(c)_i(-n)_i\Gamma(1-c-i)\Gamma(1-c-2i-b+n)}{i!(a)_i\Gamma(1-c-i-b)\Gamma(1-c-2i+n)}$$

```
> L2:=toF(L1,i):
> L3:=tor(",n);
```

$$L3 := \frac{(1-c-b)_n}{(1-c)_n}F([-n,c+b,\frac{1}{2}c-\frac{1}{2}n,\frac{1}{2}c-\frac{1}{2}n+\frac{1}{2}],[a,\frac{1}{2}c+\frac{1}{2}b-\frac{1}{2}n,\frac{1}{2}c+\frac{1}{2}b-\frac{1}{2}n+\frac{1}{2}],1)$$

```
> R0:=toF(d_sum,i);
```

$$R0 := \frac{(-1)^j\,(b)_j\,(c)_{-j}\,(-n)_j}{j!}F([c-j,-n+j],[a],1)$$

```
> gauss(");
```

$$\frac{(-1)^j\,(b)_j\,(c)_{-j}\,(-n)_j\Gamma(a)\Gamma(a-c+n)}{j!\Gamma(a-c+j)\Gamma(a+n-j)}$$

```
> toF(",j);
```

$$\frac{\Gamma(a)\Gamma(a-c+n)}{\Gamma(a-c)\Gamma(a+n)}F([b,-n,1-a-n],[1-c,a-c],-1)$$

```
> tor(",n);
```

$$\frac{(a-c)_n}{(a)_n}F([b,-n,1-a-n],[1-c,a-c],-1)$$

So the hypergeometric transformation we got here is

$\{(-1,0),(0,1),(1,-1),(1,1)\}$

$$\frac{(1-c-b)_n}{(1-c)_n}\,{}_4F_3\left(\left.\begin{matrix}-n, c+b, \frac{c}{2}-\frac{n}{2}, \frac{c}{2}-\frac{n}{2}+\frac{1}{2}\\ a, \frac{c}{2}+\frac{b}{2}-\frac{n}{2}, \frac{c}{2}+\frac{b}{2}-\frac{n}{2}+\frac{1}{2}\end{matrix}\right|1\right)$$
$$=\frac{(a-c)_n}{(a)_n}\,{}_3F_2\left(\left.\begin{matrix}b,-n,1-a-n\\ 1-c,a-c\end{matrix}\right|-1\right). \qquad (1.3.1)$$

The hypergeometric transformation we get in the same way from setting the second and third parameters nonpositive is

$\{(-1,0),(0,1),(1,-1),(1,1)\}$

$$\frac{(1+m-d)_n}{(1+m)_n}\,{}_4F_3\left(\left.\begin{matrix}d, -m-n, -\frac{m}{2}+\frac{d}{2}, -\frac{m}{2}+\frac{d}{2}+\frac{1}{2}\\ 1-a, -\frac{m}{2}-\frac{n}{2}+\frac{d}{2}, -\frac{m}{2}-\frac{n}{2}+\frac{d}{2}+\frac{1}{2}\end{matrix}\right|1\right)$$
$$=\frac{(1-a-d)_m}{(1-a)_m}\,{}_3F_2\left(\left.\begin{matrix}-n,d,a+d\\ 1+m,1-a+m\end{matrix}\right|-1\right) \qquad (1.3.2)$$

§1.4 Transformations Generated by the Maple Program

Bailey [3] obtained some hypergeometric identities by considering the product of two hypergeometric series. So far our Maple program has produced all the transformations that he got by applying Gauss's and Saalschütz's theorem and the corresponding formula numbers in his paper have been attached to each of them.

The following are transformations from Gauss's and the binomial theorem.

From $P(1,0;2,1)$:

$\{(1,-1),(0,1),(0,1)\}$

$$\frac{(1-a-c)_n}{(1-a)_n}\,{}_2F_1\left(\begin{matrix}a-n,a+c\\a-n+c\end{matrix}\middle|v\right)=(1-v)^{-a}\,{}_2F_1\left(\begin{matrix}-n,c\\1-a\end{matrix}\middle|1-v\right)$$

$\{(1,1),(0,1),(0,-1)\}$

$$\frac{(c-a)_n}{(c)_n}\,{}_2F_1\left(\begin{matrix}a,1-c+a\\1-c+a-n\end{matrix}\middle|v\right)=(1-v)^{-a}\,{}_2F_1\left(\begin{matrix}a,-n\\c\end{matrix}\middle|(1-v)^{-1}\right)$$

Transformations from Gauss's theorem.

From $P(2,1;2,1)$:

$\{(-1,-1),(0,1),(0,1),(1,0),(1,0)\}$

$$\frac{(a-c)_m}{(a)_m}\,{}_3F_2\left(\begin{matrix}-n,e,a+m-c\\a+m,a-c\end{matrix}\middle|1\right)=\frac{(a-e)_n}{(a)_n}\,{}_3F_2\left(\begin{matrix}-m,c,a+n-e\\a+n,a-e\end{matrix}\middle|1\right)\tag{1.4.1}$$

$\{(-1,1),(0,-1),(0,1),(1,0),(1,0)\}$

$$\frac{(b-1+a)_m}{(b)_m}\,{}_3F_2\left(\begin{matrix}-n,e,b-1+a+m\\a,b-1+a\end{matrix}\middle|1\right)=\frac{(a-e)_n}{(a)_n}\,{}_3F_2\left(\begin{matrix}-m,1-a-n,1-a+e\\b,1-a-n+e\end{matrix}\middle|1\right)\tag{1.4.2}$$

$\{(-1,0),(0,1),(0,1),(1,-1),(1,0)\}$

$$\frac{(1-d-c)_m}{(1-d)_m}\,{}_3F_2\left(\begin{matrix}-n,d-m,d+c\\a,d-m+c\end{matrix}\middle|1\right)=\frac{(a-d)_n}{(a)_n}\,{}_3F_2\left(\begin{matrix}-m,c,a-d+n\\1-d,a-d\end{matrix}\middle|1\right)\tag{1.4.3}$$

$\{(-1,0),(0,-1),(1,1),(1,1)\}$

$$\frac{(b-d)_n}{(b)_n}\,{}_4F_3\left(\begin{matrix}-n,d,1-b-n,1-b+d\\a,\frac{1}{2}-\frac{b}{2}-\frac{n}{2}+\frac{d}{2},1-\frac{b}{2}-\frac{n}{2}+\frac{d}{2}\end{matrix}\middle|\frac{1}{4}\right)$$
$$=\frac{(a-d)_n}{(a)_n}\,{}_4F_3\left(\begin{matrix}-n,d,1-a-n,1-a+d\\b,\frac{1}{2}-\frac{a}{2}+\frac{d}{2}-\frac{n}{2},1-\frac{a}{2}+\frac{d}{2}-\frac{n}{2}\end{matrix}\middle|\frac{1}{4}\right)\tag{1.4.4}$$

$\{(-1,1),(1,-1),(1,1)\}$

$$\frac{1}{(1-b)_n}\,{}_3F_2\left(\begin{matrix}-n,\frac{b}{2}-\frac{n}{2},\frac{b}{2}-\frac{n}{2}+\frac{1}{2}\\a,1-a+b-n\end{matrix}\middle|4\right)$$
$$=\frac{1}{(a)_n}\,{}_3F_2\left(\begin{matrix}-n,\frac{1}{2}-\frac{a}{2}-\frac{n}{2},1-\frac{a}{2}-\frac{n}{2}\\1-b,1-a+b-n\end{matrix}\middle|4\right)\tag{1.4.5}$$

$\{(-1,0),(0,1),(1,-1),(1,1)\}$

$$\frac{(1-c-b)_n}{(1-c)_n}\,{}_4F_3\left(\left.\begin{matrix}-n, c+b, \frac{c}{2}-\frac{n}{2}, \frac{c}{2}-\frac{n}{2}+\frac{1}{2}\\ a, \frac{c}{2}+\frac{b}{2}-\frac{n}{2}, \frac{c}{2}+\frac{b}{2}-\frac{n}{2}+\frac{1}{2}\end{matrix}\right|1\right) = \frac{(a-c)_n}{(a)_n}\,{}_3F_2\left(\left.\begin{matrix}b, -n, 1-a-n\\ 1-c, a-c\end{matrix}\right|-1\right) \tag{1.4.6}$$

$\{(-1,1),(0,-1),(1,0),(1,1)\}$

$$\frac{(b-1+a)_n}{(b)_n}\,{}_3F_2\left(\left.\begin{matrix}c, -n, 1-b-n\\ a, b-1+a\end{matrix}\right|-1\right) = \frac{(a-c)_n}{(a)_n}\,{}_4F_3\left(\left.\begin{matrix}-n, 1-a+c, \frac{1}{2}-\frac{a}{2}-\frac{n}{2}, 1-\frac{a}{2}-\frac{n}{2}\\ b, \frac{1}{2}-\frac{a}{2}+\frac{c}{2}-\frac{n}{2}, 1-\frac{a}{2}+\frac{c}{2}-\frac{n}{2}\end{matrix}\right|1\right) \tag{1.4.7}$$

$\{(-1,0),(0,-1),(0,1),(1,0),(1,1)\}$

$$\frac{(b-c)_n}{(b)_n}\,{}_3F_2\left(\left.\begin{matrix}d, -n, 1-b-n\\ a, 1-b+c-n\end{matrix}\right|1\right) = \frac{(a-d)_n}{(a)_n}\,{}_3F_2\left(\left.\begin{matrix}c, -n, 1-a-n\\ b, 1-a+d-n\end{matrix}\right|1\right) \tag{1.4.8}$$

$\{(-1,0),(0,-1),(0,1),(1,0),(1,1)\}$

$$\frac{(b-e)_m}{(b)_m}\,{}_3F_2\left(\left.\begin{matrix}-n, e, 1-b+e\\ a, 1-b-m+e\end{matrix}\right|1\right) = \frac{(a-e)_n}{(a)_n}\,{}_3F_2\left(\left.\begin{matrix}-m, e, 1-a+e\\ b, 1-a-n+e\end{matrix}\right|1\right) \tag{1.4.9}$$

$\{(-1,0),(1,-1),(1,2)\}$

$$\frac{(\frac{1}{2}-b)_n 2^n}{(1-2\,b)_n}\,{}_4F_3\left(\left.\begin{matrix}-n, \frac{2\,b}{3}-\frac{n}{3}, \frac{2\,b}{3}-\frac{n}{3}+\frac{1}{3}, \frac{2\,b}{3}-\frac{n}{3}+\frac{2}{3}\\ a, \frac{1}{4}+\frac{b}{2}-\frac{n}{2}, \frac{3}{4}+\frac{b}{2}-\frac{n}{2}\end{matrix}\right|\frac{27}{32}\right) = \frac{(a-b)_n}{(a)_n}\,{}_4F_3\left(\left.\begin{matrix}-\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}, \frac{1}{2}-\frac{a}{2}-\frac{n}{2}, 1-\frac{a}{2}-\frac{n}{2}\\ 1-b, 1-a+b-n, a-b\end{matrix}\right|-4\right) \tag{1.4.10}$$

$\{(-1,-1),(1,0),(1,2)\}$

$$\frac{(a-\frac{1}{2})_n 2^n}{(2\,a-1)_n}\,{}_2F_1\left(\left.\begin{matrix}b, -n\\ 2\,a+n-1\end{matrix}\right|2\right) = \frac{(a-b)_n}{(a)_n}\,{}_3F_2\left(\left.\begin{matrix}-\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}, 1-a-n\\ 1-a+b-n, a-b\end{matrix}\right|1\right) \tag{1.4.11}$$

$\{(-1,0),(0,-1),(1,0),(1,2)\}$

$$\frac{(b-\frac{1}{2})_n 2^n}{(2\,b-1)_n}\,{}_3F_2\left(\left.\begin{matrix}c, -n, 2-2\,b-n\\ a, \frac{3}{2}-b-n\end{matrix}\right|\frac{1}{2}\right) = \frac{(a-c)_n}{(a)_n}\,{}_4F_3\left(\left.\begin{matrix}-\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}, \frac{1}{2}-\frac{a}{2}-\frac{n}{2}, 1-\frac{a}{2}-\frac{n}{2}\\ b, \frac{1}{2}-\frac{a}{2}+\frac{c}{2}-\frac{n}{2}, 1-\frac{a}{2}+\frac{c}{2}-\frac{n}{2}\end{matrix}\right|1\right) \tag{1.4.12}$$

$\{(-1,1),(0,-1),(2,1)\}$

$$\frac{(b-1+a)_n}{(b)_n}\,{}_4F_3\left(\left.\begin{matrix}-\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}, \frac{1}{2}-\frac{b}{2}-\frac{n}{2}, 1-\frac{b}{2}-\frac{n}{2}\\ a, 2-b-a-n, b-1+a\end{matrix}\right|-4\right) = \frac{(a-\frac{1}{2})_n 2^n}{(2\,a-1)_n}\,{}_4F_3\left(\left.\begin{matrix}-n, \frac{2}{3}-\frac{2\,a}{3}-\frac{n}{3}, 1-\frac{2\,a}{3}-\frac{n}{3}, \frac{4}{3}-\frac{2\,a}{3}-\frac{n}{3}\\ b, \frac{3}{4}-\frac{a}{2}-\frac{n}{2}, \frac{5}{4}-\frac{a}{2}-\frac{n}{2}\end{matrix}\right|\frac{27}{32}\right) \tag{1.4.13}$$

$\{(-1,-1),(0,1),(2,1)\}$

$$\frac{(a-b)_n}{(a)_n}\,{}_3F_2\left(\begin{matrix}-\frac{n}{2},-\frac{n}{2}+\frac{1}{2},1-a-n\\1-a+b-n,a-b\end{matrix}\middle|1\right)=\frac{(a-\frac{1}{2})_n2^n}{(2\,a-1)_n}\,{}_2F_1\left(\begin{matrix}b,-n\\2\,a+n-1\end{matrix}\middle|2\right)\tag{1.4.14}$$

$\{(-1,0),(0,-1),(0,1),(2,1)\}$

$$\frac{(b-c)_n}{(b)_n}\,{}_4F_3\left(\begin{matrix}-\frac{n}{2},-\frac{n}{2}+\frac{1}{2},\frac{1}{2}-\frac{b}{2}-\frac{n}{2},1-\frac{b}{2}-\frac{n}{2}\\a,\frac{1}{2}-\frac{b}{2}+\frac{c}{2}-\frac{n}{2},1-\frac{b}{2}+\frac{c}{2}-\frac{n}{2}\end{matrix}\middle|1\right)$$
$$=\frac{(a-\frac{1}{2})_n2^n}{(2\,a-1)_n}\,{}_3F_2\left(\begin{matrix}c,-n,2-2\,a-n\\b,\frac{3}{2}-a-n\end{matrix}\middle|\frac{1}{2}\right)\tag{1.4.15}$$

$\{(-1,0),(0,-1),(2,2)\}$

$$\frac{(b-\frac{1}{2})_n}{(2\,b-1)_n}\,{}_4F_3\left(\begin{matrix}-\frac{n}{2},-\frac{n}{2}+\frac{1}{2},1-b-\frac{n}{2},\frac{3}{2}-b-\frac{n}{2}\\a,\frac{3}{4}-\frac{b}{2}-\frac{n}{2},\frac{5}{4}-\frac{b}{2}-\frac{n}{2}\end{matrix}\middle|\frac{1}{4}\right)$$
$$=\frac{(a-\frac{1}{2})_n}{(2\,a-1)_n}\,{}_4F_3\left(\begin{matrix}-\frac{n}{2},-\frac{n}{2}+\frac{1}{2},1-a-\frac{n}{2},\frac{3}{2}-a-\frac{n}{2}\\b,\frac{3}{4}-\frac{a}{2}-\frac{n}{2},\frac{5}{4}-\frac{a}{2}-\frac{n}{2}\end{matrix}\middle|\frac{1}{4}\right)\tag{1.4.16}$$

Transformations from Saalschütz and Gauss's theorem from $P(3,2;2,1)$

$\{(-1,-1),(-1,1),(0,1),(1,0),(1,0),(1,0)\}$

$$\frac{(e+f)_m(1-e-f)_n}{(a)_m(1-e-f-m)_n}\,{}_5F_4\left(\begin{matrix}-n,e,f,-\frac{n}{2}+\frac{e}{2}+\frac{f}{2}+\frac{m}{2},-\frac{n}{2}+\frac{e}{2}+\frac{f}{2}+\frac{m}{2}+\frac{1}{2}\\-n+e+f+1-a,-\frac{n}{2}+\frac{e}{2}+\frac{f}{2},-\frac{n}{2}+\frac{e}{2}+\frac{f}{2}+\frac{1}{2},a+m\end{matrix}\middle|1\right)$$
$$=\frac{(a-e)_n(a-f)_n}{(a)_n(a-e-f)_n}\,{}_4F_3\left(\begin{matrix}-m,a-e+n,a-f+n,a-e-f\\a-e,a-f,a+n\end{matrix}\middle|1\right)\tag{1.4.17}$$

$\{(-2,1),(0,1),(1,-1),(1,0),(1,0)\}$

$$\frac{(\frac{1}{2}+e)_m(\frac{1}{2}-e)_n}{(1-c)_m(\frac{1}{2}-e-m)_n}\,{}_4F_3\left(\begin{matrix}-n,e,\frac{1}{2}-n+e+m,c-m\\\frac{c}{2}-\frac{n}{2}+\frac{e}{2}+\frac{1}{4},\frac{c}{2}-\frac{n}{2}+\frac{e}{2}+\frac{3}{4},\frac{1}{2}-n+e\end{matrix}\middle|1\right)$$
$$=\frac{(-c+e+\frac{1}{2})_n(c-e+\frac{1}{2})_n}{(-c-e+\frac{1}{2})_n(c+e+\frac{1}{2})_n}\tag{1.4.18}$$
$$\times\,{}_6F_5\left(\begin{matrix}\frac{1}{2}-c+n-e,-m,-\frac{c}{2}+\frac{n}{2}+\frac{e}{2}+\frac{1}{4},-\frac{c}{2}-\frac{n}{2}-\frac{e}{2}+\frac{1}{4},\frac{c}{2}-\frac{n}{2}+\frac{e}{2}+\frac{1}{4},\frac{c}{2}+\frac{n}{2}-\frac{e}{2}+\frac{1}{4}\\1-c,\frac{c}{2}+\frac{n}{2}+\frac{e}{2}+\frac{1}{4},\frac{c}{2}-\frac{n}{2}-\frac{e}{2}+\frac{1}{4},-\frac{c}{2}-\frac{n}{2}+\frac{e}{2}+\frac{1}{4},-\frac{c}{2}+\frac{n}{2}-\frac{e}{2}+\frac{1}{4}\end{matrix}\middle|1\right)$$

$\{(-1,0),(-1,1),(0,1),(1,-1),(1,0),(1,0)\}$

$$\frac{(1+f-a)_m}{(1-d)_m(-f+a-m)_n}\,{}_3F_2\left(\begin{matrix}-n,f,d-m\\a,d-n+f+1-a\end{matrix}\middle|1\right)$$
$$=\frac{(a-d)_n}{(a)_n(a-f-d)_n}\,{}_3F_2\left(\begin{matrix}-m,a+n-d,a-f-d\\1-d,a-d\end{matrix}\middle|1\right)\tag{1.4.19}$$

$\{(-2,-1),(0,1),(1,0),(1,0),(1,1)\}$

$$(\frac{1}{2}-c-d+b)_n\,{}_4F_3\left(\begin{matrix}c,d,-n,c+d+\frac{1}{2}-b\\\frac{c}{2}+\frac{d}{2}-\frac{n}{2}+\frac{1}{4}-\frac{b}{2},\frac{c}{2}+\frac{d}{2}-\frac{n}{2}+\frac{3}{4}-\frac{b}{2},c+d+\frac{1}{2}\end{matrix}\middle|1\right)$$
$$=\frac{(-c+d+\frac{1}{2})_n(c-d+\frac{1}{2})_n}{(c+d+\frac{1}{2})_n}\tag{1.4.20}$$
$$\times\,{}_6F_5\left(\begin{matrix}b,-n,-\frac{c}{2}-\frac{d}{2}-\frac{n}{2}+\frac{1}{4},-\frac{c}{2}+\frac{d}{2}+\frac{n}{2}+\frac{1}{4},\frac{c}{2}+\frac{d}{2}-\frac{n}{2}+\frac{1}{4},\frac{c}{2}-\frac{d}{2}+\frac{n}{2}+\frac{1}{4}\\c+d-n+\frac{1}{2},\frac{c}{2}-\frac{d}{2}-\frac{n}{2}+\frac{1}{4},\frac{c}{2}+\frac{d}{2}+\frac{n}{2}+\frac{1}{4},-\frac{c}{2}+\frac{d}{2}-\frac{n}{2}+\frac{1}{4},-\frac{c}{2}-\frac{d}{2}+\frac{n}{2}+\frac{1}{4}\end{matrix}\middle|1\right)$$

$\{(-2,-1),(0,1),(1,0),(1,0),(1,1)\}$

$$\frac{(d+\frac{1}{2})_m(\frac{1}{2}-d)_n(\frac{1}{2}-d-e-m)_n}{(d+e+\frac{1}{2})_m(\frac{1}{2}-d-m)_n} \times {}_4F_3\left(\begin{matrix}-n,d,e,-n+d+\frac{1}{2}+m\\ -\frac{n}{2}+\frac{d}{2}+\frac{e}{2}+\frac{1}{4}+\frac{m}{2},-\frac{n}{2}+\frac{d}{2}+\frac{e}{2}+\frac{3}{4}+\frac{m}{2},-n+d+\frac{1}{2}\end{matrix}\middle|1\right) = \frac{(-d+e+\frac{1}{2})_n(d-e+\frac{1}{2})_n}{(d+e+\frac{1}{2})_n} \times {}_6F_5\left(\begin{matrix}-m,e,\frac{n}{2}-\frac{d}{2}+\frac{e}{2}+\frac{1}{4},\frac{n}{2}+\frac{d}{2}-\frac{e}{2}+\frac{1}{4},-\frac{n}{2}+\frac{d}{2}+\frac{e}{2}+\frac{1}{4},-\frac{n}{2}-\frac{d}{2}-\frac{e}{2}+\frac{1}{4}\\ -n+d+e+\frac{1}{2},-\frac{n}{2}-\frac{d}{2}+\frac{e}{2}+\frac{1}{4},-\frac{n}{2}+\frac{d}{2}-\frac{e}{2}+\frac{1}{4},\frac{n}{2}+\frac{d}{2}+\frac{e}{2}+\frac{1}{4},\frac{n}{2}-\frac{d}{2}-\frac{e}{2}+\frac{1}{4}\end{matrix}\middle|1\right) \tag{1.4.21}$$

$\{(-1,-1),(-1,0),(0,1),(1,0),(1,0),(1,1)\}$

$$(a-c)_n {}_3F_2\left(\begin{matrix}d,e,-n\\ d+e-n+1-a,a-c\end{matrix}\middle|1\right) = \frac{(a-d)_n(a-e)_n}{(a-e-d)_n}{}_3F_2\left(\begin{matrix}c,-n,a-e-d\\ a-e,a-d\end{matrix}\middle|1\right) \tag{1.4.22}$$

$\{(-1,-1),(-1,0),(0,1),(1,0),(1,0),(1,1)\}$

$$\frac{(a-f)_m}{(a)_m}{}_3F_2\left(\begin{matrix}-n,e,f\\ -n+e+f+1-a,a+m\end{matrix}\middle|1\right) = \frac{(a-e)_n(a-f)_n}{(a)_n(a-e-f)_n}{}_3F_2\left(\begin{matrix}-m,f,a-e+n\\ a-e,a+n\end{matrix}\middle|1\right) \tag{1.4.23}$$

$\{(-1,0),(-1,0),(0,1),(1,-1),(1,0),(1,1)\}$

$$\frac{(1-d-c)_n}{(1-d)_n}{}_5F_4\left(\begin{matrix}e,-n,d+c,\frac{d}{2}-\frac{n}{2},\frac{d}{2}-\frac{n}{2}+\frac{1}{2}\\ a,d+e-n+1-a,\frac{d}{2}+\frac{c}{2}-\frac{n}{2},\frac{d}{2}+\frac{c}{2}-\frac{n}{2}+\frac{1}{2}\end{matrix}\middle|1\right) = \frac{(a-d)_n(a-e)_n}{(a)_n(a-e-d)_n}{}_4F_3\left(\begin{matrix}c,-n,a-e-d,1-a-n\\ 1-d,1-a+e-n,a-d\end{matrix}\middle|1\right) \tag{1.4.24}$$

$\{(-1,0),(-1,0),(0,1),(1,-1),(1,0),(1,1)\}$

$$\frac{(1-d-f)_m}{(1-d)_m}{}_5F_4\left(\begin{matrix}-n,f,d-m,\frac{d}{2}+\frac{f}{2},\frac{d}{2}+\frac{f}{2}+\frac{1}{2}\\ a,d-n+f+1-a,\frac{d}{2}-\frac{m}{2}+\frac{f}{2},\frac{d}{2}-\frac{m}{2}+\frac{f}{2}+\frac{1}{2}\end{matrix}\middle|1\right) = \frac{(a-d)_n(a-f)_n}{(a)_n(a-f-d)_n}{}_4F_3\left(\begin{matrix}-m,f,a+n-d,1+f-a\\ 1-d,1-n+f-a,a-d\end{matrix}\middle|1\right) \tag{1.4.25}$$

$\{(-1,0),(-1,1),(0,1),(1,0),(2,-1)\}$

$$\frac{(\frac{3}{2}-a)_m(-\frac{1}{2}+a)_n}{(1-e)_m(-\frac{1}{2}+a-m)_n}{}_4F_3\left(\begin{matrix}-n,-\frac{1}{2}+n+a,\frac{e}{2}-\frac{m}{2},\frac{e}{2}-\frac{m}{2}+\frac{1}{2}\\ a,-n+e+\frac{3}{2}-a,-\frac{1}{2}+n+a-m\end{matrix}\middle|1\right) = \frac{(a-\frac{e}{2})_n(a-\frac{e}{2}-\frac{1}{2})_n}{(a)_n(a-e-\frac{1}{2})_n}{}_3F_2\left(\begin{matrix}-m,a-e-\frac{1}{2},2\,a-e-1+2\,n\\ 1-e,2\,a-e-1\end{matrix}\middle|1\right) \tag{1.4.26}$$

$\{(-1,0),(-1,0),(0,1),(1,1),(2,-1)\}$

$$\frac{(1-e-c)_n}{(1-e)_n}{}_6F_5\left(\begin{matrix}-n,\frac{e}{2}+\frac{c}{2},\frac{e}{2}+\frac{c}{2}+\frac{1}{2},\frac{e}{3}-\frac{n}{3},\frac{e}{3}-\frac{n}{3}+\frac{1}{3},\frac{e}{3}-\frac{n}{3}+\frac{2}{3}\\a,-n+e+\frac{3}{2}-a,\frac{e}{3}+\frac{c}{3}-\frac{n}{3},\frac{e}{3}+\frac{c}{3}-\frac{n}{3}+\frac{1}{3},\frac{e}{3}+\frac{c}{3}-\frac{n}{3}+\frac{2}{3}\end{matrix}\middle|1\right)$$
$$=\frac{(a-\frac{e}{2})_n(a-\frac{e}{2}-\frac{1}{2})_n}{(a)_n(a-e-\frac{1}{2})_n}{}_4F_3\left(\begin{matrix}c,-n,a-e-\frac{1}{2},1-a-n\\1-e,2-2\,a+e-2\,n,2\,a-e-1\end{matrix}\middle|4\right)\quad(1.4.27)$$

$\{(-1,0),(-1,0),(0,1),(1,-1),(2,1)\}$

$$\frac{(1-d-c)_n}{(1-d)_n}{}_6F_5\left(\begin{matrix}-\frac{n}{2},-\frac{n}{2}+\frac{1}{2},d+c,\frac{d}{3}-\frac{n}{3},\frac{d}{3}-\frac{n}{3}+\frac{1}{3},\frac{d}{3}-\frac{n}{3}+\frac{2}{3}\\a,d-n+\frac{3}{2}-a,\frac{d}{3}+\frac{c}{3}-\frac{n}{3},\frac{d}{3}+\frac{c}{3}-\frac{n}{3}+\frac{1}{3},\frac{d}{3}+\frac{c}{3}-\frac{n}{3}+\frac{2}{3}\end{matrix}\middle|1\right)$$
$$=\frac{(-\frac{1}{2}+a)_n(2\,a-1-2\,d)_n}{(a-\frac{1}{2}-d)_n(-1+2\,a)_n}{}_4F_3\left(\begin{matrix}c,-n,2\,a+n-1-2\,d,2-2\,a-n\\1-d,\frac{3}{2}-n-a,a-d\end{matrix}\middle|\frac{1}{4}\right)\quad(1.4.28)$$

$\{(-2,-1),(0,1),(1,0),(2,1)\}$

$$(-c+b)_n{}_3F_2\left(\begin{matrix}c,-\frac{n}{2},-\frac{n}{2}+\frac{1}{2}\\\frac{c}{2}-\frac{n}{2}+\frac{1}{2}-\frac{b}{2},\frac{c}{2}-\frac{n}{2}+1-\frac{b}{2}\end{matrix}\middle|1\right)$$
$$=(c)_n{}_4F_3\left(\begin{matrix}b,-n,\frac{c}{2}-\frac{n}{2}+\frac{1}{2},\frac{c}{2}+\frac{n}{2}\\c-n+1,-\frac{c}{2}-\frac{n}{2}+\frac{1}{2},-\frac{c}{2}+\frac{n}{2}\end{matrix}\middle|1\right)\quad(1.4.29)$$

$\{(-1,-1),(-1,0),(0,1),(1,0),(2,1)\}$

$$\frac{(a-c)_n}{(a)_n}{}_4F_3\left(\begin{matrix}d,-\frac{n}{2},-\frac{n}{2}+\frac{1}{2},1-a-n\\d-n+\frac{3}{2}-a,1-a+c-n,a-c\end{matrix}\middle|1\right)$$
$$=\frac{(-\frac{1}{2}+a)_n(2\,a-1-2\,d)_n}{(a-\frac{1}{2}-d)_n(-1+2\,a)_n}{}_3F_2\left(\begin{matrix}c,-n,2\,a+n-1-2\,d\\a-d,2\,a+n-1\end{matrix}\middle|1\right)\quad(1.4.30)$$

$\{(-1,-1),(-1,0),(0,1),(1,0),(2,1)\}$

$$\frac{(a-e)_m}{(a)_m}{}_4F_3\left(\begin{matrix}-n,\frac{e}{2},\frac{e}{2}+\frac{1}{2},1-a+e\\-n+e+\frac{3}{2}-a,1-a-m+e,a+m\end{matrix}\middle|1\right)$$
$$=\frac{(a-\frac{e}{2})_n(a-\frac{e}{2}-\frac{1}{2})_n}{(a)_n(a-e-\frac{1}{2})_n}{}_3F_2\left(\begin{matrix}-m,e,2\,a-e-1+2\,n\\a+n,2\,a-e-1\end{matrix}\middle|1\right)\quad(1.4.31)$$

$\{(-1,-1),(-1,0),(0,1),(3,1)\}$

$$\frac{(a-c)_n}{(a)_n}{}_5F_4\left(\begin{matrix}-\frac{n}{3},-\frac{n}{3}+\frac{1}{3},-\frac{n}{3}+\frac{2}{3},\frac{1}{2}-\frac{a}{2}-\frac{n}{2},1-\frac{a}{2}-\frac{n}{2}\\-n+2-a,\frac{1}{2}-\frac{a}{2}+\frac{c}{2}-\frac{n}{2},1-\frac{a}{2}+\frac{c}{2}-\frac{n}{2},a-c\end{matrix}\middle|1\right)$$
$$=\frac{(3\,a-3)_{2n}3^{-n}}{(a-1)_n(3\,a-2)_n}{}_3F_2\left(\begin{matrix}c,-n,3\,a+2\,n-3\\\frac{3\,a}{2}+\frac{n}{2}-1,\frac{3\,a}{2}+\frac{n}{2}-\frac{1}{2}\end{matrix}\middle|\frac{3}{4}\right)\quad(1.4.32)$$

§1.5 More Hypergeometric Transformations via Index Shifting

Consider the following double sum

$$\sum_{k,j} \frac{(-1)^j v^k (b)_{k+j}}{j!(a)_j (k-j)!}. \tag{1.5.1}$$

If we first sum on j and apply Gauss's theorem we get

$$ {}_2F_1\left(\begin{matrix} b, 1-a+b \\ a \end{matrix} \middle| -v\right). $$

To sum (1.5.1) on k we shift the index by setting $k = i + j$, and sum first on i. We get a ${}_1F_0$ which can be evaluated by the binomial theorem to be

$$(1-v)^{-b}{}_2F_1\left(\begin{matrix} \frac{1}{2}b, \frac{1}{2}b+\frac{1}{2} \\ a \end{matrix} \middle| -4\frac{v}{(1-v)^2}\right),$$

and the quadratic transformation obtained is

$$ {}_2F_1\left(\begin{matrix} b, 1-a+b \\ a \end{matrix} \middle| -v\right) = (1-v)^{-b}{}_2F_1\left(\begin{matrix} \frac{1}{2}b, \frac{1}{2}b+\frac{1}{2} \\ a \end{matrix} \middle| -4\frac{v}{(1-v)^2}\right).$$

So the observation here is that we can use any partitions from $P(1,0;1,1)$ to form a double sum like (1.5.1), then evaluate the sum by Gauss's and the binomial theorem via index shifting.

Similarly, if we have a double sum such as

$$\sum_{k,j} \frac{v^k (a)_{k-j} (-1)^j (2)^{-2j}}{j!\,(k-2j)!},$$

we can first sum on j and apply Gauss's theorem, to get

$$ {}_1F_0\left(\begin{matrix} 2a \\ - \end{matrix} \middle| \frac{1}{2}v\right).$$

We can also set $k = i + 2j$ and sum first on i then we get

$$(1-v)^{-a}{}_1F_0\left(\begin{matrix} a \\ - \end{matrix} \middle| -\frac{1}{4}\frac{v^2}{1-v}\right).$$

We obtain another quadratic transformation

$$ {}_1F_0\left(\begin{matrix} 2a \\ - \end{matrix} \middle| \frac{1}{2}v\right) = (1-v)^{-a}{}_1F_0\left(\begin{matrix} a \\ - \end{matrix} \middle| -\frac{1}{4}\frac{v^2}{1-v}\right).$$

Note that this transformation is trivial since both sides can be evaluated by binomial theorem. More transformations by index shifting are given in the following section.

§1.6 Transformations Generated by Maple Program via Index Shifting

Transformations from the binomial theorem and Gauss's theorem by setting $k = i + j$ from $P(0,0;2,1)$:

$\{(0,-1),(0,1),(0,1)\}$

$$\frac{(a-c)_n}{(a)_n}\,{}_2F_1\left(\left.\begin{matrix}-n,c\\1-a-n+c\end{matrix}\right|v\right)={}_2F_1\left(\left.\begin{matrix}-n,c\\a\end{matrix}\right|1-v\right) \qquad (1.6.1)$$

Transformations from the binomial and Gauss's theorem by setting $k = i + j$ from $P(1,0;1,1)$

$\{(1,1),(0,-1)\}$

$$(1-v)^{-a}\,{}_2F_1\left(\left.\begin{matrix}\frac{a}{2},\frac{a}{2}+\frac{1}{2}\\b\end{matrix}\right|-\frac{4\,v}{(1-v)^2}\right)={}_2F_1\left(\left.\begin{matrix}a,1-b+a\\b\end{matrix}\right|-v\right) \qquad (1.6.2)$$

[3, §4.09; 2, §2.11]

$\{(1,0),(0,1),(0,-1)\}$

$$(1-v)^{-a}\,{}_2F_1\left(\left.\begin{matrix}a,b\\c\end{matrix}\right|-\frac{v}{1-v}\right)={}_2F_1\left(\left.\begin{matrix}a,c-b\\c\end{matrix}\right|v\right) \qquad (1.6.3)$$

Transformations from the binomial and Gauss's theorem by setting $k = i + 2j$ from $P(1,0;0,1)$:

$\{(1,-1)\}$

$$(1-v)^{-a}\,{}_1F_0\left(\left.\begin{matrix}a\\-\end{matrix}\right|-\frac{v^2}{4-4\,v}\right)={}_1F_0\left(\left.\begin{matrix}2\,a\\\end{matrix}\right|\frac{v}{2}\right) \qquad (1.6.4)$$

$\{(1,0),(0,-1)\}$

$$(1-v)^{-a}\,{}_2F_1\left(\left.\begin{matrix}\frac{a}{2},\frac{a}{2}+\frac{1}{2}\\b\end{matrix}\right|\frac{v^2}{(1-v)^2}\right)={}_2F_1\left(\left.\begin{matrix}a,b-\frac{1}{2}\\2\,b-1\end{matrix}\right|2\,v\right) \qquad (1.6.5)$$

[3, §4.18; 2, §2.11]

Transformations from Saalschütz's and the binomial theorems by setting $k = i + j$ from $P(2,2;1,0)$:

$\{(-2,1),(1,0),(1,0)\}$

$$(1-v)^{-\frac{1}{2}+b+c}\,{}_2F_1\left(\left.\begin{matrix}b,c\\b+c+\frac{1}{2}\end{matrix}\right|4\,v\,(1-v)\right)={}_2F_1\left(\left.\begin{matrix}\frac{1}{2}-b+c,\frac{1}{2}+b-c\\b+c+\frac{1}{2}\end{matrix}\right|v\right) \qquad (1.6.6)$$

$\{(-1,0),(-1,1),(1,0),(1,0)\}$

$$(1-v)^{-a+d+c}\,{}_2F_1\left(\left.\begin{matrix}c,d\\a\end{matrix}\right|v\right)={}_2F_1\left(\left.\begin{matrix}a-c,a-d\\a\end{matrix}\right|v\right) \qquad (1.6.7)$$

[3, §2.06]

$\{(-1,0),(-1,0),(1,0),(1,1)\}$

$$(1-v)^{-d}\,{}_3F_2\left(\left.\begin{matrix} c, \frac{d}{2}, \frac{d}{2}+\frac{1}{2} \\ a, c+d+1-a \end{matrix}\right| -\frac{4\,v}{(1-v)^2}\right) = {}_3F_2\left(\left.\begin{matrix} d, a-c, 1-a+d \\ c+d+1-a, a \end{matrix}\right| v\right) \tag{1.6.8}$$

[3, §4.12, §4.07, §4.08; 2, §2.11(32), 3]

$\{(-1,0),(-1,0),(2,1)\}$

$$(1-v)^{-c}\,{}_3F_2\left(\left.\begin{matrix} \frac{c}{3}, \frac{c}{3}+\frac{1}{3}, \frac{c}{3}+\frac{2}{3} \\ a, c+\frac{3}{2}-a \end{matrix}\right| \frac{-27\,v}{4\,(1-v)^3}\right)$$
$$= {}_3F_2\left(\left.\begin{matrix} c, 2\,a-c-1, 2-2\,a+c \\ c+\frac{3}{2}-a, a \end{matrix}\right| \frac{v}{4}\right) \tag{1.6.9}$$

[3, §4.05; 2§2.11(42)]

Transformations from Saalschütz and the binomial theorems by setting $k = i + 2j$ from $P(1,2;1,0)$.

$\{(-1,0),(-1,1),(1,0)\}$

$$(1-v)^{\frac{1}{2}+c-a}\,{}_2F_1\left(\left.\begin{matrix} -\frac{1}{2}-c+a, c \\ a \end{matrix}\right| \frac{v^2}{-4+4\,v}\right) = {}_2F_1\left(\left.\begin{matrix} a-\frac{1}{2}, -1-2\,c+2\,a \\ 2\,a-1 \end{matrix}\right| v\right) \tag{1.6.10}$$

[3, §4.06]

$\{(-1,0),(-1,0),(1,1)\}$

$$(1-v)^{-c}\,{}_3F_2\left(\left.\begin{matrix} \frac{c}{3}, \frac{c}{3}+\frac{1}{3}, \frac{c}{3}+\frac{2}{3} \\ a, c+\frac{3}{2}-a \end{matrix}\right| -\frac{27\,v^2}{4\,(-1+v)^3}\right) = {}_3F_2\left(\left.\begin{matrix} c, a-\frac{1}{2}, 1-a+c \\ 2+2\,c-b2\,a, 2\,a-1 \end{matrix}\right| 4\,v\right) \tag{1.6.11}$$

[3, §4.22]

Chapter 2. Hypergeometric Transformations of Two Variables by Coefficient Extraction

It is well known that hypergeometric transformations can be obtained by extracting coefficients and using well-known identities ([2, §9.5] and [2, §9.6]). We first give a simple example in §2.1. In §2.2 we look at how the computer can generate transformations systematically by coefficient extraction. In §2.3, §2.4, and §2.5, we give the list of the identities that we have obtained from the Maple program.

§2.1 A transformation of Appell's F_1

One of the four Appell hypergeometric functions of two variables ([3, §9.1]) is

$$F_1(\alpha;\beta,\beta';\gamma;x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{m!\,n!\,(\gamma)_{m+n}}x^m y^n. \tag{2.1.1}$$

Now consider the hypergeometric function

$$(1-x)^{-\beta}(1-y)^{-\beta'}F_1\big(\gamma-\alpha;\beta,\beta';\gamma;-\frac{x}{1-x},-\frac{y}{1-y}\big).$$

It is a double sum

$$\sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}(\beta)_m(\beta')_n(\gamma-\alpha)_{m+n}}{(\gamma)_{m+n}m!\,n!}(1-x)^{(-\beta-m)}(1-y)^{(-\beta'-n)}x^m y^n. \tag{2.1.2}$$

We can expand the powers of $1-x$ and $1-y$ by the binomial theorem and we get

$$\sum_{m,n,i,j=0}^{\infty}\frac{(-1)^{m+n}(\beta)_m(\beta')_n(\gamma-\alpha)_{m+n}(\beta+m)_i(\beta'+n)_j}{(\gamma)_{m+n}m!\,n!\,i!\,j!}x^{m+i}y^{n+j}.$$

The coefficient of $x^M y^N$ in this sum is

$$\sum_{m,n=0}^{M,N}\frac{(-1)^{m+n}(\beta)_m(\beta')_n(\gamma-\alpha)_{m+n}(\beta+m)_{M-m}(\beta'+n)_{N-n}}{(\gamma)_{m+n}m!\,n!(M-m)!(N-n)!}.$$

Notice that the summand is zero when $m>M$ or $n>N$, so this is a terminating double sum on n and m. To evaluate it for each index n we first convert the sum on m into a ${}_2F_1$ hypergeometric function and evaluate it by Gauss's theorem, and then convert the sum on n into another ${}_2F_1$ and it can be evaluated by Gauss's theorem too. So we get that (2.1.2) equals

$$\sum_{M,N=0}^{\infty}\frac{(\beta)_M(\beta')_N(\alpha)_{M+N}}{(\gamma)_{M+N}M!\,N!}x^M y^N. \tag{2.1.3}$$

Hence by extracting the coefficient of (2.1.1) we get a transformation of Appell's F_1

$$F_1(\alpha;\beta,\beta';\gamma;x,y)=(1-x)^{-\beta}(1-y)^{-\beta'}F_1\big(\gamma-\alpha;\beta,\beta';\gamma;-\frac{x}{1-x},-\frac{y}{1-y}\big). \qquad [3, §9.4]$$

§2.2 A systematic way to generate transformations

After we look at the way of getting the transformation in §2.1 more closely, it is clear that the whole procedure, from a known double sum (2.1.2) to a new one (2.1.3) is just a routine which can be done by computer. So the computer can always start with an arbitrary double sum of the type (2.1.1), then perform the same routines as we did in the previous section and try to get a new double sum of the type (2.1.3). Similar to the method in chapter 1, we can use partitions with various cutoffs to generate such arbitrary double sums for the computer to start with.

Let's illustrate the process with an example. We start with a formal power series in x and y

$$(1-x)^p\,(1-y)^q \sum_{m,n} \frac{(a)_{m+n}(b)_m(c)_n}{m!\,n!\,(d)_n(e)_m} \left(-\frac{x}{1-x}\right)^m \left(-\frac{y}{1-y}\right)^n .$$

Notice that this double sum corresponds to the partition $\{(1,1),(1,0),(0,1),(0,-1),(-1,0)\}$, which has cutoff $(2,1;2,1)$. The program first applies the procedure `gather` to combine the powers of $x, y, 1-x$ and $1-y$ then uses `binexpand` to expand powers of $1-x$ and $1-y$ by the binomial theorem. After this it applies `extract` to get the coefficient of x^M and y^N, so the original sum equals

$$\sum_{M,N}\sum_{m,n} \frac{(-1)^{m+n}(a)_{m+n}(-p+m)_{-m+M}(b)_m(-q+n)_{-n+N}(c)_n}{m!\,n!\,(-m+M)!\,(-n+N)!\,(d)_n(e)_m} x^M y^N .$$

The coefficient of $x^M y^N$ is a terminating double sum in m and n. For each index n the program first converts the corresponding sum in m into a hypergeometric series as

$$\sum_n \frac{(-1)^n\,(a)_n(-p)_M(-q+n)_{-n+N}(c)_n}{n!\,M!\,(-n+N)!\,(d)_n}\, {}_3F_2\left(\left.\begin{matrix}-M, a+n, b\\ -p, e\end{matrix}\right|1\right).$$

We would like to choose the parameters so that the ${}_3F_2$ can be evaluated. Since only one parameter $a+n$ has the summation index n the numerators and denominators of the ${}_3F_2$ cannot be adjusted to make it balanced. So Saalschütz's theorem cannot be applied. The only choice is Gauss's theorem. The program needs to reduce, through cancellation, the number of both the numerator and the denominator parameters of the ${}_3F_2$ by 1. The numerator parameter, b and the denominator parameters $-p$ and e can be used for that purpose. Since canceling b with e is equivalent to canceling b with $-p$ and then setting $e=-p$ this will just give some identities which are special cases of the identities we get from setting $b=-p$. So we just have one way to do the cancellation, namely setting $b=-p$. After cancellation and evaluating the resulting ${}_2F_1$ by Gauss's theorem the coefficient of $x^M y^N$ is a hypergeometric series in n,

$$\frac{(-p)_M(-q)_N(e-a)_M}{M!\,N!\,(d)_M}\, {}_4F_3\left(\left.\begin{matrix}-N, a, c, 1-e+a\\ d, -q, 1-e-M+a\end{matrix}\right|1\right).$$

Here, since only one numerator parameter involves the summation index N and one denominator parameter involves M, the balanced condition for all M, N cannot be satisfied, so Saalschütz's theorem cannot be used. We use Gauss's theorem and need to reduce the number of both the numerator and the denominator parameters of the ${}_4F_3$ by 2. Among the numerator parameters, a, c, and $1 - e + a$, and among the denominator parameters, d and $-q$ can be used for this purpose. It seems that we can choose two parameters from the three numerator parameters in 3 ways and pair them up with the two denominators in 2 ways, so we have 6 ways of cancellation. But in fact some of these cancellations are special cases of others. For example, setting $c = d$ and $a = -q$ is equivalent to setting $c = -q$ and $a = d$, and then specializing a to be $-q$. Similarly, setting $c = d$ and $1 - e + a = -q$ is equivalent to setting $c = -q$ and $1 - e + a = d$, and then specializing c to be $-q$. The four different identities we get are listed in §2.3.

§2.3 From Gauss's and Gauss's theorem:

In this section all transformations come from applying Gauss's theorem twice. The changes of variables, which we used for forming the starting double sums in this session, are $X = -x/(1-x)$, $Y = -y/(1-y)$, and $X = -x/(1-x)(1-y)$, $Y = -y/(1-x)(1-y)$. Certain double hypergeometric series have been defined in the literature[3,4] and the following will be used later.

$$F_1(a; b, c; d; x, y) = \sum_{m,n} \frac{(a)_{m+n}(b)_m(c)_n}{m!\,n!\,(d)_{m+n}} x^m y^n$$

$$F_2(a; b, c; d, e; x, y) = \sum_{m,n} \frac{(a)_{m+n}(b)_m(c)_n}{m!\,n!\,(d)_m(e)_n} x^m y^n$$

$$F_3(a, b; c, d; e; x, y) = \sum_{m,n} \frac{(a)_m(b)_n(c)_m(d)_n}{m!\,n!\,(e)_{m+n}} x^m y^n$$

$$F_4(a, b; c, d; x, y) = \sum_{m,n} \frac{(a)_{m+n}(b)_{m+n}}{m!\,n!\,(c)_m(d)_n} x^m y^n$$

$$G_2(a, b, c, d; x, y) = \sum_{m,n} \frac{(a)_m(b)_n(d)_{m-n}(c)_{n-m}}{m!\,n!} x^m y^n$$

$$H_2(a, b, c, d; e; x, y) = \sum_{m,n} \frac{(a)_{m-n}(b)_m(c)_n(d)_n}{(e)_m m!\,n!} x^m y^n$$

The following is from the partition $\{(1,1), (1,0), (0,1), (0,-1), (-1,0)\}$.

$$(1-x)^p\,(1-y)^q\,F_2(-q; -p, c; -b+1-q, b; -\frac{x}{1-x}, -\frac{y}{1-y}) = \sum_{M,N} \frac{(-p)_M(-q)_N(-b+1)_{M-N}(1-b+c)_M}{M!\,N!\,(-b+1-q)_M(1-b+c)_{M-N}} x^M y^N \quad (2.3.1)$$

$$(1-x)^p\,(1-y)^q\,F_2(e; -p, c; q+1+e, e; -\frac{x}{1-x}, -\frac{y}{1-y}) = \sum_{M,N} \frac{(-p)_M(-q)_N(q+1)_{M-N}(1+q+c)_M}{M!\,N!\,(q+1+e)_M(1+q+c)_{M-N}} x^M y^N \quad (2.3.2)$$

$$(1-x)^p\,(1-y)^q\,F_2(e; -p, -q; a, e; -\frac{x}{1-x}, -\frac{y}{1-y}) = \sum_{M,N} \frac{(-p)_M(-q)_N(a-e)_{M-N}}{N!\,(a)_M\,(M-N)!} x^M y^N \quad (2.3.3)$$

$$(1-x)^p\,(1-y)^q\,F_2(d-1+e; -p, -q; e, d; -\frac{x}{1-x}, -\frac{y}{1-y}) = G_2(-p, -q; 1-e, -d+1; -x, -y) \quad (2.3.4)$$

Exton [4] listed this transformation in [4, §1.6.1] separately from those that can be obtained by elementary series manipulation and pointed out that it can be deduced by a Pochhammer double loop integral for G_2. We now have shown that this transformation can also be obtained by means of elementary series manipulations.

$$(1-x)^p\,(1-y)^q\,H_2(b-e,-q,-p,e,b,-\frac{y}{1-y},\frac{x}{1-x}) = H_2(-b+1,-p,-q,e,-b+1+e,x,-y) \tag{2.3.5}$$

The following is from the partition $\{(1,0),(0,1),(1,0),(0,1),(-1,-1)\}$.

$$(1-x)^p\,(1-y)^q\,F_3(-p,b;e,-q;e+b;-\frac{x}{1-x},-\frac{y}{1-y}) = F_3(-p,-q;b,e;e+b;x,y) \tag{2.3.6}$$

The following is from the partition $\{(-1,-1),(0,1),(1,0),(1,1)\}$.

$$(1-x)^p\,(1-y)^q\,F_1(-q;-p,b;a;-\frac{x}{1-x},-\frac{y}{1-y})\sum_{M,N}\frac{(-p)_M(-q)_N(a+q)_M(a-b)_{M+N}}{M!\,N!\,(a)_{M+N}(a-b)_M}x^My^N \tag{2.3.7}$$

$$(1-x)^p\,(1-y)^q\,F_1(d;-p,-q;a;-\frac{x}{1-x},-\frac{y}{1-y}) = F_1(a-d;-p,-q;a;x,y) \tag{2.3.8}$$

This transformation appears in [3, §9.4], where Bailey got it by evaluating on integral by change of variables. The following is from the partition $\{(1,-1),(-1,1),(1,0),(0,1)\}$.

$$(1-x)^p\,(1-y)^q\,G_2(-p,-q;1-a,c;\frac{x}{1-x},\frac{y}{1-y}) = F_2(a-c;-p,-q;a,1-c;x,y) \tag{2.3.9}$$

$$(1-x)^p\,(1-y)^q\,G_2(-p,1-c;1-a,c;\frac{x}{1-x},\frac{y}{1-y}) = \sum_{M,N}\frac{(-p)_M(a-c)_M(-q-1+a)_{M+N}}{M!\,N!\,(a)_M(-q-1+a)_M}x^My^N \tag{2.3.10}$$

The right hand side of the above sum can be evaluated by binomial theorem and reduced to a single sum. What we get is

$$(1-x)^p\,(1-y)^q\,G_2(-p,1-c;1-a,c;\frac{x}{1-x},\frac{y}{1-y}) = (1-y)^{q+1-a}\,{}_2F_1\left(\left.\begin{matrix}-p,a-c\\ a\end{matrix}\right|\frac{x}{1-y}\right) \tag{2.3.11}$$

This one is from the partition $\{(-1,0),(0,-1),(1,1),(1,1)\}$.

$$\begin{aligned}(1-x)^p\,(1-y)^q\,F_4(-p,-q;a,b;-\frac{x}{(1-x)\,(1-y)},-\frac{y}{(1-y)\,(1-x)})\\ = \sum_{M,N}\frac{(1-b-p)_M(-p)_M(a+q)_{M-N}(-q)_N(1-a-q)_N}{M!\,N!\,(a)_M(1-b-p)_{M-N}(b)_N}x^My^N\end{aligned} \tag{2.3.12}$$

This transformation is used by Bailey to prove another transformation. See [3, §9.6].

§**2.4 From Gauss's and Saalschütz's theorems**:

In this section the transformation comes from applying Gauss's and Saalschütz's theorem. It comes from the double sum (2.2.1) and the variable transformation $X = 4x/(1+x)^2, Y = 4y/(1+y)^2$.

$$(1+x)^{-p}(1+y)^{-q}\sum_{m,n}\frac{(\frac{p}{2})_m(\frac{p}{2}+\frac{1}{2})_m(\frac{q}{2}+\frac{1}{2})_n(2\,b-1+p-q)_n(\frac{q}{2})_n}{(b+p)_{m+n}(b)_n m!\,n!}$$
$$\times\left(\frac{4\,x}{(1+x)^2}\right)^m\left(\frac{4\,y}{(1+y)^2}\right)^n$$
$$=\sum_{M,N}\frac{(q)_N(p)_M(-b+1+q)_{N+M}(b+p-q)_M(1-b)_{M-N}}{M!\,N!\,(-b+1+q)_M(b+p-q)_{M-N}(b+p)_{N+M}}x^M\,(-y)^N \tag{2.4.1}$$

§**2.5 From Saalschütz's and Saalschütz's theorem**:

The following is from $\{(1,0),(1,0),(1,1),(-1,-1),(0,1),(0,1),(0,-1),(-1,0)\}$.

$$(1+x)^{-p}(1+y)^{-q}\sum_{m,n}\frac{(q)_{2\,n}(\frac{p}{2}+\frac{1}{2})_m(\frac{p}{2})_m(-1+q+2\,f-p)_{m+n}}{(q+f)_{m+n}(-p+q+f)_n(f)_m m!\,n!}$$
$$\times\left(\frac{4\,x}{(1+x)^2}\right)^m\left(\frac{y}{(1+y)^2}\right)^n$$
$$=\sum_{M,N}\frac{(q)_N(p)_M(1-f+p)_{M+N}(1+p-q-f)_{M-N}}{M!\,N!\,(q+f)_{M+N}(f)_{M-N}}(-x)^M\,(-y)^N \tag{2.5.1}$$

$$(1+x)^{-p}(1+y)^{-q}\sum_{m,n}\frac{(q)_{2\,n}(\frac{p}{2}+\frac{1}{2})_m(\frac{p}{2})_m(-p+q+f)_{m+n}}{(-p+q+f)_n(f)_m(q+1)_{m+n}m!\,n!}\left(\frac{4\,x}{(1+x)^2}\right)^m\left(\frac{y}{(1+y)^2}\right)^n$$
$$=\sum_{M,N}\frac{(q)_N(1-f+p)_M(p)_{M+N}(p-q)_{M-N}}{N!\;(M-N)!\,(f)_M(q+1)_{M+N}}(-x)^M\,(-y)^N \tag{2.5.2}$$

$$(1+x)^{-p}(1+y)^{-q}\sum_{m,n}\frac{(\frac{p}{2}+\frac{1}{2})_m(\frac{p}{2})_m(-1+\frac{q}{2}+h)_{m+n}(\frac{q}{2}+\frac{1}{2})_n(-1+p)_n}{(-1+\frac{q}{2}+h)_n m!\,n!\,(\frac{q}{2}+p)_{m+n}(h)_m}$$
$$\times\left(\frac{4\,x}{(1+x)^2}\right)^m\left(\frac{4\,y}{(1+y)^2}\right)^n$$
$$=\sum_{M,N}\frac{(q)_N(p)_M(1-h+p)_M(p-\frac{q}{2})_M(1-\frac{q}{2})_{M-N}(\frac{q}{2}+1)_{M+N}}{M!\,N!\,(h)_M(\frac{q}{2}+p)_{M+N}(\frac{q}{2}+1)_M(p-\frac{q}{2})_{M-N}}(-x)^M\,(-y)^N \tag{2.5.3}$$

$$(1+x)^{-p}(1+y)^{-q}\sum_{m,n}\frac{(\frac{p}{2}+\frac{1}{2})_m(\frac{p}{2})_m(-\frac{1}{2}+\frac{q}{2}+h)_{m+n}(p)_n(\frac{q}{2})_n}{(\frac{q}{2}+\frac{1}{2}+p)_{m+n}(-\frac{1}{2}+\frac{q}{2}+h)_n(h)_m m!\,n!}$$
$$\times\left(\frac{4\,x}{(1+x)^2}\right)^m\left(\frac{4\,y}{(1+y)^2}\right)^n$$
$$=\sum_{M,N}\frac{(p)_M(q)_N(1-h+p)_M(\frac{1}{2}+p-\frac{q}{2})_M(\frac{1}{2}-\frac{q}{2})_{M-N}(\frac{q}{2}+\frac{1}{2})_{M+N}}{M!\,N!\,(h)_M(\frac{q}{2}+\frac{1}{2}+p)_{M+N}(\frac{q}{2}+\frac{1}{2})_M(\frac{1}{2}+p-\frac{q}{2})_{M-N}}(-x)^M\,(-y)^N \tag{2.5.4}$$

$$(1+x)^{-p}\,(1+y)^{-q}\sum_{m,n}\frac{(\frac{p}{2}+\frac{1}{2})_m(\frac{p}{2})_m(2-2\,h+p)_n(\frac{q}{2}+\frac{1}{2})_{m+n}(\frac{q}{2})_n}{(h)_m(\frac{q}{2}+\frac{3}{2}-h)_n m!\,n!\,(\frac{q}{2}+\frac{3}{2}-h+p)_{m+n}}$$
$$\times\left(\frac{4\,x}{(1+x)^2}\right)^m\left(\frac{4\,y}{(1+y)^2}\right)^n$$
$$=\sum_{M,N}\frac{(p)_M(q)_N(1-h+p)_M(-\frac{q}{2}+\frac{3}{2}-h+p)_M(-\frac{1}{2}-\frac{q}{2}+h)_{M-N}(-\frac{1}{2}+\frac{q}{2}+h)_{M+N}}{M!\,N!\,(h)_M(\frac{q}{2}+\frac{3}{2}-h+p)_{M+N}(-\frac{1}{2}+\frac{q}{2}+h)_M(-\frac{q}{2}+\frac{3}{2}-h+p)_{M-N}}$$
$$\times\,(-x)^M\,(-y)^N \tag{2.5.5}$$

$$(1+x)^{-p}\,(1+y)^{-q}\sum_{m,n}\frac{(\frac{p}{2}+\frac{1}{2})_m(\frac{p}{2})_m(1-2\,h+p)_n(\frac{q}{2}+\frac{1}{2})_n(\frac{q}{2})_{m+n}}{(\frac{q}{2}+1-h)_n(h)_m m!\,n!\,(\frac{q}{2}+1-h+p)_{m+n}}$$
$$\times\left(\frac{4\,x}{(1+x)^2}\right)^m\left(\frac{4\,y}{(1+y)^2}\right)^n$$
$$=\sum_{M,N}\frac{(p)_M(q)_N(1-h+p)_M(-\frac{q}{2}+1-h+p)_M(-\frac{q}{2}+h)_{M-N}(\frac{q}{2}+h)_{M+N}}{M!\,N!\,(h)_M(\frac{q}{2}+1-h+p)_{M+N}(\frac{q}{2}+h)_M(-\frac{q}{2}+1-h+p)_{M-N}}$$
$$\times\,(-x)^M\,(-y)^N \tag{2.5.6}$$

§2.6 Identification of the transformations

Our program actually generates far more identities than what we have listed here. But most of them are special cases of others, and we give an example to show why this is so. We start with a double sum which corresponds to the partition $\{(-1,-1),(0,-1),(0,1),(0,1),(0,1),(1,0),(1,0)\}$. Following the same procedure as in §2.2 after first applying Gauss's theorem to the coefficient, it produces a hypergeometric series in n,

$$\frac{(q)_N(p)_M(1+p-a)_M\,(-1)^N\,x^My^N}{M!\,N!\,(a)_M}\,{}_6F_5\left(\left.\begin{matrix}-N,q+N,c,g,h,a-p\\ \frac{q}{2},\frac{q}{2}+\frac{1}{2},b,a+M,a-p-M\end{matrix}\right|1\right).$$

To apply Saalschütz's theorem the program enforces the balanced condition for the numerator and denominator parameters, which is $a+b-c-d-1=0$ and tries to reduce the ${}_6F_5$ to a ${}_3F_2$. Numerator parameters $c,g,h,$ and $a-p$, and denominator parameters $\frac{q}{2}$, $\frac{q}{2}+\frac{1}{2}$, and b can be used for the reduction. There are $\binom{4}{3}$ ways to choose the numerator parameters and 3! ways to pair them up with the denominator parameters for cancellation, hence the number of identities is expected to be 24. But in the end we just listed one identity in §2.4. Let's look at the reasons that reduce the number of identities. First, we need to eliminate identical identities due to the symmetry in numerator or denominator parameters. In this case since the numerator parameters c,g,h are three symmetric parameters, if two or all three are chosen then all permutations among them give the same identities. And since at least two of the three symmetric parameters should be chosen for cancellation we would expect four identities. Our program can eliminate identical identities due to symmetric parameters and gives the four identities as below. The program sets $h = \frac{q}{2} + \frac{1}{2}$, $g = \frac{q}{2}$, and $c = b$, and gives

$$(1+x)^{-p}(1+y)^{-q} F_3\left(\frac{p}{2}, \frac{q}{2}; \frac{p}{2}+\frac{1}{2}, \frac{q}{2}+\frac{1}{2}; q+1; \frac{4\,x}{(1+x)^2}, \frac{4\,y}{(1+y)^2}\right)$$
$$= \sum_{M,N} \frac{(p)_{N+M}(q)_N(-q+p)_{M-N}}{(M-N)!\,N!\,(q+1)_{N+M}} x^M\,(-y)^N\,.$$

The program sets $a - p = \frac{q}{2} + \frac{1}{2}, g = \frac{q}{2}, c = b$ and gets

$$(1+x)^{-p}(1+y)^{-q} F_3\left(\frac{p}{2}, \frac{q}{2}, \frac{p}{2}+\frac{1}{2}, p, \frac{q}{2}+\frac{1}{2}+p, \frac{4\,x}{(1+x)^2}, \frac{4\,y}{(1+y)^2}\right)$$
$$= \sum_{M,N} \frac{(q)_N(p)_M(\frac{q}{2}+\frac{1}{2})_{N+M}(-\frac{q}{2}+\frac{1}{2}+p)_M(\frac{1}{2}-\frac{q}{2})_{M-N}}{M!\,N!\,(-\frac{q}{2}+\frac{1}{2}+p)_{M-N}(\frac{q}{2}+\frac{1}{2})_M(\frac{q}{2}+\frac{1}{2}+p)_{N+M}} x^M\,(-y)^N\,. \qquad (2.6.2)$$

The program sets $g = \frac{q}{2} + \frac{1}{2}, a - p = \frac{q}{2}, c = b$ and gets

$$(1+x)^{-p}(1+y)^{-q} F_3\left(\frac{p}{2}, \frac{q}{2}+\frac{1}{2}, \frac{p}{2}+\frac{1}{2}, -1+p, \frac{q}{2}+p, \frac{4\,x}{(1+x)^2}, \frac{4\,y}{(1+y)^2}\right)$$
$$= \sum_{M,N} \frac{(q)_N(p)_M(-\frac{q}{2}+p)_M(\frac{q}{2}+1)_{N+M}(1-\frac{q}{2})_{M-N}}{M!\,N!\,(-\frac{q}{2}+p)_{M-N}(\frac{q}{2}+1)_M(\frac{q}{2}+p)_{N+M}} x^M\,(-y)^N\,. \qquad (2.6.3)$$

The program sets $g = \frac{q}{2} + \frac{1}{2}, c = \frac{q}{2}, a - p = b$ and gets

$$(1+x)^{-p}(1+y)^{-q} \sum_{m,n} \frac{(\frac{p}{2})_m(\frac{p}{2}+\frac{1}{2})_m(\frac{q}{2}+\frac{1}{2})_n(2\,b-1+p-q)_n(\frac{q}{2})_n}{(b+p)_{m+n}(b)_n m!\,n!}$$
$$\left(\frac{4\,x}{(1+x)^2}\right)^m \left(\frac{4\,y}{(1+y)^2}\right)^n$$
$$= \sum_{M,N} \frac{(q)_N(p)_M(-b+1+q)_{N+M}(b+p-q)_M(1-b)_{M-N}}{M!\,N!\,(-b+1+q)_M(b+p-q)_{M-N}(b+p)_{N+M}} x^M\,(-y)^N\,.$$

When we examine the four cancellations carefully it is easy to find that the first three ones are special cases of the last one. For example, the first one is specializing $b = \frac{q}{2} + \frac{1}{2}$ in the fourth cancellation, the second one is specializing $b = q - p + 1$, the third one is specializing $b = \frac{q}{2}$. So in fact we just get one identity from the original double sum.

§2.7 Trivial Transformations

There are many trivial transformations which are produced by the program. For example the following is from the partition $\{(1,-1),(1,0),(0,1),(0,1),(-1,0)\}$.

$$(1-x)^p(1-y)^q \sum_{m,n} \frac{(a)_{m-n}(1-a)_n(-p-a)_n}{m!\,n!} \left(-\frac{x}{1-x}\right)^m \left(\frac{y}{1-y}\right)^n$$
$$= \sum_{M,N} \frac{(-p-a)_M(1+q-p-a)_M}{(1)_N(1)_M(1+q-p-a)_{-N+M}} x^M\,(-y)^N \qquad (2.7.1)$$

Both sides can be evaluated by applying the binomial theorem twice and both reduce to

$$(1-y)^{q-p-a}(1-x+xy)^{p+a}.$$

§2.8 Description of the program

We have a collection of Maple procedures which does some basic operations on hypergeometric series. We list such procedures below with a brief functionality description.

`tor(summand, m)`: To convert a summand to a product of rising factorials in m.

`toF(summand, k)`: To convert hypergeometric series on index k into form (1.1.2).

`gather(summand)`: To collect all the powers of the same variable in the summand product.

`binexpand(summand)`: To expand factors of the form $(1+\alpha x)^{\beta}$ to a hypergeometric series by binomial theorem.

`extract(summand, xlist, mlist, ilist)`: To extract the coefficient of certain variables with the power specified by mlist. The variables are given by `xlist`. The index listed in `ilist` will be replaced by an expression of the index given by `mlist`.

Let's look at how to get the F_1 transformation we talked about in §2.1 by using the above Maple procedures. We start with the double sum summand (2.1.2), then apply the above procedures to get the transformation. Procedure `gauss` is to apply Gauss's theorem to evaluate a ${}_2F_1$ hypergeometric series. The whole Maple output is the following. Note that $\beta 1$ corresponds to β' in §2.1 and procedure $F1$ gives the summand of the F_1 series. Note that the output has been converted to typeset form.

```
>(1-x)^(-beta)*(1-y)^(-beta1)*F1(gamma-alpha,beta,beta1,gamma,-x/(1-x),-y/(1-y));
```

$$\frac{(-1)^{m+n}(\beta)_m(\beta 1)_n(\gamma-\alpha)_{m+n}}{(\gamma)_{m+n}m!\,n!}(1-x)^{(-\beta-m)}(1-y)^{(-\beta 1-n)}x^m y^n$$

```
>binexpand(");
```

$$\frac{(\beta 1+n)_i(\gamma-\alpha)_{m+n}(\beta)_m(\beta 1)_n\,(-1)^{m+n}\,(\beta+m)_j}{i!\,m!\,n!\,(\gamma)_{m+n}j!}y^i x^m y^n x^j$$

```
>gather(");
```

$$\frac{(\beta 1+n)_i(\beta+m)_j(\gamma-\alpha)_{m+n}(\beta)_m(\beta 1)_n\,(-1)^{m+n}}{m!\,n!\,i!\,(\gamma)_{m+n}j!}x^{m+j}y^{i+n}$$

```
>extract(",[x,y],[M,N],[i,j]);
```

$$\frac{(\beta 1+n)_{-n+N}(\beta+m)_{-m+M}(\gamma-\alpha)_{m+n}(\beta)_m(\beta 1)_n\,(-1)^{m+n}}{m!\,n!\,(-n+N)!\,(\gamma)_{m+n}\,(-m+M)!}x^M y^N$$

```
>toF(",m);
```

$$\frac{(\beta1+n)_{-n+N}(\beta)_M(\gamma-\alpha)_n(\beta1)_n\,(-1)^n\,{}_2F_1\left(\left.{\gamma-\alpha+n,-M \atop \gamma+n}\right|1\right)}{n!\,(-n+N)!\,(\gamma)_n\Gamma(M+1)}x^My^N$$

```
>gauss(");
```

$$\frac{(\beta1+n)_{-n+N}(\beta)_M(\gamma-\alpha)_n(\beta1)_n\,(-1)^n\,\Gamma(\gamma+n)\Gamma(\alpha+M)}{n!\,(-n+N)!\,(\gamma)_n\Gamma(M+1)\Gamma(\alpha)\Gamma(\gamma+n+M)}x^My^N$$

```
>toF(",n);
```

$$\frac{(\beta1)_N(\beta)_M\Gamma(\gamma)\Gamma(\alpha+M){}_2F_1\left(\left.{\gamma-\alpha,-N \atop \gamma+M}\right|1\right)}{\Gamma(N+1)\Gamma(M+1)\Gamma(\alpha)\Gamma(\gamma+M)}x^My^N$$

```
>gauss(");
```

$$\frac{(\beta1)_N(\beta)_M\Gamma(\gamma)\Gamma(M+N+\alpha)}{\Gamma(N+1)\Gamma(M+1)\Gamma(\alpha)\Gamma(\gamma+M+N)}x^My^N$$

```
>tor(",M,N);
```

$$\frac{(\beta1)_N(\beta)_M(\alpha)_{M+N}}{(1)_N(1)_M(\gamma)_{M+N}}x^My^N$$

The following carry out the systematic procedures (see §2.2) to generate the transformations.

`createsummand`: To create the partitions needed to generate the starting double sums. This is a skeleton procedure which we modify to generate partitions with different cutoffs. For example to generate partitions with cutoff $(2,1;2,1)$, we modify this procedure to become `ggcreatesummand`. The name is due to the fact that most of the partitions form a double sum that require applying Gauss's theorem twice to give a transformation. Similarly with procedure `gscreatesummand`. The `s` here refers to Saalschütz's theorem.

`get_iden`: To apply combinations of well known theorems to get the transformation. Just like `createsummand` this is a skeleton procedure and is modified to different procedures which will try applying different combinations of theorems.

Chapter 3. WZ Forms and their Change of Variables Theorem

§3.1 Introduction to WZ Forms

Wilf and Zeilberger [11] introduced WZ pairs and the WZ method for proving identities for hypergeometric series (see also [9]). In [12] Zeilberger defined WZ forms. We recall these definitions here.

Definition 1. A pair of functions $(F(a,b), G(a,b))$ is called a *WZ pair* if the following condition is satisfied:

$$F(a, b+1) - F(a,b) = G(a+1, b) - G(a,b).$$

We also call $G(a,b)$ the WZ mate of $F(a,b)$. We always assume that a and b are integers unless explicitly stated otherwise. WZ pairs can be used to prove and generate combinatorial identities. Let's give an example to get the binomial theorem by a WZ pair. Let $f(a,b) = \binom{b}{a}\frac{x^a}{(1+x)^b}$. Then its WZ mate is $g(a,b) = -\frac{a}{(1+x)(b-a+1)}f(a,b)$. We assume that both $F(a,b)$ and $G(a,b)$ are power series in x. Since $f(a,b+1) - f(a,b) = g(a+1,b) - g(a,b)$, we have

$$\sum_{a=0}^{N}(f(a,a+1) - f(a,b)) = g(N+1,b) - g(0,b).$$

Since $g(0,b)$ equals 0 and the limit of $g(N,b)$ as $N \to \infty$ is also 0, we have $\sum_{a=0}^{\infty} f(a,b)$ is independent of b as long as b is an integer. By evaluating the sum at $b = 0$ we get the binomial theorem,

$$\sum_{a=0}^{\infty}\binom{b}{a}\frac{x^a}{(1+x)^b} = 1.$$

In general, if $G(0,b) = 0$ and the limit of $G(a,b)$ as $a \to \infty$ is 0, then we have

$$\sum_{a=0}^{\infty} F(a,b+1) - \sum_{a=0}^{\infty} F(a,b) = 0.$$

That is to say the sum $\sum_{a=0}^{\infty} F(a,b)$ is independent of b. In particular we have

$$\sum_{a=0}^{\infty} F(a,b) = \sum_{a=0}^{\infty} F(a,0).$$

Definition 2 Let δa and δb be indeterminates. If $(F(a,b), G(a,b))$ is a WZ pair then we call $\omega = F(a,b)\,\delta a + G(a,b)\,\delta b$ a WZ form. A function $t(n)$ is called a hypergeometric term if $\frac{t(n+1)}{t(n)}$ is a rational function of n.

If $F(a,b)$ is a hypergeometric term in a, Gosper's indefinite summation algorithm [8] can be used to find its WZ mate $G(a,b)$ if it exists.

§3.2 Path Independence Theorem and Change of Variables Theorem for WZ Forms

Let $\omega = f(a,b)\delta a + g(a,b)\delta b$ be a WZ form. We define two type of basic steps: right steps and up steps. A right step is $(a_i, b_i), (a_i + 1, b_i)$, and an up step is $(a_i, b_i), (a_i, b_i + 1)$. We also define a path π from (a_0, b_0) to (a_m, b_m) to be a sequence of points, $(a_0, b_0), (a_1, b_1), \ldots, (a_m, b_m)$, such that $(a_i - a_{i-1}, b_i - b_{i-1})$ is either (1,0) or (0,1) for each i. Obviously π can be represented by a sequence of right steps and up steps. The *sum of ω along the two type of steps* is defined to be

$$\sum_{\text{right step}} \omega = f(a_i, b_i)$$

and

$$\sum_{\text{up step}} \omega = g(a_i, b_i).$$

The *sum of ω along π* is defined, reasonably, to be the sum of ω on each basic step. It can also be written as

$$\sum_{\pi} \omega = \sum_{i=0}^{m-1} (f(a_i, b_i)(a_{i+1} - a_i) + g(a_i, b_i)(b_{i+1} - b_i)). \tag{3.2.1}$$

In other words, a step from (a_i, b_i) to $(a_i + 1, b_i)$ contributes $f(a_i, b_i)$ to the sum and a step from (a_i, b_i) to (a_i, b_{i+1}) contributes $g(a_i, b_i)$ to the sum. Then we have the following theorems.

Theorem 3.1 *If ω is a WZ form then $\sum_\pi \omega$ depends only on the starting and ending points of π.*

Proof: Let $\omega = f(a,b)\delta a + g(a,b)\delta b$. Let π_1 be the path of an up step followed by a right step, and let π_2 be that of a right step followed by an up step. If π_1 and π_2 start and end at the same points, say (a_0, b_0) and $(a_0 + 1, b_0 + 1)$, then since ω is a WZ form we have

$$\sum\nolimits_{\pi_1} \omega = g(a_0, b_0) + f(a_0, b_0 + 1) = f(a_0, b_0) + g(a_0 + 1, b_0) = \sum\nolimits_{\pi_2} \omega.$$

Let π be an arbitrary path consisting of right steps and up steps, which starts at (p, q) and ends at $(p + r, q + s)$. If π has a subpath of type π_1 we can transform this subpath to a subpath of type of π_2 without changing its starting and ending points, hence under each such a transformation the sum along the new path is invariant and is equal to that along the original path π. We can perform a finite number of such transformations to get a path which does not have any subpath of type π_1. Let's call the obtained path π_0, which starts from (p, q) and ends at $(p + r, q + s)$ with r right steps followed by s up steps and we have $\sum_\pi \omega = \sum_{\pi_0} \omega$. Since given the starting and ending points π_0 is uniquely determined we have proved the theorem.

We call this theorem the path independence theorem for WZ forms.

Theorem 3.2 *Suppose that $\omega = f(a,b)\delta a + g(a,b)\delta b$ is a WZ form. Then for any positive integers r, s, t, u*

$$\left[\sum_{(ra+sb,ta+ub)}^{(r(a+1)+sb,t(a+1)+ub)} \omega\right]\delta a + \left[\sum_{(ra+sb,ta+ub)}^{(ra+s(b+1),ta+u(b+1))} \omega\right]\delta b$$

is a WZ form.

Proof: Let

$$F(a,b) = \sum_{(ra+sb,ta+ub)}^{(r(a+1)+sb,t(a+1)+ub)} \omega$$

and

$$G(a,b) = \sum_{(ra+sb,ta+ub)}^{(ra+s(b+1),ta+u(b+1))} \omega.$$

We sum ω along two paths starting from $(ra+sb, ta+ub)$ and ending at $(r(a+1)+s(b+1), t(a+1)+u(b+1))$. By Theorem 3.1 the sum is independent of the chosen paths so we may choose the path to pass through the point $(r(a+1)+sb, t(a+1)+ub)$. We have:

$$\begin{aligned}\left[\sum_{(ra+sb,ta+ub)}^{(r(a+1)+s(b+1),t(a+1)+u(b+1))} \omega\right] &= \left[\sum_{(ra+sb,ta+ub)}^{(r(a+1)+sb,t(a+1)+ub)} \omega\right] + \left[\sum_{(r(a+1)+sb,t(a+1)+ub)}^{(r(a+1)+s(b+1),t(a+1)+u(b+1))} \omega\right]. \\ &= F(a,b) + G(a+1,b)\end{aligned}$$

If we choose the path to pass through the point $(ra+s(b+1), ta+u(b+1))$ we have

$$\begin{aligned}\left[\sum_{(ra+sb,ta+ub)}^{(r(a+1)+s(b+1),t(a+1)+u(b+1))} \omega\right] &= \left[\sum_{(ra+sb,ta+ub)}^{(ra+s(b+1),ta+u(b+1))} \omega\right] + \left[\sum_{(ra+s(b+1),ta+u(b+1))}^{(r(a+1)+s(b+1),t(a+1)+u(b+1))} \omega\right]. \\ &= G(a,b) + F(a,b+1)\end{aligned}$$

So $(F(a,b), G(a,b))$ is a WZ pair and ω is a WZ form.

We call this theorem the change of variables theorem for WZ forms.

In (3.2.1) we have defined the sum of a WZ form along the paths which consist of right and up steps. Now define two more types of basic step: a left step as $(a_i, b_i), (a_i - 1, b_i)$, and a down step as $(a_i, b_i), (a_i, b_i - 1)$. We define the sum of the WZ form ω along these two steps by

$$\sum_{\text{left step}} \omega = -f(a_i - 1, b_i)$$

$$\sum_{\text{down step}} \omega = -g(a_i, b_i - 1).$$

With this definition we can extend the definition of the *sum of ω along any path π*, which is a sequence of the four type of basic steps, to be the sum of ω on each step of ω. It is obvious that the change of variables theorem generalizes to such paths as π. Moreover, we can define the sum of ω along $(a_i, b_i), (a_i+1, b_i+1)$ to be $f(a_i, b_i) + g(a_i+1, b_i)$, along $(a_i, b_i), (a_i+1, b_i-1)$ to be $f(a_i, b_i) - g(a_i+1, b_i-1)$, along $(a_i, b_i), (a_i-1, b_i-1)$ to be $-f(a_i-1, b_i) - g(a_i-1, b_i-1)$ and along $(a_i, b_i), (a_i-1, b_i+1)$ to be $-f(a_i-1, b_i) + g(a_i-1, b_i)$. With such definition, the type of steps of π can to extended to include the four types of diagonal steps and the path independence theorem still holds on such paths.

§3.3 Examples of Path Independence Theorem

In this section we start with the WZ pair for Gauss's theorem with numerator parameters differing by a half. We first show how to get Gauss's theorem by summing this WZ form on a path, then we'll show how summing along other paths gives other identities.

Let $F_1, \cdots, F_n$ be n functions of variables $a_1, \cdots, a_n$. For any i, j, we can think of (F_i, F_j) as functions of a_i and a_j with other variables fixed. If for all i, j, (F_i, F_j) is a WZ pair then we call $\omega = F_1 \delta a_1 + \cdots + F_n \delta a_n$ a WZ form in $a_1, \cdots, a_n$. In all the cases that we are interested in, it is easy to show that if for all i (F_1, F_i) is a WZ pair then for all i, j, (F_i, F_j) is also a WZ pair. We define the sum of ω along a step $(a_1, \cdots, a_i, \cdots, a_n), (a_1, \cdots, a_i+1, \cdots, a_n)$ to be $F_i(a_1, \cdots, a_i, \cdots, a_n)$. We also define the sum of ω along a step $(a_1, \cdots, a_i, \cdots, a_n), (a_1, \cdots, a_i-1, \cdots, a_n)$ to be $-F_i(a_1, \cdots, a_i - 1, \cdots, a_n)$. With this definition, the path independence theorem holds true for $n-$variable WZ forms and the proof is similar to the path independence theorem in two variables. For the sum of ω on an arbitrary step, we can always define it to be the sum along a path which has the same starting and ending point and consists only of the two types of steps on which we just defined the sum of ω.

The following is Gauss's theorem with numerator parameters differing by a half:

$$ {}_2F_1\left(\left.\begin{matrix} \frac{c}{2}, \frac{c}{2}+\frac{1}{2} \\ c+\frac{1}{2}+b \end{matrix}\right| 1\right) = \frac{2^{c+2\,b-1}\Gamma(b)\Gamma(c+\frac{1}{2}+b)}{\sqrt{\pi}\Gamma(c+2\,b)}, \tag{3.3.1}$$

where c is a nonpositive integer. If we expand the hypergeometric series in (3.3.1) with summation index a then divide each summand by the right hand side of (3.3.1) then (3.3.1) may be written as

$$\sum_a F_a = 1$$

where

$$F_a = \frac{\sqrt{\pi}\Gamma(c+2\,b)\Gamma(c+2\,a)2^{1-2a-2b-c}}{\Gamma(a+c+\frac{1}{2}+b)\Gamma(c)\Gamma(1+a)\Gamma(b)}.$$

With Gosper's algorithm we can find F_b, and F_c such that (F_a, F_b) and (F_a, F_c) are WZ pairs, and it follows that (F_b, F_c) is a WZ pair, so $F_a\delta a + F_b\delta b + F_c\delta c$ is a WZ form. We can leave out the constant factor $2\sqrt{\pi}$ in F_a and the WZ form in three variables we get is

$$\begin{aligned}
&\frac{2^{-2a-2b-c}\Gamma(c+2\,b)\Gamma(c+2\,a)}{\Gamma(1+a)\Gamma(b)\Gamma(c)\Gamma(a+c+\frac{1}{2}+b)}\delta a\\
&\quad+\frac{2^{-2a-2b-c}\Gamma(c+2\,b)\Gamma(c+2\,a)}{\Gamma(a)\Gamma(b+1)\Gamma(c)\Gamma(a+c+\frac{1}{2}+b)}\delta b\\
&\quad-\frac{2^{1-2a-2b-c}\Gamma(c+2\,b)\Gamma(c+2\,a)}{\Gamma(a)\Gamma(b)\Gamma(1+c)\Gamma(a+c+\frac{1}{2}+b)}\delta c.
\end{aligned} \tag{3.3.2}$$

Now we show how we can derive (3.3.1) when c is a nonpositive integer, say $-n$, with this WZ form using the path independence theorem. Suppose n is a nonnegative even integer. We pick a path from $(0,b,-n)$ to $(\frac{n}{2}+1,b,-n)$ with $(1,0,0)$ steps, then from $(\frac{n}{2}+1,b,-n)$ to $(\frac{n}{2}+1,b,0)$ with all $(0,0,1)$ steps, then from $(\frac{n}{2}+1,b,0)$ to $(0,b,0)$ with all $(-1,0,0)$ steps, then back to the starting point $(0,b,-n)$ with all $(0,0,-1)$ steps. The sum of the WZ form from $(0,b,-n)$ to $(\frac{n}{2}+1,b,-n)$ is

$$\frac{\Gamma(2\,b-n)2^{n-2\,b}}{\Gamma(-n+\frac{1}{2}+b)\Gamma(b)}\,{}_2F_1\left(\left.\begin{matrix}-\frac{n}{2},-\frac{n}{2}+\frac{1}{2}\\ -n+\frac{1}{2}+b\end{matrix}\right|1\right). \tag{3.3.3}$$

The sum of the WZ form from $(\frac{n}{2}+1,b,-n)$ to $(\frac{n}{2}+1,b,0)$ is

$$\sum_{c=-n}^{-1}\frac{2^{-c-2\,b}\Gamma(c+2\,b)(\frac{c}{2})_{\frac{n}{2}+1}(\frac{c}{2}+\frac{1}{2})_{\frac{n}{2}+1}}{\Gamma(b+1)\Gamma(\frac{n}{2}+1)\Gamma(c+\frac{1}{2}+b)(c+\frac{1}{2}+b)_{\frac{n}{2}+1}}.$$

Since $(\frac{c}{2})_{\frac{n}{2}+1}(\frac{c}{2}+1)_{\frac{n}{2}+1}$ is always 0, so the sum above is equal to 0. The sum on the steps from $(\frac{n}{2}+1,b,0)$ to $(0,b,0)$ is

$$-\sum_{a=0}^{\frac{n}{2}}\frac{\Gamma(2\,b)2^{-2\,b}(0)_a(\frac{1}{2})_a}{\Gamma(b)\Gamma(b+\frac{1}{2})(1)_a(b+\frac{1}{2})_a}.$$

The summand is 0 except when $a=0$ so the sum of the WZ form by the duplication formula for the gamma function is

$$-\frac{\Gamma(2\,b)2^{-2\,b}}{\Gamma(b+\frac{1}{2})\Gamma(b)}=-\frac{1}{2\sqrt{\pi}}.$$

The sum on steps from $(0,b,0)$ back to $(0,b,-n)$ is

$$-\sum_{c=0}^{-n}\frac{2^{-c-2\,b}\Gamma(c+2\,b)(\frac{c}{2})_0(\frac{c}{2}+\frac{1}{2})_0}{\Gamma(b+1)\Gamma(0)\Gamma(c+\frac{1}{2}+b)(c+\frac{1}{2}+b)_0}.$$

The sum is 0 due to $\Gamma(0)$ in the denominator of the summand. By the path independence theorem, the sum on the whole path is 0, so we get

$$\frac{\Gamma(c+2\,b)2^{-c-2\,b}}{\Gamma(c+\frac{1}{2}+b)\Gamma(b)}\,{}_2F_1\left(\left.\begin{matrix}\frac{c}{2},\frac{c}{2}+\frac{1}{2}\\ c+\frac{1}{2}+b\end{matrix}\right|1\right)=\frac{1}{2\sqrt{\pi}}.$$

After dividing all the extra factors to leave only the ${}_2F_1$ on the left side of the equation, we get (3.3.1). Similarly, we can prove the case when n is odd.

We use the term segment to refer to the connected section of a path which consists of the most possible same type of steps. So the path we chose above has four segments. Our observation is that the sum of the WZ form gives a hypergeometric series with some coefficient on one segment of the path. The sum on the parallel segment is the evaluation of that hypergeometric series with the coefficient at one of the parameters of the hypergeometric series equal to 0. As a result, the hypergeometric series has only one nonzero term which is 1. Thus the evaluation is a constant. Moreover, the terms of the sum of the WZ form on the other two parallel segments are all 0, hence, indicating the value of the hypergeometric series is a constant. In our following examples, all the paths we choose give us an evaluation of a hypergeometric series in the same manner.

Now we look at the sum of the WZ forms on different type of steps. The sum on step $(a, b, c+i), (a, b, c+i+1)$ is

$$-\frac{2^{1-c-i-2a-2b}\Gamma(c+i+2\,b)\Gamma(c+i+2\,a)}{\Gamma(c+i+1)\Gamma(a)\Gamma(a+c+i+\frac{1}{2}+b)\Gamma(b)}.$$

When c equals 0, summing on i gives a hypergeometric series

$$-\frac{\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})}{2\,\pi\,\Gamma(a+b+\frac{1}{2})}\,{}_2F_1\left(\begin{matrix}2\,b, 2\,a\\ a+b+\frac{1}{2}\end{matrix}\middle|\frac{1}{2}\right). \tag{3.3.4}$$

By setting a to be a nonpositive integer, say $-n$, we get a terminating hypergeometric series

$$-\frac{(\frac{1}{2}-b)_n}{2\sqrt{\pi}(\frac{1}{2})_n}\,{}_2F_1\left(\begin{matrix}2\,b, -2\,n\\ -n+b+\frac{1}{2}\end{matrix}\middle|\frac{1}{2}\right). \tag{3.3.5}$$

Furthermore we pick the following path which shows that the terminating hypergeometric series sums to a constant value. The path goes from $(-n, b, 0)$ to $(-n, b, 2n+1)$ with all $(0, 0, 1)$ steps which gives us (3.3.5), then to $(0, b, 2n+1)$ with $(1, 0, 0)$ steps on which the WZ form is all 0, then to $(0, b, 0)$ with all $(0, 0, -1)$ steps, the sum of the WZ form on which is the negative of the evaluation of (3.3.4) at $a = 0$, then back to $(-n, b, 0)$ with all $(-1, 0, 0)$ steps on which the WZ form is again all 0. Thus we have

$${}_2F_1\left(\begin{matrix}2\,b, -2\,n\\ -n+b+\frac{1}{2}\end{matrix}\middle|\frac{1}{2}\right) = \frac{(\frac{1}{2})_n}{(\frac{1}{2}-b)_n}.$$

Now we sum (3.3.2) on the step $(a+i, b-i, c), (a+i+1, b-i-1, c)$. We get

$$-\frac{\Gamma(c+2\,b-2-2\,i)\Gamma(c+2\,a+2\,i)\,(-b+2\,i+1+a)\,2^{2-2\,a-2\,b-c}}{\Gamma(c)\Gamma(a+i+1)\Gamma(b-i)\Gamma(a-\frac{1}{2}+c+b)}.$$

When $a = 0$, summing on i we get

$$-\frac{\Gamma(c+2\,b-2)\,(1-b)\,2^{2-2\,b-c}}{\Gamma(b)\Gamma(-\frac{1}{2}+c+b)}\,{}_4F_3\left(\begin{matrix}\frac{c}{2}, \frac{c}{2}+\frac{1}{2}, 1-b, \frac{3}{2}-\frac{b}{2}\\ \frac{3}{2}-\frac{c}{2}-b, 2-\frac{c}{2}-b, \frac{1}{2}-\frac{b}{2}\end{matrix}\middle|-1\right).$$

We set c to be a nonpositive integer $-n$ to make the above ${}_4F_3$ terminating

$$\frac{2^{-1+n}(\frac{3}{2}-b)_n}{(3-2\,b)_n\sqrt{\pi}}\,{}_4F_3\left(\left.\begin{matrix}\frac{3}{2}-\frac{b}{2},-\frac{n}{2},-\frac{n}{2}+\frac{1}{2},1-b\\ \frac{1}{2}-\frac{b}{2},\frac{3}{2}+\frac{n}{2}-b,2+\frac{n}{2}-b\end{matrix}\right|-1\right).$$

To prove the above is a constant we choose the path from $(0,b,-n)$ to $(\frac{n}{2}+1,b-\frac{n}{2}-1,-n)$ then to $(\frac{n}{2}+1,b-\frac{n}{2}-1,0)$ then to $(0,b,0)$ then back to $(0,b,-n)$. Note that we assumed that n is even. When n is odd, we can just use $(\frac{n+1}{2}+1,b,-n)$ and $(\frac{n+1}{2}+1,b,0)$ instead. The hypergeometric evaluation we get here is

$${}_4F_3\left(\left.\begin{matrix}-\frac{n}{2},-\frac{n}{2}+\frac{1}{2},1-b,\frac{3}{2}-\frac{b}{2}\\ \frac{3}{2}+\frac{n}{2}-b,2+\frac{n}{2}-b,\frac{1}{2}-\frac{b}{2}\end{matrix}\right|-1\right)=\frac{2^{-n}(3-2\,b)_n}{(\frac{3}{2}-b)_n}.$$

Lastly, we sum (3.3.2) on $(a+i,b+i,c-i),(a+i+1,b+i+1,c-i-1)$. We get

$$\frac{(2\,a+2\,b+3\,i+c)\,\Gamma(c+i+2\,b)\Gamma(c+i+2\,a)2^{-2\,a-c-1-2\,b-3\,i}}{\Gamma(c-i)\Gamma(b+1+i)\Gamma(a+i+1)\Gamma(a+c+i+\frac{1}{2}+b)}.$$

If we set $a=0$ and c to be a nonpositive integer $-n$ and sum on i we get

$$\frac{2^{-1+n}(\frac{1}{2}-b)_n}{\sqrt{\pi}(-2\,b)_n}\,{}_4F_3\left(\left.\begin{matrix}1+n,-\frac{n}{3}+\frac{2\,b}{3}+1,-n+2\,b,-n\\ b+1,-\frac{n}{3}+\frac{2\,b}{3},-n+b+\frac{1}{2}\end{matrix}\right|-\frac{1}{8}\right).$$

The path that can prove the above a constant is from $(0,0,-n)$ to $(n+1,n+1,-2n-1)$, then to $(n+1,n+1,-n-1)$, then to $(0,0,0)$, then back to $(0,0,-n)$. The hypergeometric evaluation we get is

$${}_4F_3\left(\left.\begin{matrix}1+n,-\frac{n}{3}+\frac{2\,b}{3}+1,-n+2\,b,-n\\ b+1,-\frac{n}{3}+\frac{2\,b}{3},-n+b+\frac{1}{2}\end{matrix}\right|-\frac{1}{8}\right)=\frac{2^{-n}(-2\,b)_n}{(\frac{1}{2}-b)_n}.$$

The change of variables theorem gives the connection between various forms of a hypergeometric evaluation and their WZ forms. That is, the WZ forms for the different forms of the same hypergeometric evaluation can be derived from one another via the change of variable theorem. For example, another form of (3.3.1) is the following which is obtained by replacing c by $c-b$ in (3.3.1)

$${}_2F_1\left(\left.\begin{matrix}\frac{c-b}{2},\frac{c-b}{2}+\frac{1}{2}\\ c+\frac{1}{2}\end{matrix}\right|1\right)=\frac{2^{c+b-1}\Gamma(b)\Gamma(c+\frac{1}{2})}{\sqrt{\pi}\Gamma(c+b)}, \tag{3.3.6}$$

The WZ form we get for (3.3.6) as we got (3.3.2) from (3.3.1) is

$$\begin{aligned}&\frac{2^{-2a-c-b}\Gamma(c-b+2\,a)\Gamma(b+c)}{\Gamma(c-b)\Gamma(1+a)\Gamma(b)\Gamma(a+c+\frac{1}{2})}\\ &\quad\frac{2^{1-2a-c-b}\Gamma(-1-b+c+2\,a)\Gamma(b+c)}{\Gamma(a-\frac{1}{2}+c)\Gamma(c-b)\Gamma(b+1)\Gamma(a)}\\ &\quad-\frac{2^{1-2a-c-b}\Gamma(c-b+2\,a)\Gamma(b+c)}{\Gamma(a)\Gamma(1-b+c)\Gamma(b)\Gamma(a+c+\frac{1}{2})}.\end{aligned} \tag{3.3.7}$$

If we use change of variables theorem to compute the new WZ form for (3.3.2) under the change of variables $a \to a, b \to b, c \to c - b$ we also get (3.3.7).

§3.4 WZ forms of linear hypergeometric transformations

Finding the WZ mate for a linear hypergeometric transformation is not trivial. For example, we take Euler's transformation

$$(1-x)^{b+c-d}\,{}_2F_1\left(\begin{matrix} b, c \\ d \end{matrix} \middle| x\right) = {}_2F_1\left(\begin{matrix} d-b, d-c \\ d \end{matrix} \middle| x\right). \tag{3.4.1}$$

We divide the left side summand by the right side and get

$$F(a,b) = \frac{(1-x)^{b+c-d}\dfrac{(b)_a(c)_a}{(d)_a}\dfrac{x^a}{a!}}{{}_2F_1\left(\begin{matrix} d-b, d-c \\ d \end{matrix} \middle| x\right)}. \tag{3.4.2}$$

Gosper's algorithm cannot find a WZ mate $G(a,b)$ for $F(a,b)$, such that $F(a,b+1) - F(a,b) = G(a+1,b) - G(a,b)$. In fact, $F(a,b)$ is not even a hypergeometric term. If we make a change of variables $c = d + m$, assuming that m is an integer, then the summand becomes

$$F(a,b) = \frac{(1-x)^{b+m}\dfrac{(b)_a(d+m)_a}{(d)_a}\dfrac{x^a}{a!}}{{}_2F_1\left(\begin{matrix} d-b, -m \\ d \end{matrix} \middle| x\right)}, \tag{3.4.3}$$

which can also be written as

$$F(a,b) = \frac{\Gamma(b+a)x^a(1-x)^{b+m}(d+m)_a}{\Gamma(b)\Gamma(a+1)_2F_1\left(\begin{matrix} d-b, -m \\ d \end{matrix} \middle| x\right)}.$$

Now if we let $m = 1$, the denominator is a polynomial in b, so $F(a,b)$ is a hypergeometric term. Applying Gosper's algorithm we get the WZ mate for (3.4.3)

$$-\frac{\Gamma(b+a)x^a(1-x)^{b+1}(d+a)_3F_2\left(\begin{matrix} d-b, d+a-1, -1 \\ d, d+a \end{matrix} \middle| x\right)}{\Gamma(b+1)\Gamma(a)\,d\,{}_2F_1\left(\begin{matrix} d-b, -1 \\ d \end{matrix} \middle| x\right){}_2F_1\left(\begin{matrix} d-b-1, -1 \\ d \end{matrix} \middle| x\right)}. \tag{3.4.4}$$

The WZ mate turns out to have a nice formula and it motivated us to look at more values of m in the hope that we can find a general form for the WZ mate for (3.4.3). When $m = 2$ we have the WZ mate for (3.4.3)

$$-\frac{\Gamma(b+a)x^a(1-x)^{b+2}(d+a)_2\,{}_3F_2\left(\begin{matrix} d-b, d+a-1, -2 \\ d, d+a \end{matrix} \middle| x\right)}{\Gamma(b+1)\Gamma(a)(d)_2\,{}_2F_1\left(\begin{matrix} d-b, -2 \\ d \end{matrix} \middle| x\right){}_2F_1\left(\begin{matrix} d-b-1, -2 \\ d \end{matrix} \middle| x\right)}. \tag{3.4.5}$$

When $m = 3$ we have

$$-\frac{\Gamma(b+a)x^a(1-x)^{b+3}(d+a)_3\ {}_3F_2\left(\begin{matrix} d-b, d+a-1, -3 \\ d, d+a \end{matrix}\middle| x\right)}{\Gamma(b+1)\Gamma(a)(d)_3\ {}_2F_1\left(\begin{matrix} d-b, -3 \\ d \end{matrix}\middle| x\right) {}_2F_1\left(\begin{matrix} d-b-1, -3 \\ d \end{matrix}\middle| x\right)}. \tag{3.4.6}$$

Hence, we claim that

$$G(a,b) = -\frac{\Gamma(b+a)x^a(1-x)^{b+m}(d+a)_m\ {}_3F_2\left(\begin{matrix} d-b, d+a-1, -m \\ d, d+a \end{matrix}\middle| x\right)}{\Gamma(b+1)\Gamma(a)(d)_m\ {}_2F_1\left(\begin{matrix} d-b, -m \\ d \end{matrix}\middle| x\right) {}_2F_1\left(\begin{matrix} d-b-1, -m \\ d \end{matrix}\middle| x\right)} \tag{3.4.7}$$

is the WZ mate of $F(a,b)$ in (3.4.3). To prove it we need to verify that

$$F(a,b+1) - F(a,b) = G(a+1,b) - G(a,b),$$

which is the following equation

$$\begin{aligned}
&\frac{(1-x)^{b+m+1}\frac{(b+1)_a(d+m)_a}{(d)_a}\frac{x^a}{a!}}{{}_2F_1\left(\begin{matrix} d-b-1, -m \\ d \end{matrix}\middle| x\right)} - \frac{(1-x)^{b+m}\frac{(b)_a(d+m)_a}{(d)_a}\frac{x^a}{a!}}{{}_2F_1\left(\begin{matrix} d-b, -m \\ d \end{matrix}\middle| x\right)} \\
&= -\frac{\Gamma(b+a+1)x^{a+1}(1-x)^{b+m}(d+a+1)_m\ {}_3F_2\left(\begin{matrix} d-b, d+a, -m \\ d, d+a+1 \end{matrix}\middle| x\right)}{\Gamma(b+1)\Gamma(a+1)(d)_m\ {}_2F_1\left(\begin{matrix} d-b, -m \\ d \end{matrix}\middle| x\right) {}_2F_1\left(\begin{matrix} d-b-1, -m \\ d \end{matrix}\middle| x\right)} \\
&\quad + \frac{\Gamma(b+a)x^a(1-x)^{b+m}(d+a)_m\ {}_3F_2\left(\begin{matrix} d-b, d+a-1, -m \\ d, d+a \end{matrix}\middle| x\right)}{\Gamma(b+1)\Gamma(a)(d)_m\ {}_2F_1\left(\begin{matrix} d-b, -m \\ d \end{matrix}\middle| x\right) {}_2F_1\left(\begin{matrix} d-b-1, -m \\ d \end{matrix}\middle| x\right)}.
\end{aligned} \tag{3.4.9}$$

This turns out to be a contiguous relation,

$$\begin{aligned}
&(1-x)^{b+m+1}\frac{(b+1)_a(d+m)_a}{(d)_a}\frac{x^a}{a!}{}_2F_1\left(\begin{matrix} d-b, -m \\ d \end{matrix}\middle| x\right) \\
&\quad - (1-x)^{b+m}\frac{(b)_a(d+m)_a}{(d)_a}\frac{x^a}{a!}{}_2F_1\left(\begin{matrix} d-b-1, -m \\ d \end{matrix}\middle| x\right) \\
&\quad = -\frac{\Gamma(b+a+1)x^{a+1}(1-x)^{b+m}(d+a+1)_m}{\Gamma(b+1)\Gamma(a+1)(d)_m}\ {}_3F_2\left(\begin{matrix} d-b, d+a, -m \\ d, d+a+1 \end{matrix}\middle| x\right) \\
&\qquad + \frac{\Gamma(b+a)x^a(1-x)^{b+m}(d+a)_m}{\Gamma(b+1)\Gamma(a)(d)_m}\ {}_3F_2\left(\begin{matrix} d-b, d+a-1, -m \\ d, d+a \end{matrix}\middle| x\right).
\end{aligned}$$

This relation can be proved by first dividing both sides by $x^a(1-x)^{b+m}$ then comparing the coefficient of x on both sides. Note that the proof does not require m to be an integer, so we can write the WZ pair as

$$F(a,b) = \frac{(1-x)^{b+c-d}\, x^a\Gamma(b+a)\Gamma(c+a)\Gamma(d)}{\Gamma(a+1)\Gamma(b)\Gamma(c)\Gamma(d+a){}_2F_1\left(\begin{matrix} d-b, -c+d \\ d \end{matrix}\middle| x\right)}$$

and

$$G(a,b) = -\frac{(1-x)^{b+c-d}\, x^a \Gamma(b+a)\Gamma(c+a)\Gamma(d)}{\Gamma(a)\Gamma(b+1)\Gamma(c)\Gamma(d+a)} \frac{{}_3F_2\left(\begin{matrix} d-b, d+a-1, -c+d \\ d, d+a \end{matrix}\middle| x\right)}{{}_2F_1\left(\begin{matrix} d-b, -c+d \\ d \end{matrix}\middle| x\right) {}_2F_1\left(\begin{matrix} d-b-1, -c+d \\ d \end{matrix}\middle| x\right)}.$$

It is easy to see that, if $(f(a,b),\ g(a,b))$ is a WZ pair, then $(-f(-a-1,b),\ g(-a,b))$ is also a WZ pair [5]. For example, the WZ pair for binomial theorem is

$$f(a,b) = \frac{\Gamma(b+1)x^a}{\Gamma(a+1)\Gamma(b-a+1)\,(1+x)^b},$$

and

$$g(a,b) = -\frac{a\Gamma(b+1)x^a}{(b-a+1)\,\Gamma(a+1)\Gamma(b-a+1)\,(1+x)^{b+1}}.$$

So the following is a new WZ pair,

$$f'(a,b) = -f(-a-1,b) = -\frac{\Gamma(b+1)x^{-a-1}}{\Gamma(-a)\Gamma(b+a+2)\,(1+x)^b},$$

and

$$g'(a,b) = g(-a,b) = \frac{a\Gamma(b+1)x^{-a}}{(b+a+1)\,\Gamma(-a+1)\Gamma(b+a+1)\,(1+x)^{b+1}}.$$

When a is a nonnegative integer, $f'(a,b)$ is zero. To make a more interesting WZ pair out of $(f'(a,b), g'(a,b))$ we recall another method that a new pair of WZ form is derived from an existing WZ pair in [5]. If $p(a,b)$ is a function such that $p(a+1,b) = p(a,b+1) = p(a,b)$ for all complex a and b for which $p(a,b)$ is defined then $(p(a,b)F(a,b), p(a,b)G(a,b))$ is a WZ-pair. Clearly $(\frac{F(a,b)}{p(a,b)}, \frac{G(a,b)}{p(a,b)})$ is also a WZ-pair. One such $p(a,b)$ is $(-1)^{sa+tb}[\sin\pi(sa+tb)]/\pi$, where $(-1)^\alpha$ is defined to be $e^{i\pi\alpha}$. By the reflection formula for the gamma function, we have $\Gamma(sa+tb)\Gamma(1-sa-tb) = \pi/\sin\pi(sa+tb)$. So $p(a,b)$ can also be expressed as

$$p(a,b) = \frac{(-1)^{sa+tb}}{\Gamma(sa+tb)\Gamma(1-sa-tb)}.$$

So we can multiply both $f'(a,b)$ and $g'(a,b)$ by

$$\frac{\Gamma(-a)\Gamma(1+a)}{(-1)^{-a}}$$

and the new WZ pair we get is

$$f''(a,b) = -\frac{(-1)^{-a}\,\Gamma(b+1)\Gamma(a+1)x^{-a-1}}{\Gamma(b+a+2)\,(1+x)^b}$$

and

$$g''(a,b) = -\frac{(-1)^{-a}\,\Gamma(b+1)\Gamma(a+1)x^{-a}}{\Gamma(b+a+2)\,(1+x)^{1+b}}.$$

Here when a is a nonnegative integer $p(a,b)$ is not defined. So we first define $f''(a,b)$ and $g''(a,b)$ where $p(a,b)$ is well defined, then extend the definition to any (a,b) by taking the limit. Similarly, we want to get a new WZ pair from the WZ pair we just got for Euler's transformation by taking the WZ pair $(-F(-a-1,b), G(-a,b))$ and multiplying both functions by

$$P(a,b) = \frac{\Gamma(-a)\Gamma(a+1)\,(-1)^{b-a}}{(-1)^{-a}\,\Gamma(-b+a+1)\Gamma(b-a)}.$$

The new WZ pair we obtained is

$$F'(a,b) = \frac{(-1)^b x^{-a-1}\,(1-x)^{b+c-d}\,\Gamma(d)\Gamma(a+1)\Gamma(c-a-1)}{\Gamma(2-b+a)\Gamma(c)\Gamma(b)\Gamma(d-a-1)\,{}_2F_1\left(\left.{d-b,-c+d \atop d}\right| x\right)}$$

and

$$G'(a,b) = -\frac{(-1)^b x^{-a}\,(1-x)^{b+c-d}\,\Gamma(d)\Gamma(a+1)\Gamma(c-a)}{\Gamma(d-a)\Gamma(b+1)\Gamma(c)\Gamma(-b+a+1)}\,\frac{{}_3F_2\left(\left.{d-b,d-a-1,-c+d \atop d,d-a}\right| x\right)}{{}_2F_1\left(\left.{d-b,-c+d \atop d}\right| x\right)\,{}_2F_1\left(\left.{d-b-1,-c+d \atop d}\right| x\right)}.$$

Notice that if a is a nonnegative integer, $F'(a,0)$ equals 0 and when $b > a+1$ $F'(a,b)$ equals 0 too, so the sum of $G'(a,b)$ on b is independent of n and since

$$\sum_{b=0}^{\infty} G'(0,b) = -1$$

we get the following identity

$$\sum_{b=0}^{a} \binom{a}{b} (x-1)^b \frac{{}_3F_2\left(\left.{d-b,d-c,d-a-1 \atop d,d-a}\right| x\right)}{{}_2F_1\left(\left.{d-b,d-c \atop d}\right| x\right)\,{}_2F_1\left(\left.{d-b-1,d-c \atop d}\right| x\right)} = (1-x)^{d-c}\frac{(1-c)_a}{(1-d)_a}x^a. \quad (3.4.10)$$

This identity is a dual of Euler's transformation in the sense of [5] and [11].

Now we want to try out the same method for Pfaff's transformation,

$${}_2F_1\left(\left.{b,c \atop d}\right| \frac{x}{x-1}\right) = (1-x)^b\,{}_2F_1\left(\left.{b,d-c \atop d}\right| x\right).$$

We make the substitution $c = d+m$ and divide the left hand summand by the right hand side, so we get

$$\begin{aligned} F(a,b) &= \frac{(-1)^a (b)_a (d+m)_a x^a}{(d)_a a!\,(1-x)^{a+b}\,{}_2F_1\left(\left.{b,-m \atop d}\right| x\right)} \\ &= \frac{(-1)^a \Gamma(a+b) x^a (d+m)_a}{\Gamma(a+1)\Gamma(b)(d)_a(1-x)^{a+b}\,{}_2F_1\left(\left.{b,-m \atop d}\right| x\right)} \qquad (3.4.11) \end{aligned}$$

When $m = 1$ we get the WZ mate for (3.4.11) as

$$\frac{(-1)^{a+1}\Gamma(b+a)x^a(-S(a,b,d,1)x + d(d+a))}{\Gamma(a)\Gamma(b+1)d^2(1-x)^{a+b}\,{}_2F_1\left(\left.\begin{smallmatrix} b,-1 \\ d \end{smallmatrix}\right| x\right)\,{}_2F_1\left(\left.\begin{smallmatrix} b+1,-1 \\ d \end{smallmatrix}\right| x\right)},$$

where $S(a,b,d,1) = bd + d + ba - b$.

When $m = 2$ we get the WZ mate as

$$\begin{aligned}&(-1)^{a+1}\Gamma(b+a)x^a \\ &\times \frac{S(a,b,d,2)x^2 - 2(d+1)(d+1+a)S(a,b,d,1)x + d(d+1)(d+a)(d+a+1)}{\Gamma(a)\Gamma(b+1)(d)_2(d)_2(1-x)^{a+b}\,{}_2F_1\left(\left.\begin{matrix} b,-2 \\ d \end{matrix}\right| x\right)\,{}_2F_1\left(\left.\begin{matrix} b+1,-2 \\ d \end{matrix}\right| x\right)},\end{aligned}$$

where $S(a,b,d,2) = a^2b + a^2b^2 - b^2a + 4abd + ba + 2ab^2d + b^2d^2 + 2d^2 + 3bd^2 - bd + 2d - b^2d - 2b$.

When $m = 3$ we get the WZ mate as

$$\frac{(-1)^{a+1}\Gamma(b+a)x^a A}{\Gamma(a)\Gamma(b+1)(d)_3(d)_3(1-x)^{a+b}\,{}_2F_1\left(\left.\begin{matrix} b,-3 \\ d \end{matrix}\right| x\right)\,{}_2F_1\left(\left.\begin{matrix} b+1,-3 \\ d \end{matrix}\right| x\right)},$$

where

$$\begin{aligned}A =& - S(a,b,d,3)x^3 + 3(d+2)(d+2+a)S(a,b,d,2)x^2 \\ &- 3(d+1)(d+2)(d+2+a)(d+a+1)S(a,b,d,1)x \\ &+ d(d+1)(d+2)(d+2+a)(d+a+1)(d+a)\end{aligned}$$

and

$$\begin{aligned}S(a,b,d,3) =& - 14bd - 12b + 12d - b^3d - 9b^2d + 15bd^2 \\ &+ 3b^2d^2 + 6a^2b + 6a^2b^2 + 18d^2 + 2a^3b + 3a^3b^2 + a^3b^3 \\ &+ 6d^3 + 11bd^3 + b^3d^3 + 6b^2d^3 - 9b^2a + 4ba - b^3a \\ &+ 18abd^2 + 3ab^3d^2 + 15ab^2d^2 + 27abd + 3a^2b^3d \\ &+ 12a^2b^2d + 9a^2bd + 9ab^2d.\end{aligned}$$

After some careful study of such $S(a,b,d,i)$ we find a formula for them:

$$S(a,b,d,i) = (d)_i i! \left(\sum_{j=0}^{i} \frac{(b)_j(a+d-1)_j}{(d)_j\, j!} \right).$$

Now we claim that

$$\begin{aligned}G(a,b) =& \frac{(-1)^{a+1}\Gamma(b+a)x^a(d+a)_m}{\Gamma(a)\Gamma(b+1)(d)_m(1-x)^{a+b}} \\ &\times \frac{\displaystyle\sum_{i=0}^{m} \frac{(-m)_i}{(d+a)_i} \left(\sum_{j=0}^{i} \frac{(b)_j(a+d-1)_j}{(d)_j\, j!} \right) x^i}{{}_2F_1\left(\left.\begin{matrix} b,-m \\ d \end{matrix}\right| x\right)\,{}_2F_1\left(\left.\begin{matrix} b+1,-m \\ d \end{matrix}\right| x\right)}\end{aligned}$$

is the WZ mate of $F(a,b)$ in (3.4.11). To prove this we need to prove that

$$\frac{(-1)^a(b+1)_a(d+m)_a x^a}{(d)_a a!\,(1-x)^{a+b+1}{}_2F_1\left(\begin{matrix}b+1,-m\\d\end{matrix}\middle|\,x\right)} - \frac{(-1)^a(b)_a(d+m)_a x^a}{(d)_a a!\,(1-x)^{a+b}{}_2F_1\left(\begin{matrix}b,-m\\d\end{matrix}\middle|\,x\right)}$$
$$= \frac{(-1)^{a+2}\Gamma(b+a+1)x^{a+1}(d+a+1)_m}{\Gamma(a+1)\Gamma(b+1)(d)_m(1-x)^{a+b+1}} \times \frac{\sum_{i=0}^{m}\frac{(-m)_i}{(d+a+1)_i}\left(\sum_{j=0}^{i}\frac{(b)_j(a+d)_j}{(d)_j\,j!}\right)x^i}{{}_2F_1\left(\begin{matrix}b,-m\\d\end{matrix}\middle|\,x\right){}_2F_1\left(\begin{matrix}b+1,-m\\d\end{matrix}\middle|\,x\right)}$$
$$- \frac{(-1)^{a+1}\Gamma(b+a)x^a(d+a)_m}{\Gamma(a)\Gamma(b+1)(d)_m(1-x)^{a+b}} \times \frac{\sum_{i=0}^{m}\frac{(-m)_i}{(d+a)_i}\left(\sum_{j=0}^{i}\frac{(b)_j(a+d-1)_j}{(d)_j\,j!}\right)x^i}{{}_2F_1\left(\begin{matrix}b,-m\\d\end{matrix}\middle|\,x\right){}_2F_1\left(\begin{matrix}b+1,-m\\d\end{matrix}\middle|\,x\right)}.$$

After multiplying both sides by the product of the two contiguous hypergeometric series in the denominator we get the following

$$\frac{(-1)^a(b+1)_a(d+m)_a x^a}{(d)_a a!\,(1-x)^{a+b+1}}{}_2F_1\left(\begin{matrix}b,-m\\d\end{matrix}\middle|\,x\right) - \frac{(-1)^a(b)_a(d+m)_a x^a}{(d)_a a!\,(1-x)^{a+b}}{}_2F_1\left(\begin{matrix}b+1,-m\\d\end{matrix}\middle|\,x\right)$$
$$= \frac{(-1)^{a+2}\Gamma(b+a+1)x^{a+1}(d+a+1)_m}{\Gamma(a+1)\Gamma(b+1)(d)_m(1-x)^{a+b+1}} \times \sum_{i=0}^{m}\frac{(-m)_i}{(d+a+1)_i}\left(\sum_{j=0}^{i}\frac{(b)_j(a+d)_j}{(d)_j\,j!}\right)x^i$$
$$- \frac{(-1)^{a+1}\Gamma(b+a)x^a(d+a)_m}{\Gamma(a)\Gamma(b+1)(d)_m(1-x)^{a+b}} \times \sum_{i=0}^{m}\frac{(-m)_i}{(d+a)_i}\left(\sum_{j=0}^{i}\frac{(b)_j(a+d-1)_j}{(d)_j\,j!}\right)x^i. \tag{3.4.12}$$

We multiply both sides by $\frac{(-1)^a x^a}{(1-x)^{a+b+1}}$ and compare the coefficient of x^n of the two sides. The left hand side coefficient is

$$\frac{(b+1)_a(d+m)_a(b)_n(-m)_n}{(d)_a a!\,(d)_n n!} - \frac{(b)_a(d+m)_a(b+1)_n(-m)_n}{(d)_a a!\,(d)_n n!} + \frac{(b)_a(d+m)_a(b+1)_{n-1}(-m)_{n-1}}{(d)_a a!\,(d)_{n-1}\,(n-1)!} \tag{3.4.13}$$

On the right hand side the coefficient is

$$\begin{aligned}
&\frac{\Gamma(b+a+1)(a+d+1)_m(-m)_{n-1}}{(d)_m\Gamma(a+1)\Gamma(b+1)(d+a+1)_{n-1}}\sum_{j=0}^{n-1}\frac{(b)_j(a+d)_j}{(d)_j\, j!}\\
&\quad+\frac{\Gamma(b+a)(a+d)_m(-m)_n}{(d)_m\Gamma(a)\Gamma(b+1)(d+a)_n}\sum_{j=0}^{n}\frac{(b)_j(a+d-1)_j}{(d)_j\, j!}\\
&\quad-\frac{\Gamma(b+a)(a+d)_m(-m)_{n-1}}{(d)_m\Gamma(a+1)\Gamma(b+1)(d+a)_{n-1}}\sum_{j=0}^{n-1}\frac{(b)_j(a+d-1)_j}{(d)_j\, j!}.
\end{aligned} \tag{3.4.14}$$

Let's denote the second summation by $T(a-1,n)$. The coefficient can be written as

$$\begin{aligned}
&\frac{\Gamma(b+a+1)(a+d+1)_m(-m)_{n-1}}{(d)_m\Gamma(a+1)\Gamma(b+1)(d+a+1)_{n-1}}\left(T(a,n)-\frac{(b)_n(d+a)_n}{(d)_n\, n!}\right)\\
&+\frac{\Gamma(b+a)(a+d)_m(-m)_n}{(d)_m\Gamma(a)\Gamma(b+1)(d+a)_n}T(a-1,n)\\
&-\frac{\Gamma(b+a)(a+d)_m(-m)_{n-1}}{(d)_m\Gamma(a+1)\Gamma(b+1)(d+a)_{n-1}}\left(T(a-1,n)-\frac{(b)_n(d+a-1)_n}{(d)_n\, n!}\right).
\end{aligned}$$

In the following we try to simplify this coefficient by expressing $T(a,n)$ in term of $T(a-1,n)$. When b is a negative integer and a is not a nonnegative integer,

$$f(j,a)=\frac{\Gamma(d-b)\Gamma(-a)\Gamma(b+j)\Gamma(d+a+j)}{\Gamma(b)\Gamma(d+a)\Gamma(-b-a)\Gamma(d+j)\, j!}$$

has a WZ mate,

$$g(j,a)=-\frac{\Gamma(-a-1)\Gamma(d+a+j)\Gamma(b+j)\Gamma(d-b)}{\Gamma(d+a+1)\Gamma(j)\Gamma(d+j-1)\Gamma(-b-a)\Gamma(b)}.$$

That is to say

$$f(j,a)-f(j,a-1)=g(j+1,a-1)-g(j,a-1).$$

If we divide the equation above by $\Gamma(-a)$ we get the following

$$\begin{aligned}
&\frac{\Gamma(b+j)\Gamma(d+a+j)\Gamma(d-b)}{\Gamma(j+1)\Gamma(d+j)\Gamma(b)\Gamma(d+a)\Gamma(-b-a)}+\frac{\Gamma(b+j)\Gamma(d+a-1+j)\Gamma(d-b)a}{\Gamma(j+1)\Gamma(d+j)\Gamma(b)\Gamma(d+a-1)\Gamma(-b-a+1)}\\
&=-\frac{\Gamma(j+1+b)\Gamma(d+a+j)\Gamma(d-b)}{\Gamma(-b-a+1)\Gamma(j+1)\Gamma(d+j)\Gamma(b)\Gamma(d+a)}+\frac{\Gamma(b+j)\Gamma(d+a-1+j)\Gamma(d-b)}{\Gamma(j)\Gamma(d+j-1)\Gamma(-b-a+1)\Gamma(b)\Gamma(d+a)}.
\end{aligned}$$

This equation holds true for any b and a. If we sum both sides of the equation on j from 0 to n, then the right hand side has only one term left and the left hand side gives

$$\frac{\Gamma(d-b)T(a,n)}{\Gamma(d)\Gamma(-b-a)}+\frac{a\Gamma(d-b)T(a-1,n)}{\Gamma(d)\Gamma(-b-a+1)}=-\frac{\Gamma(d+a+n)\Gamma(b+n+1)\Gamma(d-b)}{\Gamma(-b-a+1)\Gamma(n+1)\Gamma(d+n)\Gamma(b)\Gamma(d+a)}.$$

Solving the above equation for $T(a,n)$, we get that $T(a,n)$ equals

$$-\frac{\Gamma(-b-a)(aT(a-1,n)\Gamma(d+a)\Gamma(n+1)\Gamma(d+n)\Gamma(b)+\Gamma(d+a+n)\Gamma(b+n+1)\Gamma(d))}{\Gamma(-b-a+1)\Gamma(d+a)\Gamma(n+1)\Gamma(d+n)\Gamma(b)}.$$

Thus, we can have expressed $T(a,n)$ in term of $T(a-1,n)$, and we can write (3.4.14) as

$$\begin{aligned}
&\frac{\Gamma(b+a+1)(a+d+1)_m(-m)_{n-1}}{(d)_m\Gamma(a+1)\Gamma(b+1)(d+a+1)_{n-1}} \\
&\times\Bigg(-\frac{\Gamma(-b-a)(aT(a-1,n)\Gamma(d+a)\Gamma(n+1)\Gamma(d+n)\Gamma(b)+\Gamma(d+a+n)\Gamma(b+n+1)\Gamma(d))}{\Gamma(-b-a+1)\Gamma(d+a)\Gamma(n+1)\Gamma(d+n)\Gamma(b)} \\
&\quad -\frac{(b)_n(d+a)_n}{(d)_n\, n!}\Bigg)+\frac{\Gamma(b+a)(a+d)_m(-m)_n}{(d)_m\Gamma(a)\Gamma(b+1)(d+a)_n}T(a-1,n) \\
&\quad -\frac{\Gamma(b+a)(a+d)_m(-m)_{n-1}}{(d)_m\Gamma(a+1)\Gamma(b+1)(d+a)_{n-1}}(T(a-1,n)-\frac{(b)_n(d+a-1)_n}{(d)_n\, n!}).
\end{aligned} \tag{3.4.16}$$

The coefficient of $T(a-1,n)$ in (3.4.16) is

$$\begin{aligned}
&-\frac{\Gamma(b+a+1)(a+d+1)_m(-m)_{n-1}\Gamma(-b-a)a}{(d)_m\Gamma(a+1)(a+d+1)_{n-1}\Gamma(b+1)\Gamma(-b-a+1)} \\
&\quad+\frac{\Gamma(b+a)(d+a)_m(-m)_n}{(d)_m\Gamma(a)\Gamma(b+1)(d+a)_n} \\
&\quad-\frac{\Gamma(b+a)(d+a)_m(-m)_{n-1}}{(d)_m\Gamma(a)\Gamma(b+1)(d+a)_{n-1}}.
\end{aligned}$$

The above reduces to 0 when we simplify it in Maple. Moreover, simplifying the difference (3.4.13) − (3.4.16) in Maple gives 0. Hence we proved that the contiguous relation in (3.4.12) is true. Therefore, we have proved that

$$\begin{aligned}
G(a,b)=&\frac{(-1)^{a+1}\Gamma(b+a)x^a(d+a)_m}{\Gamma(a)\Gamma(b+1)(d)_m(1-x)^{a+b}} \\
&\times\frac{\displaystyle\sum_{i=0}^{m}\frac{(-m)_i}{(d+a)_i}\left(\sum_{j=0}^{i}\frac{(b)_j(a+d-1)_j}{(d)_j\, j!}\right)x^i}{{}_2F_1\left(\begin{matrix}b,-m\\d\end{matrix}\middle|\,x\right){}_2F_1\left(\begin{matrix}b+1,-m\\d\end{matrix}\middle|\,x\right)}
\end{aligned}$$

is the WZ mate of

$$F(a,b)=\frac{(-1)^a\Gamma(a+b)x^a(d+m)_a}{\Gamma(a+1)\Gamma(b)(d)_a(1-x)^{a+b}{}_2F_1\left(\begin{matrix}b,-m\\d\end{matrix}\middle|\,x\right)}.$$

§3.5 Symmetric Function and Path Independence Theorem of WZ forms

So far all the WZ forms we studied are for hypergeometric functions. In this section we use the symmetric function WZ pair defined in [7] and apply the path independence theorem to prove the q-binomial identity in [6].

Let us define the complete symmetric functions h_n in the variables $u_1,u_2,\cdots,u_m$ by

$$h_n(u_0,u_1,\cdots,u_m)=\sum_{0\le i_1\le i_2\le\cdots\le i_n\le m}u_{i_1}u_{i_2}\cdots u_{i_n},$$

with $h_0(u_0, u_1, \cdots, u_m) = 1$ and $h_n(u_0, u_1, \cdots, u_m) = 0$ for $n < 0$. We set

$$H_n(j,k) = h_n(x_j, x_{j+1}, \cdots, x_k),$$

with $H_0(j,k) = 1$ for all j and k, $H_n(j,k) = 0$ for $n < 0$ and $H_n(j, j-1) = 0$ for $n \neq 0$. If we weight a step from (a,b) to $(a+1,b)$ with the weight x_b, where the x_i are indeterminates, and assign vertical steps the weight 1, then the sum of the weights of all paths from (α, β) to (γ, δ) is $H_{\gamma-\alpha}(\beta, \delta)$. Given $s \geq 0, t \geq 0$, we look at all the paths in the region of $\mathbf{Z}\times$ $\mathbf{Z}$: $\{(a,b)|0 \leq a \leq s, 0 \leq b \leq t\}$, which start at $(0,0)$ and end at (s,t), and have steps $(0,1)$ and $(1,0)$. Let's name this region R. For all positive $a < s, b < t$ the set of paths that go into the point (a,b) is the same as the set of paths that go out of the point (a,b). So we can express the weight of the set of paths that go through point (a,b) in two ways. One way is the weight of all paths going through step $(a-1,b),(a,b)$ or $(a,b-1),(a,b)$. The other way is the weight of all paths going through step $(a,b),(a+1,b)$ or $(a,b),(a,b+1)$. The weight of all paths going through step $(a-1,b),(a,b)$ is $H_{a-1}(0,b)x_bH_{s-a}(b,t)$, and the weight of all paths going through step $(a,b-1),(a,b)$ is $H_a(0,b-1)H_{s-a}(b,t)$. On the other hand, the weight of all paths going through step $(a,b),(a+1,b)$ is $H_a(0,b)x_bH_{s-a-1}(b,t)$ and the weight of all paths going through step $(a,b),(a,b+1)$ is $H_a(0,b)H_{s-a}(b+1,t)$. Hence we have

$$\begin{aligned} &H_{a-1}(0,b)x_bH_{s-a}(b,t) + H_a(0,b-1)H_{s-a}(b,t) \\ &\quad = H_a(0,b)x_bH_{s-a-1}(b,t) + H_a(0,b)H_{s-a}(b+1,t), \end{aligned} \tag{3.5.1}$$

Now we look at the boundary points of R. When $a = 0, b = 0$ the left side of (3.5.1) is

$$H_{-1}(0,0)x_0H_s(0,t) + H_0(0,-1)H_s(0,t) = H_s(0,t).$$

The right side is

$$H_0(0,0)x_0H_{s-1}(0,t) + H_0(0,0)H_s(1,t) = x_0H_{s-1}(0,t) + H_s(1,t).$$

The left side equals the weight of all paths going from $(0,0)$ to (s,t), while the right side equals the weight of all paths going through step $(0,0),(1,0)$ or $(0,0,),(0,1)$. Clearly, the left side equals the right side. Similarly, we can prove that (3.5.1) holds true at all other boundary points. Thus (3.5.1) holds true for all (a,b) in the region R. Now we define

$$f(a,b) = H_a(0,b-1)H_{s-a}(b,t)$$

$$g(a,b) = -H_{a-1}(0,b)x_bH_{s-a}(b,t).$$

Then (3.5.1) may be written as

$$f(a, b+1) - f(a, b) = g(a+1, b) - g(a, b).$$

Then $f(a,b)\delta a + g(a,b)\delta b$ is a WZ form in the sense that path independence theorem holds for paths within the rectangle whose vertices are $(a,b), (a+1,b), (a+1,b+1), (a,b+1)$. Thus $f(a,b)\delta a+g(a,b)\delta b$ is a WZ form in the region R, $\mathbf{Z}\times\mathbf{Z}$: $\{(a,b)|0 \le a \le s, 0 \le b \le t\}$. Moreover path independence theorem holds on all paths in $\{(a,b)|0 \le a \le s+1, 0 \le b \le t+1\}$, in another words, the sum of the WZ form along any path within this region depends only on the starting and ending points of the path.

We also find that the weight of a northwest step from $(a+1, b-1)$ to (a, b) is $-f(a,b) + g(a+1, b-1) = -H_a(0, b-1)H_{s-a}(b-1, t)$.

The q-binomial coefficient is defined to be

$$\begin{bmatrix} s \\ t \end{bmatrix} = \prod_{i=0}^{t-1} \frac{q^{s-i} - 1}{q^{t-i} - 1}.$$

Note that the above is 0 if $t < 0$. If we set $x_i = q^i$ then we have [10, p.303]

$$H_n(j,k) = h_n(x_j, x_{j+1}, \cdots, x_k) = q^{j(k-j+1)} h_n(1, q, \cdots, q^{k-j}) = q^{j(k-j+1)} \begin{bmatrix} k-j+n \\ n \end{bmatrix}. \qquad (3.5.2)$$

Now we prove the q-Vandermonde variation in Gessel [6] via the path independence theorem. In the proof we always evaluate the symmetric functions by (3.5.2).

Theorem 3.3 Let a_k and b_k be nonnegative integers, $a_0 b_0 = 0$, $a_k b_k \neq 0$ for $k > 0$, $(a_{k+1} - a_k, b_{k+1} - b_k)$ is $(0,1), (1,0)$, or $(1,1)$. Then for $s \ge a_0$, $t \ge b_0$ we have

$$\sum_{k=0}^{\infty} q^{a_k b_k} \begin{bmatrix} s - a_k + b_{k+1} - 1 \\ s - a_k \end{bmatrix} \begin{bmatrix} t - b_k + a_{k+1} - 1 \\ t - b_k \end{bmatrix} = \begin{bmatrix} s+t \\ s \end{bmatrix}.$$

Proof: Let $f(a,b) = H_a(0, b-1)H_{s-a}(b,t)$ and $g(a,b) = -H_{a-1}(0,b)x_b H_{s-a}(b,t)$. Then $\omega = f(a,b)\delta a + g(a,b)\delta b$ is a WZ form for $0 \le a \le s$ and $0 \le b \le t$, and the sum of ω over any path in the region $[0, s+1] \times [0, t+1]$ depends only on the ending and starting point. Now, we look at the path whose vertices are $(s+1-a_k, b_k)$. Obviously this path intersects the region $[0, s+1] \times [0, t+1]$. Let P be the path whose vertices are the points $(s+1-a_k, b_k)$ which are in the region. Clearly the starting point of P is $(s+1-a_0, b_0)$. Let's assume the ending point of P is $(s+1-a_n, b_n)$. Note that either $s+1-a_n = 0$ or $b_n = t+1$. We take two paths from $(s+1, 0)$ to $(0,0)$. The first path is all west steps, and the second goes from $(s+1, 0)$ to $(s+1-a_0, b_0)$, then along the path P to

$(s+1-a_n, b_n)$, then from $(s+1-a_n, b_n)$ to $(0, t+1)$, finally all south steps back to $(0,0)$. The sum of ω on the first path is $-f(0,0) = -H_s(0,t) = \begin{bmatrix} s+t \\ t \end{bmatrix}$. Now we look at the sum of ω on the second path. The path from $(s+1, 0)$ to $(s+1-a_0, b_0)$ will be all west steps if $b_0 = 0$ or all north steps if $a_0 = 0$. If it consists of all west steps, the first coordinate i of each step has the property that $s+1 \geq i \geq s+1-a_0 \geq 1$ so the sum of ω on such a west step is $-f(i,0) = 0$. So the sum of ω on all these west steps is 0. If all steps on this path are north, the sum of ω on each north step is $g(s+1, b)$ and $g(s+1, b) = -H_s(0,b)x_bH_{-1}(b,t) = 0$. So, we have the sum of ω along the path from $(s+1, 0)$ to $(s+1-a_0, b_0)$ is 0. The path from $(s+1-a_n, b_n)$ to $(0, t+1)$ can either be all west steps if $b_n = t+1$ or all north steps if $a_n = 0$. If it consists of all west steps, the sum of ω on such a step is $-f(a, t+1)$. Since $a_k \geq 1$ when $k > 0$ and $(a_n, b_n) \neq (a_0, b_0)$, a_n is less than $s+1$, hence $-f(a, t+1) = 0$ and the sum of ω on all the west steps is 0. If the path consists of all north steps, we have $g(0, b) = 0$, so the sum of ω on all the north steps is 0. Thus, we have the sum of ω along the path from (a_n, b_n) to $(0, t+1)$ is 0. Clearly the sum of ω along the path from $(0, t+1)$ to $(0,0)$ is 0. Therefore, we have proved that the sum of ω on the second path is equal to the sum of ω on path P. Now we look at the sum of ω on the steps of P. There are three types of step on P. When $(a_{k+1} - a_k, b_{k+1} - b_k)$ is $(1, 0)$, we have a west step in P: $(s+1-a_k, b_k)$, $(s+1-a_{k+1}, b_{k+1})$. The sum of ω on this step is

$$\begin{aligned} -f(s+1-a_{k+1}, b_{k+1}) =& -H_{s+1-a_{k+1}}(0, b_{k+1}-1)H_{a_{k+1}-1}(b_{k+1}, t) \\ =& -q^{(a_{k+1}-1)b_{k+1}} \begin{bmatrix} s+1-a_{k+1}+b_{k+1}-1 \\ s+1-a_{k+1} \end{bmatrix} \begin{bmatrix} t-b_{k+1}+a_{k+1}-1 \\ a_{k+1}-1 \end{bmatrix} \\ =& q^{a_kb_k} \begin{bmatrix} s-a_k+b_{k+1}-1 \\ s-a_k \end{bmatrix} \begin{bmatrix} t-b_k+a_{k+1}-1 \\ t-b_k \end{bmatrix}. \end{aligned}$$

When $(a_{k+1} - a_k, b_{k+1} - b_k)$ is $(0, 1)$, we have a north step in P: $(s+1-a_k, b_k)$, $(s+1-a_{k+1}, b_{k+1})$. The sum of ω on this step is

$$\begin{aligned} g(s+1-a_{k+1}, b_{k+1}) =& -H_{s-a_k}(0, b_k-1)H_{a_{k+1}-1}(b_{k+1}, t) \\ =& -q^{(a_{k+1}-1)b_{k+1}} \begin{bmatrix} s+1-a_{k+1}+b_{k+1}-1 \\ s+1-a_{k+1} \end{bmatrix} \begin{bmatrix} t-b_{k+1}+a_{k+1}-1 \\ a_{k+1}-1 \end{bmatrix} \\ =& q^{a_kb_k} \begin{bmatrix} s-a_k+b_{k+1}-1 \\ s-a_k \end{bmatrix} \begin{bmatrix} t-b_k+a_{k+1}-1 \\ t-b_k \end{bmatrix}. \end{aligned}$$

When $(a_{k+1} - a_k, b_{k+1} - b_k)$ is $(1, 1)$, we have a northwest step in P: $(s+1-a_k, b_k)$, $(s+1-a_{k+1}, b_{k+1})$. The sum of ω on this step is

$$\begin{aligned} -H_a(0, b-1)H_{s-a}(b-1, t) =& -q^{a_kb_k} \begin{bmatrix} s+1-a_{k+1}+b_{k+1}-1 \\ s+1-a_{k+1} \end{bmatrix} \begin{bmatrix} t-b_{k+1}+1+a_{k+1}-1 \\ a_{k+1}-1 \end{bmatrix} \\ =& q^{a_kb_k} \begin{bmatrix} s-a_k+b_{k+1}-1 \\ s-a_k \end{bmatrix} \begin{bmatrix} t-b_k+a_{k+1}-1 \\ t-b_k \end{bmatrix}. \end{aligned}$$

Hence we proved the identity.

References

[1] T. Amdeberhan and D. Zeilberger, Hypergeometric series acceleration via the WZ Method, Electronic Journal of Combinatorics, 4 (1997).

[2] Bateman Manuscript Project–Higher Transcendental Functions, Vol. 1 (McGraw Hill, 1954).

[3] W. Bailey, Generalized Hypergeometric Series, (Cambridge University Press, New York, 1972).

[4] H. Exton, Multiple Hypergeometric Functions and Applications, (Ellis Horwood Ltd, 1976).

[5] I. M. Gessel, Finding identities with the WZ Method, J. Symbolic Computation, 20 (1995) 537–566.

[6] I. M. Gessel, Some generalized Durfee square identities, Discrete Math., 49 (1984), 41–44.

[7] I. M. Gessel, Combinatorial aspects of the WZ method, unpublished manuscript, (2001).

[8] R. W. Gosper Jr., Decision procedure for indefinite hypergeometric series, SIAM J. Math. Anal., 12 (1978) 295–308.

[9] M. Petkovsek, H.S. Wilf, and D. Zeilberger, $A = B$, (A.K. Peters, Ltd., Wellesley, Mass., 1996).

[10] R. P. Stanley, Enumerative Combinatorics, Vol. 2, (Cambridge University Press, 1999).

[11] H.S. Wilf, and D. Zeilberger, Rational functions certify combinatorial identities, J. Amer. Math. Soc. 3 (1990) 147–158.

[12] D. Zeilberger, Closed form (pun intended!) Contemporary Mathematics, 143 (1993) 579–607.