Topics in Consumption Theory

1. Graphs and Motivation
2. Intertemporal Elasticity of Substitution
3. The General Model
4. Consumption as a Random Walk
5. Certainty Equivalence
6. Precautionary Saving

Intertemporal Elasticity of Substitution

- Let

$$\sigma(c(t), c(s)) \equiv -\frac{\Delta\% \left[\frac{c(t)}{c(s)}\right]}{\Delta\% \left[\frac{u'(c(t))}{u'(c(s))}\right]}$$

- Letting $s \longrightarrow t$, we obtain

$$\sigma(c(t)) = \frac{u'(c)}{cu''(c)},$$

  the instantaneous elasticity of substitution

- High elasticity: consumption responds a lot to Δr
- Inverse of $\sigma(c)$ is the coefficient of relative risk aversion (CRRA)

An Useful Family of Utility Functions

- Often use *constant intertemporal elasticity of substitution* (CES) utility function.

$$u(c) = \begin{cases} \frac{c^{1-\theta}-1}{1-\theta} & \theta > 0, \theta \neq 1 \\ \ln(c) & \theta = 1 \end{cases}$$

- $\sigma(c) = \frac{1}{\theta}$, independent of c
- CRRA $= \theta$.

The General Model

Consider the following model:

$$\max_{c_t} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to:

$$a_{t+1} = (1 + r_t)(a_t + y_t - c_t)$$

where

$$\begin{aligned}
a_t &= \text{beginning of period } t \text{ assets} \\
y_t &= \text{period } t \text{ income} \\
c_t &= \text{period } t \text{ consumption} \\
r_t &= \text{period } t \text{ return of assets} \\
\beta &\equiv \frac{1}{1+\delta} \text{ subjective discount factor} \\
\delta &= \text{subjective discount rate (or rate of pure time preference)}
\end{aligned}$$

with $u(c_t)$ increasing and concave. y_t and r_t may be random variables.

Bellman Formulation

$$V(a, y, r) = \max_{c} \{u(c) + \beta EV(a', y', r')\}$$

subject to

$$a' = (1 + r)(a + y - c)$$

- Let's make our life easier
  - Set $y_t = \bar{y} = 0$ (diversifiable labor income case)
  - Set $r_t = r$ (constant interest rate)
- Now there is no uncertainty

- Bellman equation can be written as

$$V(a) = \max_c \left\{ u(a - \frac{a'}{1+r}) + \beta V(a') \right\}$$

- first-order condition is

$$u'\left(a - \frac{a'}{1+r}\right)\frac{-1}{1+r} + \beta V'(a') = 0$$

- The envelop condition is

$$V'(a) = u'\left(a - \frac{a'}{1+r}\right)$$

- These two conditions imply the following Euler equation

$$u'(c_t) = (1+r)\beta u'(c_{t+1})$$

or

$$u'(c_t) = \frac{1+r}{1+\delta} u'(c_{t+1})$$

- Hence

$$\delta = r \Rightarrow c_t \text{ constant}$$
$$\delta > r \text{ (impatient)} \Rightarrow c_t \text{ downward sloping}$$
$$\delta < r \text{ (patient)} \Rightarrow c_t \text{ upward sloping}$$

- If we assume that u is CES:

$$\frac{c_{t+1}}{c_t} = (\beta(1+r))^{1/\theta}$$

- Hence

$$\Delta \ln(c_{t+1}) = \frac{\ln(\beta) + \ln(1+r)}{\theta} \approx \frac{r - \delta}{\theta}$$

Guess and Verify a Solution

- Guess solution of the form

$$V(a) = G + H \ln(a)$$

- Can show $H = \frac{1}{1-\beta}$ and $c = \frac{\delta}{1+\delta} A$
- Consumption is linear in wealth.
- Consumption/saving decision depends only on subjective discount rate δ, not on r.

Certainty Equivalence

Four Assumptions

1. Quadratic utility

$$u(c) = c - \frac{b}{2}c^2$$

- Thus marginal utility is linear in consumption.
- Simplifies the math considerably
- Unfortunate consequences: 1) $u'(0) < \infty$ so negative consumption may be optimal; 2) may hit bliss point after which $u'(c) < 0$.

2. $r_t = r = \delta$ thus $\beta = \frac{1}{1+r}$.

3. y_t an i.i.d. random variable, observe before c_t decision

4. Infinite horizon

- Define $x_t \equiv a_t + y_t$ to be "cash on hand"
- So Bellman equation is

$$V(x) = \max_c \{u(c) + \beta EV(x')\}$$

subject to:

$$x' = (1+r)(x-c) + y'$$

- Use the first-order condition and the envelop condition to derive the Euler equation:

$$u'(c_t) = \beta(1+r)E_t u'(c_{t+1})$$

or

$$E_t u'(c_{t+1}) = \frac{1}{\beta(1+r)} u'(c_t)$$

- Marginal utility of consumption follows a Markov process.

- Since utility is quadratic, and $\beta(1+r) = 1$, this Euler equation becomes:

$$c_t = E_t c_{t+1}$$

- Let

$$\epsilon_t \equiv C_t - E_{t-1} C_t.$$

- Then

$$C_t = C_{t-1} + \epsilon_t$$

  with

$$E_{t-1} \epsilon_t = 0$$

- Hence c_t follows a random walk.
- Citation: Hall, Robert E. (1978) "Stochastic Implications of the Life Cycle-Permanent Income Hypothesis: Theory and Evidence" *Journal of Political Economy* 86(6): 971-988.

Explicit Expressions for c_t and Δc_t

- Use inter-temporal budget constraint:

$$a_{t+1} = (1+r)(a_t + y_t - c_t)$$

recursively to derive

$$\sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t c_t = a_0 + \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t y_t.$$

- Take E_0 on both sides to obtain *expected* budget constraint.

$$\sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t E_0 c_t = a_0 + \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t E_0 y_t.$$

- Since $E_0 c_t = c_0$, this expression becomes

$$c_0 = \frac{r}{1+r} \left(a_o + E_0 \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t y_t\right)$$

- In general

$$c_t = \frac{r}{1+r}\left(a_t + \sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s E_t y_{t+s}\right)$$

- Interpretation: Certainty Equivalence
  - Replace stochastic variables by their expected values.
  - Consumption does not change if the variance of y changes.
- Consumption linear in total wealth
  - constant of proportionality: $\frac{r}{1+r}$.
  - first term (a_t): financial wealth
  - second term (EPDV of y): human wealth

- With a bit of patience can show that:

$$\Delta c_t = \frac{r}{1+r} \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s \left[E_t y_{t+s} - E_{t-1} y_{t+s}\right] \tag{1}$$

- Hence Δc equals the change in expected annuity value from wealth, i.e. change in permanent income.
- This result can be interpreted as a modern version of the Friedman's Permanent Income Hypothesis (PIH).
- If we work with finite horizons, it can be interpreted as a modern version of Modigliani's Life-cycle Theory (LCT).

Some Particular Income Processes: y_t i.i.d.

- Assume

$$y_t + \mu + e_t$$

with e_t i.i.d. with zero mean.

- Then from equation (1) we get

$$\Delta c_t = \frac{r}{1+r} e_t$$

- Hence

$$\frac{\partial \Delta c_t}{\partial e_t} = \frac{r}{1+r} << 1.$$

- Transitory shocks: consumption adjusts by just the annuity value of the shock.

Some Particular Income Processes: y_t follows AR(1)

- Assume

$$y_t - \mu = \rho(y_{t-1} - \mu) + e_t$$

- Assume $0 \leq \rho \leq 1$ denotes the first-order autocorrelation.

  captures the persistence of shocks

  $\rho = 0$ corresponds to i.i.d. case (all shocks transitory)

  $\rho = 1$ corresponds to random walk (all shocks permanent.)

- Then for $s \geq 0$

$$E_t[y_{t+s} - \mu] = \rho^s(y_t - \mu) \Rightarrow E_t[y_{t+s}] = \mu + \rho^s(y_t - \mu)$$

- Hence from equation (1)

$$\begin{aligned}\Delta c_t &= \frac{r}{1+r}\sum_{s\geq 0}\left(\frac{1}{1+r}\right)^s\left(\rho^s(y_t-\mu)-\rho^{s+1}(y_{t-1}-\mu)\right)\\ &= \frac{r}{1+r}\sum_{s\geq 0}\left(\frac{1}{1+r}\right)^s\rho^s e_t\\ &= \frac{r}{1+r-\rho}e_t\end{aligned}$$

- Hence

$$\frac{\partial\Delta c_t}{\partial e_t}=\frac{r}{1+r-\rho}.$$

- Varies from $\frac{r}{1+r}$ (for $\rho=0$) to 1 (for $\rho=1$).
- Consumption responds more to more persistence shocks.
- Adjusts fully to shocks for $\rho=1$, $\Delta c_t=e_t$.

Intuition

- Expected contribution of e_t to y_{t+s} is $\rho^s e_t$.
- Individuals discount future income, hence value $t+s$ income by $\beta^s \rho^s e_t$.
- Hence by certainty equivalence

$$\Delta c_t = \frac{r}{1+r} \sum_{s \geq 0} \beta^s \rho^s e_t = \frac{r}{1+r-\rho} e_t$$

- This extend consumption smoothing intuition to stochastic case: increase consumption by annuity value of present discounted value of expected additional income associated with current shock e_t.

Precautionary Saving

- Under certainty-equivalence, more income uncertainty (e.g. a mean preserving spread) does not affect saving ...
- Hmm ....
- It turns out that $u''' > 0$ is necessary and sufficient to obtain precautionary motive for saving

An Informal Derivation

Use two results from decision making under uncertainty

A. Jensen's Inequality

If

$f : \mathcal{R} \rightarrow \mathcal{R}$

f is strictly convex

x is a random variable with $var(x) > 0$.

then $E[f(x)] > f[E(x)]$

B. Mean-preserving spread and $E[f(x)]$

If

$f : \mathcal{R} \rightarrow \mathcal{R}$

f is strictly convex

y is a random variable symmetric w.r.t. 0.

y is independent of x

$z = x + y$ (mean preserving spread)

then $E[f(z)] > E[f(x)]$

- Assume $r = \delta = 0$
- Assume mean and variance of c_{t+1} (conditional on t) is given exogenously (this is an informal derivation).
- Consider mean preserving spread on c_{t+1} conditional on t.
- Then

$$E_t[C_{t+1}^{post}] = E_t[C_{t+1}^{pre}]$$
$$Var_t[C_{t+1}^{post}] > Var_t[C_{t+1}^{pre}]$$

- Then from Euler equation

$$u'(c_t) = E_t u'(c_{t+1}).$$

- Hence if u' is convex ($u''' > 0$):

$$\begin{aligned} u'(c_t^{post}) &= E_t[u'(c_{t+1}^{post}] \text{ from Euler equation} \\ &> E_t[u'(c_{t+1}^{pre}] \text{ from Property B} \\ &> u'(E_t[c_{t+1}^{pre}]) \text{ from Property A} \end{aligned}$$

- Hence

$$c_t^{post} < E_t[c_{t+1}^{pre}]$$

- By contrast, easy to show that with certainty equivalence:

$$c_t^{C.E.,post} = E_t[c_{t+1}^{C.E.,pre}]$$

- Since we assumed the right hand side of both expressions above are equal, we have shown that precautionary saving, as defined as:

$$\begin{aligned} s_t^{precautionary} &\equiv s_t^{post} - s_t^{C.E.,post} \\ &= c_t^{C.E.,post} - c_t^{post} \end{aligned}$$

  is strictly positive when $u''' > 0$.

- Similar argument shows, informally, that precautionary saving is negative if $u''' < 0$.

What can we say about the sign of u'''*?*

- No theoretical guides for sign of u'''.
- Hard to measure u''' directly from micro studies.
- Note: CES utility functions have $u''' > 0$.