Inter-temporal Choice

1. A Two-Period Fisher model
2. A Two-Period Model of Capital
3. A T-period Optimal Growth Model
4. An Infinite Horizon Model
5. Dynamic Programming
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7. Three Methods for Solving the Bellman Equation

Inter-temporal Choice: Irving Fisher

- Assumptions
  1. no uncertainty
  2. no money, so no inflation
  3. no taxes,
- Consider an environment in which agents live for two periods:

- We assume an individual gets utility from consumption (C_t) in each period. He or she is an utility maximizer so he or she wishes to:

$$\max_{C_1, C_2} U(C_1, C_2)$$

- For now we assume he or she can borrow and lend freely across the two periods at an interest rate r. The agent faces the following budget constraint:

$$C_1 + \frac{C_2}{1+r} = Y_1 + \frac{Y_2}{1+r}$$

- Note everything in this equation is units of "goods in period 1". The term $\frac{1}{1+r}$ is relative price of goods between the two periods and converts the units from the second period into the first.

- This budget line equates the present value of lifetime consumption (PVLC) to the present value of lifetime resources (PVLR).

The agent's constrained maximization problem

- The individual's problem once again is:

$$\max_{C_1,C_2} U(C_1, C_2)$$

subject to:

$$C_1 + \frac{C_2}{1+r} = Y_1 + \frac{Y_2}{1+r}$$

To solve the problem we substitute the constraint into the objective function so agent's problem becomes:

$$\max_{C_2} U\left(Y_1 + \frac{Y_2}{1+r} - \frac{C_2}{1+r}, C_2\right)$$

- We take the first derivative with respect to C_2:

$$U_1(C_1, C_2)\left(\frac{-1}{1+r}\right) + U_2(C_1, C_2) = 0$$

  where U_1 is derivative of U with respect to the first argument, and where U_2 is derivative of U with respect to the second argument.

- Rearrange terms:

$$\frac{U_1(C_1, C_2)}{U_2(C_1, C_2)} = 1 + r$$

- So the agent chooses C_1 and C_2 such that the marginal rate of substitution between consumption today for consumption tomorrow is equal to the interest rate $1 + r$.

An example with a specific utility function

- Set $U(C_1, C_2) = \ln(C_1) + \beta \ln(C_2)$ where $\beta < 1$
- Why is $\beta < 1$?
  - We assume agents discount future consumption at the rate β. That is, they prefer current consumption to future consumption all other things held equal.
- We write the agent's problem as

$$\max_{C_1, C_2} \ln(C_1) + \beta \ln(C_2)$$

subject to:

$$C_1 + \frac{C_2}{1+r} = Y_1 + \frac{Y_2}{1+r}$$

Solving

1. Substitute the constraint into the objective function so the problem becomes:

$$\max_{C_2} \ln\left(Y_1 + \frac{Y_2}{1+r} - \frac{C_2}{1+r}\right) + \beta \ln(C_2)$$

2. Take the derivative with respect to C_2:

$$\frac{1}{Y_1 + \frac{Y_2}{1+r} - \frac{C_2}{1+r}}\left(\frac{-1}{1+r}\right) + \frac{\beta}{C_2} = 0$$

3. Do some algebra

$$C_2 = (1+r)\left(\frac{\beta}{1+\beta}\right)\left(Y_1 + \frac{Y_2}{1+r}\right)$$

4. Using the budget constraint to solve for C_1 yields

$$C_1 = \left(\frac{1}{1+\beta}\right)\left(Y_1 + \frac{Y_2}{1+r}\right)$$

5. Saving in this model economy is

$$\begin{aligned} S &= Y_1 - C_1 \\ &= Y_1 - \left(\frac{1}{1+\beta}\right)\left(Y_1 + \frac{Y_2}{1+r}\right) \end{aligned}$$

So what's the point of all this math?

- We just derived consumption and saving functions.
- Note consumption in both periods depend on PVLR.
- Really the agent's problem boils down to how to split total lifetime income between the two periods.

- If $S > 0$, this person is a lender, $S < 0$, this person is a borrower.
- Note that for ln utility

$$\frac{C_2}{C_1} = \frac{(1+r)\left(\frac{\beta}{1+\beta}\right)\left(Y_1 + \frac{Y_2}{1+r}\right)}{\left(\frac{1}{1+\beta}\right)\left(Y_1 + \frac{Y_2}{1+r}\right)} = \beta(1+r)$$

- So if

$$\beta > \frac{1}{1+r}$$

the agent consumes more in the second period than in the first period.

- If

$$\beta < \frac{1}{1+r}$$

the agent consumes more in the first period than in the second period.

- If

$$\beta = \frac{1}{1+r}$$

the agent consumes equal amounts in both periods.

Irving Fisher's Two-Period Capital Theory

- Add production to the model.

  Recall our agent lives for two periods: 1 and 2. This agent gets utility from consumption in each period:

$$\max_{C_1, C_2} U(C_1, C_2)$$

- But let's assume for the time being that instead of being handed income each period, this agent has access to a production function:

$$AF(K)$$

  Assume the agent comes into this world with a certain a given capital stock (K_1). The agent produces $AF(K_1)$ stuff during the initial time period. Of that stuff she can either eat it or save it.

- In particular, assume $AF(K) = AK^\alpha$

  There are two ways to interpret this production function.

  1. The production function is Cobb-Douglas, and we set N=1,
  2. We are thinking about a per-capita output

$$\begin{aligned} \frac{Y}{N} &= \frac{AK^\alpha N^{1-\alpha}}{N} \\ &= AK^\alpha N^{-\alpha} \\ &= A\left(\frac{K}{N}\right)^\alpha \\ &= Ak^\alpha \end{aligned}$$

where k is the capital to labor ratio. In this second case we want to think of C as representing per capita consumption and K as representing the capital to labor ratio.

- So in period 1, the agent's budget constraint is:

$$C_1 + K_2 = AF(K_1)$$

- In period 2, there is no saving decision. The agent just eats everything he or she produces:

$$C_2 = AF(K_2)$$

- So the agent's problem is

$$\max_{C_1, C_2} U(C_1, C_2)$$

subject to:

$$C_1 + K_2 = AF(K_1)$$
$$C_2 = AF(K_2)$$

- Relative price of K_2 to C_1 is 1. That is, the price of capital p_K is 1.

How do we solve this problem?

1. Substitute the constraints into the utility function:

$$\max_{K_2} U(AF(K_1) - K_2, AF(K_2))$$

   The choice variable is now K_2 which is the saving from period 1 to period 2.

2. Get the first-order necessary condition for a maximum by taking a derivative with respect to K_2:

$$U_1(C_1, C_2)(-1) + U_2(C_1, C_2)AF'(K_2) = 0$$

3. rearrange terms

$$\frac{U_1(C_1, C_2)}{U_2(C_1, C_2)} = AF'(K_2)$$

This pins down the interest rate

- recall we just derived:

$$\frac{U_1(C_1, C_2)}{U_2(C_1, C_2)} = 1 + r$$

- We can also solve the one period problem of a firm maximizing the value of production (in terms of goods).

$$\max_K AF(K) - (1 + r)K$$

This implies

$$AF'(K) = 1 + r$$

The deterministic one-sector growth model

- no labor (marginal utility of leisure set to 0)
- complete depreciation of capital

finite horizon case

$$\max_{\{c_t, k_{t+1}\}_0^{T-1}} \sum_{t=0}^{T} \beta^t U(c_t)$$

subject to

$$\begin{aligned} c_t + k_{t+1} &= f(k_t) \\ k_0 &\ \text{given} \\ k_t &\geq 0 \end{aligned}$$

- substituting the constraint into the utility function yields:

$$\max_{\{k_{t+1}\}_0^{T-1}} \sum_{t=0}^{T} \beta^t U(f(k_t) - k_{t+1})$$

given the initial conditions stated above.

- Taking first-order conditions yields

$$\beta^t U'(f(k_t) - k_{t+1})(-1) + \beta^{t+1} U'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}) = 0$$

  – holds for $t = 0, 1, 2, ...T - 1$

  – at time T, $k_{t+1} = 0$

Pick functional forms

- $U(c_t) = \ln c_t$
- $f(k_t) = Ak^{\alpha}$.
- The first order condition then becomes

$$\begin{aligned} \beta^t \frac{1}{Ak_t^{\alpha} - k_{t+1}}(-1) + \beta^{t+1} \frac{\alpha A k_{t+1}^{\alpha-1}}{Ak_{t+1}^{\alpha} - k_{t+2}} &= 0 \quad \text{for t=0,1,2, ...T-1} \\ k_{T+1} &= 0 \end{aligned}$$

- Still have T non-linear equations and T unknowns.

Trick – change the variables

- Define

$$X_t = \frac{k_{t+1}}{Ak_t^\alpha}$$

so X_t is the savings rate. Substituting back into the first order condition yields

$$X_{t+1} = 1 + \alpha\beta - \frac{\alpha\beta}{X_t}$$

- We know $X_T = 0$, so we can solve the difference equation by backwards substitution, then use k_0 to get the capital stock. Working backwards yields:

$$X_t = \alpha\beta\frac{1-(\alpha\beta)^{T-t+1}}{1-(\alpha\beta)^{T-t+2}}$$

Infinite horizon case

$$\max_{\{k_{t+1}\}_0^\infty} \sum_{t=0}^{\infty} \beta^t \ln(Ak_t^\alpha - k_{t+1})$$

with the transversality condition:

$$\lim_{T\to\infty} \beta^T \left(\frac{\alpha A k_T^{\alpha-1}}{Ak_T^\alpha - k_{T+1}}\right) k_T = 0$$

- If we take first-order conditions, we get an infinite number of non-linear second-order difference equations and infinite number of unknowns.
- That's tough to solve.
- But note that in the finite time model that if T is large

$$X_t \approx \alpha\beta$$

or

$$k_{t+1} \approx \alpha\beta A k_t^\alpha$$

conjecture: the solution to this infinite horizon problem has the form

$$k_{t+1} = h(k_t), \quad t = 0, 1, ...$$

In particular

$$k_{t+1} = \alpha\beta A k_t^{\alpha}.$$

So this suggests another approach.

- Define a function

$$V(k_0) = \max_{k_{t+1}} \sum_{t=0}^{\infty} \beta^t \ln(Ak_t^\alpha - k_{t+1}).$$

- We can then go forward one period and define:

$$V(k_1) = \max_{k_{t+1}} \sum_{t=1}^{\infty} \beta^t \ln(Ak_t^\alpha - k_{t+1}).$$

- Given this function $V(k_1)$, the planner's problem in period 0 would be:

$$\max_{k_1} [\ln(Ak_0^\alpha - k_1) + \beta V(k_1)].$$

- If the function $V(k_1)$ known, we could solve for the policy function h such that:

$$k_1 = h(k_0) \quad \text{and}$$
$$c_0 = Ak_0^\alpha - h(k_0).$$

- But note: $V(k_0)$ is the maximized value of the original problem. Thus $V(k_0)$ must be the maximized value of the one-period time-zero problem as written in previous overhead:

$$V(k_0) = \max_{k_1}[\ln(Ak_0^\alpha - k_1) + \beta V(k_1)].$$

  - Now we have the problem in a recursive structure.
  - If we go out t+1 periods we get

$$V(k_t) = \max_{k_{t+1}}[\ln(Ak_t^\alpha - k_{t+1}) + \beta V(k_{t+1})].$$

- Drop the subscripts:

$$V(k) = \max_{k'}[\ln(Ak^\alpha - k') + \beta V(k')],$$

- The solution to this Bellman equation is:

$$V(k) = \ln(Ak^\alpha - h(k)) + \beta V(h(k))$$

  This equation is a *functional equation* and can be solved for the pair of unknown functions $V(k)$ and $h(k)$.

The general deterministic control problem

$$\max_{\{u_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t r(x_t, u_t)$$

subject to

$$x_{t+1} \in g(x_t, u_t)$$
$$x_0 \text{ given}$$

- We let $r(x_t, u_t)$ denote the single period return function.
- We let $g(x_t, u_t)$ denote the set of constraints the determine the feasible choices of u_t given x_t.
- u_t are called *control* variables.
- x_t are called *state* variables.

- We define

$$V^*(x_0) \equiv \max_{\{u_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t r(x_t, u_t) \text{ subject to } x_{t+1} \in g(x_t, u_t)$$

for a given x_0.

- So in the general set-up we write down the Bellman equation as:

$$V(x) = \max_u \{r(x, u) + \beta V(x')\} \tag{1}$$

subject to:

$$x' \in g(x, u)$$

Properties of the Bellman Equation
Most standard discussions of dynamic programming assume the following:

1. r is concave and bounded, and
2. the set $\{(x_{t+1}, x_t) : x_{t+1} \in g(x_t, u_t), u_t \in R^k\}$ is convex and compact.

Then we can show:

1. $V(x)$ is a monotonically increasing function (Stokey-Lucas-Prescott Theorem 4.7).
2. $V(x)$ is strictly concave and $h(x)$ is a continuous single-valued function (SLP Theorem 4.8).
3. (*The Principle of Optimality*) The solution to (1) is $V^*(x_0)$.
4. This solution is approached in the limit as $j \to \infty$ by iterations on:

$$V_{j+1}(x) = \max_u \{r(x, u) + \beta V_j(x')\}$$

   subject to

$$x' \in g(x, u)$$

   Note we are iterating "backwards."

5. (*Benveniste and Scheinkman*) The limiting value function V is differentiable with

$$V'(x) = \frac{\partial r}{\partial x}[x, h(x)] + \beta \frac{\partial g}{\partial x}[x, h(x)] V'(g[x, h(x)])$$

   This is Theorem 4.10 in SLP.

Note that if we go back to our special case of the one sector growth model and apply Benveniste and Scheinkman, we get:

$$V'(k) = \frac{\alpha A k^{\alpha-1}}{Ak^{\alpha} - k'}.$$

Take the FOC for the right-hand side of (1) yields:

$$0 = \frac{-1}{Ak^{\alpha} - k'} + \beta V'(k')$$

These two derivatives imply:

$$\frac{1}{Ak^{\alpha} - k'}(-1) + \beta \frac{\alpha A k'^{\alpha-1}}{Ak'^{\alpha} - k''} = 0$$

Solving the Bellman equation

1. Value function iteration (working backwards)
2. Guess and verify
3. Policy iteration

Method 1: Value function iteration (working backwards)

We are looking for a fixed point of a concave functional equation. So we can:

1. Start off with a bounded and continuous initial $V_0(x)$.
2. Solve the one period problem

$$V_1(x) = \max_u \{r(x,u) + \beta V_0(x')\}$$

   subject to

$$x' \in g(x,u)$$

3. Take the value $V_1(x)$ that solves the above maximization problem and solve

$$V_2(x) = \max_u \{r(x,u) + \beta V_1(x')\}$$

   subject to

$$x' \in g(x,u)$$

4. Repeat until the $h_j(x)$ "stops changing" and/or $V_j(x)$ and $V_{j+1}(x)$ are "close."

Back to the OSGM

1. Choose $V_0(k) = 0$.

2. Set $V_1(k) = \max_{k'} \{\ln(Ak^\alpha - k') + \beta 0\}$

3. The solution is to set $k' = 0$ so $V_1(k) = \ln(Ak^\alpha) = \ln A + \alpha \ln k$.

4. Set $V_2(k) = \max_{k'} \{\ln(Ak^\alpha - k') + \beta (\ln A + \alpha \ln k')\}$

5. The solution is to set $k' = \frac{\beta\alpha}{1+\beta\alpha} Ak^\alpha$ so

$$\begin{aligned} V_2(k) &= \ln\left(Ak^\alpha - \frac{\beta\alpha}{1+\beta\alpha}Ak^\alpha\right) + \beta\left(\ln A + \alpha\ln\left(\frac{\beta\alpha}{1+\beta\alpha}Ak^\alpha\right)\right) \\ &= \ln\left(\frac{A}{1+\beta\alpha}\right) + \beta\ln A + \alpha\beta\ln\left(\frac{\beta\alpha A}{1+\beta\alpha}\right) + (\alpha+\alpha^2\beta)\ln k \end{aligned}$$

6. Repeat enough times until you see that the value function follows a geometric sequence that converges to

$$V(k) = = \frac{1}{1-\beta}\left[\ln(A(1-\beta\alpha)) + \frac{\beta\alpha}{1-\beta\alpha}\ln(A\beta\alpha)\right] + \frac{\alpha}{1-\alpha\beta}\ln k.$$

The associated policy function is

$$k' = \beta\alpha A k^\alpha.$$

Method 2: Guess and Verify

- Guess a functional form for $V(k)$.
- Verify the functional form of type guessed solves the Bellman equation and deduce the values for the coefficients.

There are only a handful of models for which you can use this method. How do you form a good guess?

Back to the OSGM II

1. Guess $V(k) = E + F\ln(k)$.

2. Now verify

$$E + F\ln(k) = \max_{k'} \{\ln(Ak^\alpha - k') + \beta(E + F\ln(k'))\}$$

- Take the first-order condition

$$0 = \frac{-1}{Ak^\alpha - k'} + \beta F \frac{1}{k'}$$

Solving for k'

$$k' = \frac{\beta F}{1+\beta F} Ak^\alpha$$

- Substitute back in to the Bellman equation

$$\begin{aligned} E + \ln k &= \ln\left(Ak^\alpha - \frac{\beta F}{1+\beta F}Ak^\alpha\right) + \beta\left(E + F\ln\left(\frac{\beta F}{1+\beta F}Ak^\alpha\right)\right) \\ &= \ln\left(\frac{A}{1+\beta F}\right) + \alpha\ln k + \beta E + \beta F \ln\left(\frac{\beta F A}{1+\beta F}\right) + \beta F\alpha\ln k. \end{aligned}$$

- Equate coefficients on the $\ln k$ terms

$$F \ln k = \alpha \ln k + \beta F \alpha \ln k$$

$$F = \frac{\alpha}{1 - \beta\alpha} \tag{2}$$

- Equate coefficients on the constant term

$$E = \ln\left(\frac{A}{1 + \beta F}\right) + \beta E + \beta F \ln\left(\frac{\beta A F}{1 + \beta F}\right)$$

- Use equation (2) to substitute out F
- Do a lot of algebra, and solve for E

Method 3: Policy Function Iteration

1. Pick a feasible policy, $u = h_0(x)$, and compute the value of sticking with that policy forever:

$$V_{h_0}(x) = \sum_{t=0}^{\infty} \beta^t r(x_t, h_0(x_t)), \quad \text{where} \quad x_{t+1} = g(x_t, h_0(x_t)).$$

2. Choose a new policy function $h_1(x)$ that maximizes the following two period problem:

$$\max_u \{r(x,u) + \beta V_{h_0}(g(x, h_0(x)))\}$$

3. repeat steps 1 and 2 with the updated policy function until the policy function "stops changing."

Back to OSGM III

1. Pick feasible policy function $k_{t+1} = h_0(x) = \frac{1}{2}Ak_t^\alpha$
2. Compute

$$\begin{aligned} V_{h_0}(x) &= \sum_{t=0}^{\infty} \beta^t \ln\left(Ak_t^\alpha - \frac{1}{2}Ak_t^\alpha\right) \\ &= \sum_{t=0}^{\infty} \beta^t \ln\left(\frac{1}{2}Ak_t^\alpha\right) \\ &= \sum_{t=0}^{\infty} \beta^t \left(\ln\left(\frac{1}{2}A\right) + \alpha \ln k_t\right) \end{aligned}$$

Note

$$\begin{aligned} k_t &= \frac{1}{2}Ak_{t-1}^\alpha \\ &= \frac{1}{2}A\left[\frac{1}{2}Ak_{t-2}^\alpha\right]^\alpha \\ &= \left(\frac{1}{2}\right)^{1+\alpha} A^{1+\alpha}k_{t-2}^{\alpha^2} \end{aligned}$$

repeated recursive substitution yields:

$$k_t = Dk_0^{\alpha^t}$$

where D denotes a constant term so,

$$\ln k_t = \ln D + \alpha^t k_0.$$

Therefore

$$\begin{aligned} V_{h_0}(x) &= \sum_{t=0}^{\infty} \beta^t \left(\ln(\frac{1}{2}A) + \alpha \ln D + \alpha^{t+1} \ln k_0 \right) \\ &= \text{constant term} + \frac{\alpha}{1-\beta\alpha} \ln k_0 \end{aligned}$$

3. Evaluate the two period problem:

$$\max_{k'} \left\{ \ln(Ak^\alpha - k') + \beta \left[\text{constant term} + \frac{\alpha}{1-\beta\alpha} \ln k' \right] \right\}.$$

Taking the first-order condition yields:

$$\frac{-1}{Ak^\alpha - k'} + \frac{\beta\alpha}{1-\beta\alpha} \frac{1}{k'} = 0.$$

Thus

$$k' = \alpha\beta A k^\alpha.$$

The Howard improvement algorithm converges in a single step!