Markov Chains

- A Markov chain is a special type of stochastic process.
- It has the property that at any given time t when the current state x_t and all previous states $x_{t-1}, x_{t-2}, \ldots$ are known, the probabilities of all future states only depend on the current state x_t and do not depend on the earlier states $x_{t-1}, x_{t-2}, \ldots$.
- Formally, a stochastic process $\{x_t\}$ is said to have the Markov property if for all $k \geq 1$ and at all t

$$Prob(x_{t+1}|x_t, x_{t-1}, x_{t-2}, \ldots) = Prob(x_{t+1}|x_t)$$

- An AR(1) process has this property. For now let's consider discrete states. For example, the weather can either be sunny or rainy. A worker can either be employed or unemployed.

- We need three objects

  1. A list of the discrete states (a $n \times 1$ vector)
  2. A $n \times n$ transition matrix whose elements are the probabilities of moving from one value of the state to another.
  3. A $n \times 1$ vector of initial probabilities.

- Example: two states of the world – sunny or rainy.

  So the vector of states is $\begin{bmatrix} \text{sunny} \\ \text{rainy} \end{bmatrix}$

- Let's assume that if it is sunny today there is a 70 percent chance it will be sunny tomorrow and a 30 percent chance it will be rainy tomorrow.

  If it is rainy today, there is 40 percent chance it will be sunny tomorrow and a 60 percent chance it will be rainy tomorrow.

  So then the transition matrix is $P = \begin{bmatrix} .70 & .30 \\ .40 & .60 \end{bmatrix}$

- Note two things that hold for all transition matrices

  – Each element of the matrix is

  $$P_{i,j} = Prob(x_{t+1} = e_j | x_t = e_i)$$

  – Each row of the transition matrix must sum to one.

- Let's assume there is a 75 percent chance that the initial condition is sunny and and 25 percent chance the initial condition is rainy.

  So $\pi_0 = \begin{bmatrix} .75 \\ .25 \end{bmatrix}$

- The unconditional probability of sun and rain in period 1 is: $\pi_1' = \pi_0' P$
- The unconditional probability of sun and rain in period 2 is: $\pi_2' = \pi_1' P = \pi_0' P P = \pi_0' P^2$
- Likewise, the probability of sun and rain in period k is: $\pi_k' = \pi_0' P^k$

Stationary Distributions

- The unconditional probability distributions evolve

$$\pi'_{t+1} = \pi_t P$$

- The unconditional distribution is called *stationary* or *invariant* if it satisfies

$$\pi_{t+1} = \pi_t$$

In other words ...

$$\pi' = \pi' P$$

or

$$\pi'(I - P) = 0.$$

Transposing both sides

$$(I - P')\pi = 0$$

So π is the eigenvector associated with the unit eigenvalue of P'.

Asymptotic Stationary

- Consider the limit

$$\lim_{t \to \infty} \pi_t = \pi_\infty$$

- As long every element of P is greater than zero, π_∞ solves

$$(I - P')\pi = 0$$

- The ‘quick and dirty’ way to compute the stationary distribution is raise P to a large number.

Absorbing States

- An absorbing state is a state such that one you get in, you can't get out.
- Example: two states, life and death. Death is an absorbing state.
- Consider the transition matrix

$$P = \begin{bmatrix} .999 & .001 \\ 0 & 1 \end{bmatrix}$$

Expectations

- Suppose income can either be 'high' with a value of 10 or 'low' with a value of 5.
- Let the transition matrix $P = \begin{bmatrix} .70 & .30 \\ .40 & .60 \end{bmatrix}$
- We can compute the conditional expectation of income next period given the state this period.
  - If income is high this period, expected income next period is:

$$(.70 \times 10) + (.30 \times 5) = 7 + 1.5 = 8$$

  - If income is low this period, expected income next period is:

$$(.40 \times 10) + (.60 \times 5) = 4 + 3 = 7$$

- Using the matrix P^k, expectations k periods ahead can be computed.