

Problem Set 2: Due in class Friday, September 22, 2006

1. Consider the problem of an agent who lives for two periods and has preferences expressed by the utility function

$$\ln C_1 + \beta \ln C_2.$$

At the beginning of the first period, the agent is endowed with K_1 beans. During both periods, the agent has access to a Cobb-Douglas production function which transforms beans, K_t , and labor, N_t into more beans:

$$Y_t = A_t K_t^\alpha N_t^{1-\alpha}.$$

The agent supplies one unit of labor each period, so $N = 1$ each period. After harvesting the beans in the first period, the agent chooses how many beans to consume this period, C_1 , and how many beans to save next period, K_2 .

- (a) Assume $A_1 = A_2 = \bar{A}$. Solve for the agent's consumption and saving functions (C_1 , C_2 , and K_2).
- (b) Now suppose everyone knows that in the future, capital will become twice as productive as before. That is, $A_2 = 2 \cdot A_1 = 2 \cdot \bar{A}$. What effect does this increase in productivity have on the equilibrium interest rate? How does this increased productivity affect the agent's consumption and saving plans? Describe the income and substitution effects.
2. Consider the optimal growth problem with preferences

$$\sum_{t=0}^{\infty} \beta^t U(c_t),$$

where $0 < \beta < 1$ and $U(c_t) = \ln(c_t)$. The technology is

$$k_{t+1} = (1 + \alpha)(k_t - c_t)$$

where $\alpha > 0$.

- (a) State the Bellman equation for this growth problem. What is(are) the state variable(s)? What is(are) the control variable(s)?
- (b) Show that the value function has the form $v(k) = A + D \ln(k)$. Find the values of A and D.
- (c) Find the optimal policy function. Find the optimal growth rate of the capital stock.

3. Consider an economy in which a single agent has preferences $\sum_{t=0}^{\infty} \beta^t \ln(c_t)$, where $0 < \beta < 1$. The agent is endowed with one unit of time each period. He can allocate the fraction u_t to the production of the consumption good, and $1 - u_t$ to the accumulation of human capital. If he begins period t with x_t units of human capital, his production technology is:

$$c_t = x_t u_t \qquad x_{t+1} = \delta x_t^\alpha (1 - u_t)^\gamma.$$

- State this agent's Bellman equation. What is(are) the state variable(s)? What is(are) the control variable(s)?
 - What is the economic interpretation of the value function?
 - Show that the Bellman equation has a solution of the form: $v(x) = A + B \ln(x)$. Find B . (If you have lots of free time, find A .) Find the optimal policy.
 - For what values of the parameters α , β , γ , and δ will this agent's consumption exhibit sustained growth? For these parameter values, what is the growth rate?
4. A firm maximizes present value cash flows, with future earnings discounted at the rate β . Income at time t is given by $p_t \cdot q_t$ where p_t is the price of the good, and q_t is the quantity produced. The firm behaves competitively and therefore takes prices as given. It knows that prices evolve according to the law of motion given by $p_{t+1} = f(p_t)$.

Total or gross production depends on the amount of capital, k_t , and labor, n_t , and on the square of the difference between current ratio of sales to investment, x_t , and the previous-period ratio. This last feature captures the notion that changes in the ratio of sales to investment require some reallocation of resources within the firm and consequently reduce the level of efficiency. It is assumed that the wage rate is constant and equal to w . Capital depreciates at the rate δ . The firm's problem is

$$\max \sum_{t=0}^{\infty} \beta^t [p_t q_t - w n_t]$$

subject to

$$\begin{aligned} q_t + x_t &\leq g \left[k_t, n_t, \left(\frac{q_t}{x_t} - \frac{q_{t-1}}{x_{t-1}} \right)^2 \right] \\ k_{t+1} &= (1 - \delta)k_t + x_t \\ p_{t+1} &= f(p_t) \end{aligned}$$

with $0 < \beta < 1$, $0 < \delta < 1$. The initial conditions $k_0 > 0$ and $q_{-1}/x_{-1} > 0$ are given.

Assume that $g(\cdot)$ is bounded, increasing in the first two arguments and decreasing in the third.

- Formulate the firm's problem recursively; that is, state the firm's Bellman equation. Identify the state and controls, and indicate the laws of motion of the state variables.
- What is the economic interpretation of the value function?