Some Notes on Solving a RBC Model

- We want to solve and estimate model of the type:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left(u(c_t) + v(1-h_t)\right)$$

subject to

$$\begin{aligned} c_t + i_t &= \lambda_t k_t^{\theta} h_t^{1-\theta} \\ k_{t+1} &= (1-\delta)k_t + i_t \\ \lambda_{t+1} &= (1-\rho) + \rho\lambda_t + \epsilon_{t+1} \end{aligned}$$

- Let's assume $u(c_t) = \log c_t$, $v(1-h_t) = A\log(1-h_t)$.

- We are going to follow Kydland and Prescott (1982) by replacing this problem with a linear quadratic approximation of this model.
- We are going to take a Taylor-approximation around the non-stochastic steady-state of this model.
- So we are do this in three steps

  1. Solve the model for a steady state.
  2. Substitute the constraint into the objective function.
  3. Take a Taylor approximation of the objective function

Step 1: Solve for a steady state

- Set of the non-stochastic Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\log c_t + A \log(1 - h_t) + \mu_t(\lambda_t k_t^{\theta} h_t^{1-\theta} + (1-\delta)k_t - c_t - k_{t+1})\right]$$

- Taking first-order conditions yields

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} &= \frac{1}{c_t} - \mu_t = 0 \\
\frac{\partial \mathcal{L}}{\partial k_{t+1}} &= -\mu_t + \beta \mu_{t+1} \left[(1-\delta) + \lambda_{t+1} \theta k_{t+1}^{\theta-1} h_{t+1}^{1-\theta}\right] = 0 \\
\frac{\partial \mathcal{L}}{\partial h_t} &= \frac{-A}{1-h_t} + \mu_t (1-\theta) \lambda_t k_t^{\theta} h_t^{-\theta} = 0 \\
\frac{\partial \mathcal{L}}{\partial \mu_t} &= \lambda_t k_t^{\theta} h_t^{1-\theta} + (1-\delta) k_t - c_t - k_{t+1} = 0
\end{aligned}$$

- Impose the steady-state

$$
\begin{aligned}
\frac{1}{c} &= \mu \\
\mu &= \beta\mu\left[(1-\delta) + \lambda\theta k^{\theta-1} h^{1-\theta}\right] \\
\frac{A}{1-h_t} &= \mu(1-\theta)\lambda k^{\theta} h^{-\theta} \\
\lambda k^{\theta} h^{1-\theta} + (1-\delta)k - c - k &= 0
\end{aligned}
$$

- Do some algebra and you get

$$
\begin{aligned}
k &= \left[\frac{1-\beta(1-\delta)}{\beta\lambda\theta}\right]^{\frac{1}{\theta-1}} h \\
k &= \phi^{\frac{1}{\theta-1}} h \\
c &= \phi^{\frac{1}{\theta-1}}\left[\lambda\phi^{\theta} - \delta\right] h \\
h &= \frac{\frac{(1-\theta)\lambda\phi}{A}}{\lambda\phi^{\theta} - \delta + (1-\theta)\lambda\theta/A}
\end{aligned}
$$

Step 2: Substitute the constraint into the objective function

So the problem becomes:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left[\log\left[\lambda_t k_t^{\theta} h_t^{1-\theta} - i_t\right] + A\log(1-h_t)\right]$$

subject to:

$$\begin{aligned} k_{t+1} &= (1-\delta)k_t + i_t \\ \lambda_{t+1} &= (1-\rho) + \rho\lambda_t + \epsilon_{t+1} \end{aligned}$$

So now the problem has the following form

$$\max_{u_t} E_0 \sum_{t=0}^{\infty} \beta^t r_t(z_t)$$

subject to:

$$x_{t+1} = Ax_t + Bu_t + Cw_{t+1}$$

where

$$\begin{aligned} x_t & \quad \text{is the vector of state variables} \\ u_t & \quad \text{is the vector of control variables} \\ w_{t+1} & \quad \text{is a vector of shocks} \\ z_t &= [x_t' \, ; \;\; u_t'] \end{aligned}$$

and A, B, and C are matrices.
In particular w_{t+1} is a martingale difference sequence with $Ew_{t+1}w_{t+1}' = I$.

In our case

$$x_t = \begin{bmatrix} 1 \\ \lambda_t \\ k_t \end{bmatrix} \tag{1}$$

$$u_t = \begin{bmatrix} i_t \\ h_t \end{bmatrix} \tag{2}$$

$$z_t = \begin{bmatrix} 1 \\ \lambda_t \\ k_t \\ i_t \\ h_t \end{bmatrix} \tag{3}$$

Writing out the law of motion for the state variable.

$$\begin{bmatrix} 1 \\ \lambda_{t+1} \\ k_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1-\rho & \rho & 0 \\ 0 & 0 & 1-\delta \end{bmatrix} \begin{bmatrix} 1 \\ \lambda_t \\ k_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i_t \\ h_t \end{bmatrix} + \begin{bmatrix} 0 \\ \sigma_\epsilon \\ 0 \end{bmatrix} w_{t+1} \tag{4}$$

Replace the $r(z_t)$ function with a Taylor approximation.

$$r(z_t) \approx \hat{r}(z_t) = r(z^{ss}) + r(z - z^{ss})\frac{\partial r}{\partial z} + \frac{1}{2}(z - z^{ss})\frac{\partial^2 r}{\partial z \partial z'}(z - z^{ss})$$

In matrix form

$$\hat{r}(z_t) = z' M z$$

where

$$\begin{aligned} M \;=\; & e\left(r(z^{ss}) - \frac{\partial r(z^{ss})'}{\partial z^{ss}} z^{ss} + \frac{1}{2} z^{ss\prime} \frac{\partial^2 r(z^{ss})'}{\partial z^{ss2}} z^{ss}\right) e' + \\ & \frac{1}{2}\left(e\frac{\partial r(z^s s)'}{\partial z^{ss}} + \frac{\partial r(z^{ss})}{\partial z^{ss}} e' - e z^{ss\prime} \frac{\partial^2 r(z^{ss})}{\partial z^{ss2}} - \frac{\partial^2 r(z^{ss})}{\partial z^{ss2}} z e' + \frac{\partial^2 r(z^{ss})'}{\partial z^{ss2}}\right) \end{aligned}$$

where e is a vector of zeros with 1 in the element corresponding to the constant term in x_t.

So we are going to generate an approximate solution to original problem by solving the following linear quadratic problem

$$\max_{u_t} E_0 \sum_{t=0}^{\infty} \beta^t z_t' M z_t$$

subject to:

$$x_{t+1} = Ax_t + Bu_t + Cw_{t+1}$$

Let's map this problem into a optimal linear regulator formulation

$$\hat{r}(z) = z' M z \tag{5}$$

$$= \begin{bmatrix} x & u \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \tag{6}$$

$$= \begin{bmatrix} x & u \end{bmatrix} \begin{bmatrix} Q & W \\ W' & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \tag{7}$$

We can write the problem as:

$$\max_{u_t} E_0 \sum_{t=0}^{\infty} \beta^t \left[x_t' Q x_t + u_t' R u_t + 2x_t' W u_t\right]$$

subject to:

$$x_{t+1} = Ax_t + Bu_t + Cw_{t+1}$$

The most straight-forward way to solve this problem is via dynamic programming. Let $V(x)$ be the optimal value associated with the program starting from an initial state vector $x_0 = x$.

$$V_{j+1}(x_t) = \max_{u} \{x_t' Q x_t + u_t' R u_t + 2x_t' W u_t + \beta E_t V(x_{t+1})\}$$

subject to:

$$x_{t+1} = Ax_t + Bu_t + Cw_{t+1}$$

Can solve via backward recursions. Initialize $V_0(x) = 0$ This will yield the following quadratic form.

$$V_{j+1}(x_t) = x_t' P_j x_t + \rho_j$$

where

$$\begin{aligned} P_{j+1} &= Q + \beta A' P_j A - (\beta A' P_j B + W)(R + \beta B' P_j B + W)(R + \beta B' P_j B)^{-1}(\beta B' P_j A + W') \\ \rho_{j+1} &= \beta \rho_j + \beta trace P_j C C' \end{aligned}$$

This first equation is known as a matrix Riccati equation.
By backward recursions, the value function will converge to:

$$V(x_t) = x_t' P x_t + \rho$$

If we substitute P_j into the value function and take the derivative with respect to u we can solve the decision rule

$$u_t = -F_j x_t$$

where

$$F_j = (R + \beta B' P_j B)^{-1}(\beta B' P_j A + W')$$

If we iterate on P we will get a fixed point, so

$$F = (R + \beta B'PB)^{-1}(\beta B'PA + W')$$

Notice the decision rule does not depend on C. Certainty Equivalence! But the value function depends on C.
Return to our law of motion for the state variables

$$x_{t+1} = Ax_t + Bu_t + Cw_{t+1}$$

Substitute

$$\begin{aligned} x_{t+1} &= Ax_t + B(-Fx_t) + Cw_{t+1} \\ &= (A - BF)x_t + Cw_{t+1} \end{aligned}$$

Call $A_o = (A - BF)$.
Now we have

$$x_{t+1} = A_o x_t + Cw_{t+1}$$

We can augment this system with a second matrix equation

$$y_t = Gx_t$$

where y is a vector of variables we observe and care about.