Stochastic Dynamic Programming

1. Recap from deterministic dynamic programming
2. The stochastic control problem
3. A simple stochastic growth model
4. Steady-state distributions

Recap

Recall the general form of the deterministic control problem:

$$\max_{\{u_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t r(x_t, u_t) \tag{1}$$

subject to

$$x_{t+1} \in g(x_t, u_t)$$
$$x_0 \text{ given}$$

where

- $r(x_t, u_t)$ is the single period return function.
- $g(x_t, u_t)$ is the set of constraints the determine the feasible choices of u_t given x_t.
- u_t are the *control* or choice variables.
- x_t are the *state* variables.

Modify problem to permit uncertainty of particular kinds.

$$\max_{\{u_t\}_0^\infty} E_0 \sum_{t=0}^{\infty} \beta^t r(x_t, u_t) \tag{2}$$

subject to

$$x_{t+1} \in g(x_t, u_t, \epsilon_{t+1})$$
$$x_0 \text{ given}$$

- We let ϵ_t be a sequence of independently and identically distributed random variables
- We assume that at time t, the current and past history of the state variables $\{x_t, x_{t-1}, ..., x_0\}$ are known. We assume all x_{t+j} for $j \geq 1$ are unknown at date t.
- The operator E_t takes the expectations of a random variable given the information known at time t.
- Finally we assume the shock ϵ_{t+1} is realized after the control variable u_t has been chosen.

Properties of the Bellman Equation

We can write the Bellman equation for the stochastic control as:

$$V(x) = \max_{u} \{r(x,u) + \beta E[V(x')|x]\} \tag{3}$$

subject to

$$x' \in g(x,u,\epsilon')$$

If the one-period return function is bounded, and the constraint set is convex and compact, then:

1. $V(x)$ is a monotonically increasing function
2. $V(x)$ is strictly concave and $h(x)$ is a continuous single-valued function
3. (*The Principle of Optimality*) The solution to the Bellman equation is $V^*(x_0)$ where

   $$V^*(x_0) \equiv \max_{\{u_t\}_0^\infty} E \sum_{t=0}^{\infty} \beta^t r(x_t,u_t) \;\text{ subject to } x_{t+1} \in g(x_t,u_t,\epsilon_{t+1})$$

   for a given x_0.
4. This solution is approached in the limit as $j \rightarrow \infty$ by iterations on:

   $$V_{j+1}(x) = \max_{u} \{r(x,u) + \beta E[V_j(x')|x]\}$$

   subject to

   $$x' \in g(x,u,\epsilon')$$
5. The limiting value function V is differentiable.

A Simple Stochastic Growth Model

$$\max_{\{c_t, k_{t+1}\}_0^\infty} E \sum_{t=0}^{\infty} \beta^t \ln(c_t) \tag{4}$$

subject to:

$$c_t + k_{t+1} \leq A_t k_t^\alpha$$

$$k_{t+1} \geq 0$$

and we are going to let $\{A_t\}$ be a sequence of i.i.d. random variables.

So what are the objects of choice in this model? They are set of contingency plans:

$$c_0, c_1(A_1), c_2(A_1, A_2), ..., c_t(A_1, A_2, ..., A_t), ...$$
$$k_1, k_2(A_1), k_2(A_1, A_2), ..., k_{t+1}(A_1, A_2, ..., A_t), ...$$

Assume A_t can take on two values each period: high or low.

$$\begin{aligned} E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t) &= \sum_{t=0}^{\infty} \beta^t E_0 \ln(c_t) \\ &= \sum_{t=0}^{\infty} \beta^t E_0 \ln(c_t(A_1, A_2, ..., A_t)) \end{aligned}$$

So at period 1

$$E_0 \ln(c_1) = \text{prob}(A_1 = \text{high}) \ln(c_1(A_{\text{high}})) + \text{prob}(A_1 = \text{low}) ln(c_1(A_{\text{low}}))$$

In period 2

$$\begin{aligned} E_0 \ln(c_2) = \; & (\text{prob}(A_1 = \text{high}) \times \text{prob}(A_2 = \text{high})) \ln(c_2(A_{\text{high}}, A_{\text{high}})) + \\ & (\text{prob}(A_1 = \text{high}) \times \text{prob}(A_2 = \text{low})) \ln(c_2(A_{\text{high}}, A_{\text{low}})) + \\ & (\text{prob}(A_1 = \text{low}) \times \text{prob}(A_2 = \text{high})) \ln(c_2(A_{\text{low}}, A_{\text{high}})) + \\ & (\text{prob}(A_1 = \text{low}) \times \text{prob}(A_2 = \text{low})) \ln(c_2(A_{\text{low}}, A_{\text{low}})) \end{aligned}$$

So at date t there are 2^t possible shock paths.

We write the Bellman equation as:

$$V(k, A) = \max_{c,k'} \{\ln c + \beta E[V(k', A')]\}$$

subject to

$$\begin{aligned} c + k' &\leq Ak^{\alpha} \\ k &\geq 0 \\ A' &\text{ is i.i.d.} \end{aligned}$$

We assume A takes on n values $A^1, A^2, ..., A^n$ with probabilities $\pi_1, \pi_2, ..., \pi_n$. We place the usual restriction on π: $\pi_i \geq 0 \;\; \forall i$ and $\Sigma_i \pi_i = 1$.

$$E[V(k', A')] = \sum_{j=1}^{n} \pi_j V(k', A'^j)$$

If we substitute the constraint into the objective, we can write the Bellman equation as:

$$V(k, A) = \max_{k'} \{\ln (Ak^{\alpha} - k') + \beta E[V(k', A')]\}$$

Taking the first-order condition yields:

$$0 = \frac{-1}{Ak^{\alpha} - k'} + \beta E[V_1(k', A')]$$

Applying Benvensite-Scheinkman (the envelop condition) yields:

$$V_1(k, A) = \frac{\alpha A k^{\alpha-1}}{Ak^{\alpha} - k'}$$

So we get

$$0 = \frac{-1}{Ak^{\alpha} - k'} + \beta E\left[\frac{\alpha A' k'^{\alpha-1}}{A' k'^{\alpha} - k''}\right]$$

We can guess and verify that the solution to this problem is of the form:

$$V(k, A) = F + G \ln k + H \ln A$$

and the optimal policy rule is:

$$k' = \alpha\beta A k^{\alpha}$$

So $k' = h(k, A)$ is a stochastic difference equation, so capital follows a Markov process.

We know in the deterministic case we solve for a steady state of capital:

$$k' = \alpha\beta A k^{\alpha}$$

Taking logs:

$$\ln k' = \ln(\alpha\beta A) + \alpha \ln k$$

$$\ln(k_{ss}) = \frac{\ln(\alpha\beta A)}{1 - \alpha}$$

We can say some things about the stationary distribution of capital:

$$k' = \alpha\beta A k^{\alpha}$$

Taking logs:

$$\ln k' = \ln(\alpha\beta) + \ln(A) + \alpha \ln k$$

Assume $\ln(A_t)$ is an i.i.d. random variable whose $E[\ln A_t] = 0$, and $\mathrm{var}[\ln A_t] = E[(\ln A_t - E \ln A_t)^2] = \sigma_A^2$.
Then we get

$$E[\ln k_{t+1}] = E\left[\ln(\alpha\beta) + \ln A + \alpha \ln k_t\right]$$

$$E[\ln k_{t+1}] = \ln(\alpha\beta) + \alpha E[\ln k_t]$$

To compute the variance of the distribution of capital:

$$\begin{aligned} \mathrm{var}[\ln k_{t+1}] &= \mathrm{var}[\alpha \ln(k_t) + \ln A_t] \\ &= \mathrm{var}(\alpha \ln(k_t)) + \mathrm{var}(\ln A_t) \\ &= \alpha^2 \mathrm{var}(\ln k_t) + \sigma_A^2 \end{aligned}$$

So

$$\mathrm{var}(\ln k_t)) \rightarrow \frac{\sigma_A^2}{1-\alpha^2} \text{ as } t \rightarrow \infty$$