The Solow Growth Model

1. Set-Up
2. Dynamics, Simulations and Steady-States
3. Comparative Dynamics
4. Golden Rule
5. Convergence

Set-Up

- Closed economy, single good produced each period, Y_t.
- Discrete time (in contrast to the Romer text).
- There are two factors of production, labor, N_t and K_t.
- Total factor productivity, A, is a constant.
- There is a constant returns-to-scale production function. In particular, we are going to assume that the production function is Cobb-Douglas:

$$\begin{aligned} Y_t &= AF(K_t, N_t) \\ &= AK_t^{1-\alpha} N_t^{\alpha} \end{aligned}$$

- We have the standard expenditure identity for a closed economy without government:

$$Y_t = C_t + I_t.$$

- The law of motion for capital is:

$$K_{t+1} = (1-\delta)K_t + I_t$$

  for a given K_0.

- Let s be the fraction of income that is saved (invested) each period. so

$$I_t = sY_t$$

  and

$$C_t = (1-s)Y_t.$$

- So the law of motion of capital can be written as:

$$K_{t+1} = (1-\delta)K_t + sY_t.$$

- Assume N grows over time:

$$N_{t+1} = (1+n)N_t$$

- Let y_t = per capital output. So $y_t \equiv \frac{Y_t}{N_t}$.

$$\begin{aligned} y_t &= \frac{AF(K_t, N_t)}{N_t} \\ &= AF(\frac{1}{N_t}K_t, \frac{1}{N_t}N_t) \\ &= AF(\frac{K_t}{N_t}, 1) \\ &= AF(k_t, 1) \\ &= AF(k_t) \end{aligned}$$

where k_t is the capital-to-labor ratio.

- In the case of a Cobb-Douglas production function:

$$\begin{aligned} y_t &= \frac{Y_t}{N_t} \\ &= \left(\frac{1}{N_t}\right) AK_t^{\alpha} N_t^{1-\alpha} \\ &= A\left(\frac{K_t^{\alpha}}{N_t^{\alpha}}\right)\left(\frac{N_t^{1-\alpha}}{N_t^{1-\alpha}}\right) \\ &= Ak_t^{\alpha} \end{aligned}$$

- The law of motion for the capital-to-labor ratio, k_t:

$$\begin{aligned}
k_{t+1} &= \frac{K_{t+1}}{N_{t+1}} \\
&= \frac{(1-\delta)K_t + I_t}{(1+n)N_t} \\
&= \frac{(1-\delta)K_t + sY_t}{(1+n)N_t} \\
&= \frac{(1-\delta)K_t + sAK_t^{\alpha}N^{1-\alpha}}{(1+n)N_t} \\
&= \frac{(1-\delta)}{(1+n)}\frac{K_t}{N_t} + \frac{sA}{(1+n)}\frac{K_t^{\alpha}}{N_t^{\alpha}} \\
&= \frac{(1-\delta)}{(1+n)}k_t + \frac{s}{(1+n)}Ak_t^{\alpha}
\end{aligned}$$

Let's do some simulations

- Assign values to δ, s, n, A, and α. Given a starting value for k we can simulate this economy.
- Let's use 1959 as a starting value. Use data from Economic Report of the President and BEA So

$$k_{1959} = \frac{K_{1959}}{N_{1959}} = \frac{411.7}{115.3} = 3.57$$

- Set $A = TFP_{1959} = 2.87$. Set $\delta = 0.08$, $s = .10$, $n = 0.015$, and $\alpha = 0.36$
- Just to make sure you understand what we are doing we are setting $k_0 = 3.57$ and using the law of motion for k_t we just derived to compute values for k_1, k_2, k_3, ...

Experiment

- After about 75 years the capital-to-labor ratio stops growing.
- Forget above doubling every 30. This model does not predict continued growth in per capita output.
- We do get sustained growth in total output. But we are way off from the data.

What's going on here?

- Go back to our law of motion for k_t.

$$k_{t+1} = \frac{(1-\delta)}{(1+n)}k_t + \frac{sA}{(1+n)}k_t^{\alpha}$$

- Note that when $k_t = 0$, $k_{t+1} = 0$.
- Take a derivative:

$$\frac{dk_{t+1}}{dk_t} = \frac{(1-\delta)}{(1+n)} + \frac{sA\alpha}{1+n}k_t^{\alpha-1}$$

- Since $1-\alpha < 0$, we know when $k_t = 0$, $dk_{t+1}/dk_t = \infty$
- Also since $1-\alpha < 0$, when $k_t = \infty$, $k_t^{\alpha-1} = 0$. Therefore $dk_{t+1}/dk_t = (1-\delta)/(1+n) < 1$.

Steady-state

- To compute the steady state capital-to-labor ratio we need to do a little algebra:

$$\begin{aligned} k^* &= \frac{(1-\delta)}{1+n}k^* + \frac{sA}{1+n}k^{*\alpha} \\ (1+n)k^* &= (1-\delta)k^* + sAk^{*\alpha} \\ (n+\delta)k^* &= sAk^{*\alpha} \end{aligned}$$

- Figure 1.2 in Romer plots the RHS and LHS of this last equation.

$$\begin{aligned} (n+\delta)k^* &= sAk^{*\alpha} \\ \frac{n+\delta}{sA} &= k^{*\alpha-1} \\ k^* &= \left(\frac{n+\delta}{sA}\right)^{\frac{1}{\alpha-1}} \end{aligned}$$

Let get the exponent positive

$$k^* = \left(\frac{sA}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

- Growth does occur while $k_t < k^*$.
- But eventually growth slows to zero.

Comparative statics/dynamics

- A change in s.
- A change in δ
- A change in n
- A one-time change in A
- Natural and unnatural disasters
- Growth effects versus level effects

Growth in A

- So neither growth in capital nor growth in labor can sustain output growth over a long period time. What's left? A.
- At this point you should be thinking there are decreasing marginal benefits to increasing capital and labor – but not A.
- So it is going to turn out that growth in A can get us sustained growth in output per worker.
- But what the heck is A? We just computed it as a residual.

Saving and the Golden Rule

- What determines the saving rate, s? We assumed:

$$I_t = sY_t$$

and

$$C_t = (1 - s)Y_t.$$

- So what is the optimal s?
- Suppose we want to maximize steady-state consumption.

$$\begin{aligned} c^* &= (1 - s)y^* \\ &= (1 - s)Ak^{*\alpha} \\ &= (1 - s)A\left(\frac{sA}{n + \delta}\right)^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

- Easier question: What steady-state level of the capital-to-labor ratio, k^*, maximizes steady-state consumption per worker.
- Define two more variables.
  - $c_t \equiv C_t/N_t$. So c_t denotes consumption per worker.
  - $i^* \equiv I_t/N_t$. So i_t denotes investment per worker.
- Then we can write the expenditure identity as:

$$\begin{aligned} Y_t &= C_t + I_t \\ \frac{Y_t}{N_t} &= \frac{C_t}{N_t} + \frac{I_t}{N_t} \\ y_t &= c_t + i_t \end{aligned}$$

So in steady-state:

$$y^* = c^* + i^*$$

- We can get the law-of-motion for K_t in per-worker (lower case terms) as well. You can fill in the intermediate steps ...

$$\begin{aligned} K_{t+1} &= (1-\delta)K_t + I_t \\ &\vdots \\ (1+n)k_{t+1} &= (1-\delta)k_t + i_t \end{aligned}$$

So in steady-state:

$$(1+n)k^* = (1-\delta)k^* + i^*$$

Solving for i^* yields:

$$i^* = (n+\delta)k^*$$

- If we substitute in for y^* and i^* are per-worker expenditure identity becomes:

$$Ak^{*\alpha} = c^* + (n+\delta)k^*$$

or

$$c^* = Ak^{*\alpha} - (n+\delta)k^*$$

- Now remember why did all this algebra. We wanted to find the k^* that maximizes c^*. So we want take the derivative of the above equation with respect to k^*. So the k^* that maximizes c^* is the k^* that satisfies:

$$\alpha A k^{*\alpha-1} - \delta = n$$

or

$$\text{MPK} - \delta = n.$$

- Graphically, this k^* is the capital-to-labor ratio that maximizes the difference between the production function and steady-state investment line.

s	k^*	y^*	δk^*	c^*	MPK	MPK - δ
0	0	0	0	0	∞	∞
0.1	54.9	52.2	4.4	47.0	0.2850	0.2050
0.2	147.8	70.2	11.8	56.2	0.1425	0.0625
0.3	263.9	83.6	21.1	58.5	0.0950	0.0150
0.4	398.0	94.5	31.8	56.7	0.0713	-0.0087
0.5	547.4	104.0	43.8	52.0	0.0570	-0.0230
0.6	710.2	112.5	56.8	45.0	0.0475	-0.0325
0.7	885.2	120.1	70.8	36.0	0.0407	-0.0393
0.8	1071.3	127.2	85.7	25.4	0.0356	-0.0444
0.9	1267.6	133.8	101.4	13.4	0.0317	-0.0483
1.0	1473.5	140.0	117.9	0	0.0285	-0.0515

Table 1: A Numerical Comparison of Steady-States

Why should we poor people make sacrifices for those who will in any case live in luxury in the future?

Robert Solow

Convergence

- Unconditional convergence: Poor countries should eventually catch up to rich countries.
  - This should occur if saving rates, population growth rates, and production functions are the same everywhere.
  - This really hinges on whether there is free flowing international borrowing and lending.
  - Why isn't all new investment occurring in the Third World?
  - Is there too little capital mobility in the world?
- Conditional convergence: Living standards will converge in countries with similar characteristics: s, n, δ, α.
  - There would be convergence except different countries have different parameters in the Solow model.

Evidence of Convergence

- In the case of Post-WW2 Japan, Germany, and England, much capital was destroyed in the war so reasonable to think of them being far away from steady-state so high GDP growth in decades following war partially explained as transitioning to steady-state.
- For developed countries, Solow model predicts we would already be at or near steady state.
- If one does not account for differing population growth rates and saving rates, there is simply no evidence for convergence.
- If one accounts for differing saving and population growth rates, convergence might be occurring, although there is a small sample problem. The theory predicts that countries like Japan and Germany in 1950 will have higher growth rates than US.
- US states also seem to be converging. (Poor states grow faster than rich ones.) Right now the South is growing faster than the North. Evidence of conditional convergence since the U.S. states have similar characteristics.
- After the Civil War, average per capita income in South was 40% of the average per capita income in North. Now it is about the same.