Quantum Information and Entanglement in Holographic Theories

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Contents

1 Entanglement entropy & mutual information

Entropy is extensive: if systems A, B are decoupled

$$\rho(AB) = \rho(A) \otimes \rho(B)\,, \qquad S(AB) = S(A) + S(B)$$

Entropy is subadditive: if

$$\rho(AB) \neq \rho(A) \otimes \rho(B) \qquad \text{(where } \rho(A) \equiv \mathrm{tr}_B\, \rho(AB),\ \rho(B) \equiv \mathrm{tr}_A\, \rho(AB)\text{)}$$

then

$$S(AB) < S(A) + S(B)$$

$S(A), S(B)$ are called *entanglement entropies*, because they can be non-zero even if $\rho(AB)$ is pure. Example:

$$\rho(AB) = \frac{1}{2}\left(|00\rangle + |11\rangle\right)\left(\langle 00| + \langle 11|\right)$$

$$\rho(A) = \rho(B) = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$

$$S(AB) = 0\,, \qquad S(A) = S(B) = \ln 2$$

In general, if $\rho(AB)$ is pure, then $S(A) = S(B)$. (Note that there can be entanglement entropy without entanglement, or even correlation!)

Mutual information:

$$I(A:B) = S(A) + S(B) - S(AB)$$

measures correlation (both quantum & classical). In above example,

$$I(A:B) = 2\ln 2$$

Classical example:

$$\rho(AB) = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right)$$

$$\rho(A) = \rho(B) = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$

$$S(AB) = S(A) = S(B) = \ln 2\,, \qquad I(A:B) = \ln 2$$

Mutual information bounds correlators between normalized operators (Wolf, Verstraete, Hastings, Cirac '07):

$$\left(\langle \mathcal{O}_A \mathcal{O}_B\rangle - \langle \mathcal{O}_A\rangle\langle \mathcal{O}_B\rangle\right)^2 \leq 2I(A:B)$$

Mutual information increases under adjoining other systems to A or B:

$$I(A:BC) \geq I(A:B)$$

i.e.

$$S(AB) + S(BC) \geq S(B) + S(ABC)$$

(Strong subadditivity of entropy)

In a QFT, we can take subsystems $A, B, \ldots$ to be spatial regions. The EE is UV-divergent:

$$S(A) = \epsilon^{2-d}\mathrm{area}(\partial A) + \cdots$$

or, in two dimensions,

$$S(A) = -\frac{c_{\mathrm{UV}}}{6}\ln\epsilon\,\#(\partial A) + \cdots$$

Mutual information:

$$I(A:B) = \epsilon^{2-d}\mathrm{area}(\partial A \cap \partial B) + \cdots$$

$$I(A:B) = -\frac{c_{\mathrm{UV}}}{6}\ln\epsilon\,\#(\partial A \cap \partial B) + \cdots\,,$$

In two & three dimensions, mutual information for separated regions is finite. Subleading terms are very difficult to compute in practice—even in free QFTs.

These are very useful non-local observables:

- Again, $I(A:B)$ bounds all possible two-point functions between A & B. (EE knows a lot about the system!)
- EE can be used as an order parameter (e.g. to detect topological phase transitions)
- EE is the most efficient method for numerically extracting central charges of fixed points in two dimensions
- ...

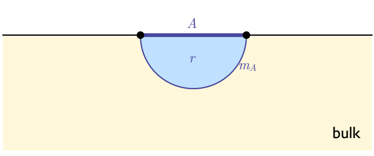
2 Ryu-Takayanagi formula

Ryu-Takayanagi formula for EE

- of a spatial region
- in a holographic theory
- dual to classical Einstein gravity ("large N, strong coupling")
- in a state described in the bulk by a static, classical field configuration ($\Rightarrow$ distinguished constant-time surfaces):

$$S(A) = \frac{1}{4G_N}\min_{m\sim A}\Big(\mathrm{area}(m)\Big)$$

($m \sim A$ means $\exists$ bulk region r s.t. $\partial r = m \cup A$; area is w.r.t. spatial, Einstein-frame metric). Call minimizer m_A



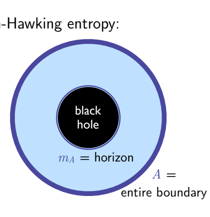
Simple, elegant, easy to compute! If true, is a deep yet simple statement about quantum gravity.

Checks:

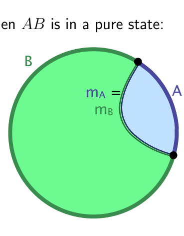
- reproduces EE of interval in vacuum of $1+1$ CFT on $\mathbf{R}$:

$$S([u,v]) = \frac{c}{3}\ln\left(\frac{v-u}{\epsilon}\right)$$

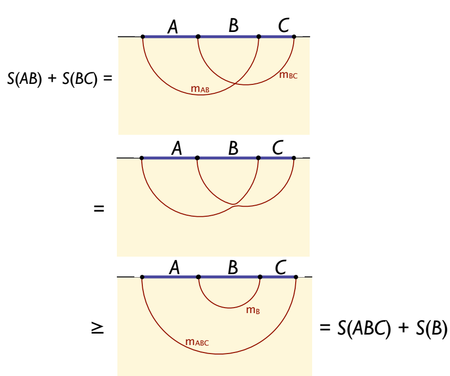
  (Holzhey, Larsen, Wilczek '94)
- reproduces UV divergent part of EE (proportional to area of ∂A) in arbitrary dimension
- reproduces Bekenstein-Hawking entropy:



- obeys $S(A) = S(B)$ when AB is in a pure state:



- obeys strong subadditivity: $S(AB) + S(BC) \geq S(B) + S(ABC)$ (MH + Takayanagi '07)



Comments:

- Fursaev '06 claimed a derivation of the RT formula, but it was incorrect (MH '10)
- RT formula applies equally whether A is connected or disconnected (there was confusion early on about this due to an error in the EE literature, but it has been cleared up)
- Conjecture for corrections (α' = classical higher-derivative; G_N = quantum):

$$S(A) = \frac{1}{4G_N}\min_{m\sim A}\Big(\mathrm{area}(m) + \mathcal{O}(\alpha')\Big) + \mathcal{O}(G_N^0)$$

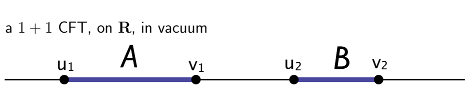
  See Hung, Myers, Smolkin '10 & de Boer, Kulaxizi, Parnachev '10
- Conjecture for covariant generalization by Hubeny, Rangamani, Takayanagi '07. Has been applied to various systems. However, not clear whether it satisfies all the tests of RT

3 Two intervals

(MH '10) Ryu-Takayanagi formula makes many new predictions. An interesting one is a novel phase transition in the mutual information between separated regions as a function of their separation. The simplest case—where we can try to test it by a first-principles calculation—is two intervals on a line in a two-dimensional CFT.

3.1 Prediction

Let $\mathcal{C}$ be a $1+1$ CFT, on $\mathbf{R}$, in vacuum



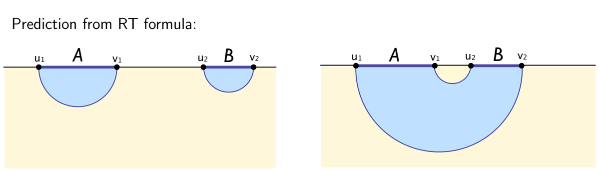
EE of 2 intervals: much harder to compute from first principles than of 1 interval; depends on whole theory $\mathcal{C}$, not just central charge c. Unknown even for *free boson*! (Calabrese, Cardy, Tonni)

Mutual information is UV-finite & conformally invariant:

$$I(x) \equiv I(A:B) = S(A) + S(B) - S(AB)$$

$$x = \frac{(v_1-u_1)(v_2-u_2)}{(u_2-u_1)(v_2-v_1)}$$

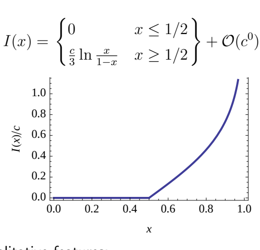
Prediction from RT formula:



$x \leq 1/2$ $\qquad$ $x \geq 1/2$

$m_{AB} = m_A \cup m_B$ $\qquad$ $m_{AB} \neq m_A \cup m_B$

$$I(x) = \begin{Bmatrix} 0 & x \leq 1/2 \\ \frac{c}{3}\ln\frac{x}{1-x} & x \geq 1/2 \end{Bmatrix} + \mathcal{O}(c^0)$$



(Recall $G_N \sim 1/c$.) Qualitative features:

- phase transition at $x = 1/2$
- $I(x) = 0$ for $x \leq 1/2$

(Expect both features to be preserved by higher-derivative classical corrections, but not by quantum corrections)

3.2 Tests

Let's try to compute mutual information of 2 intervals from first principles. Replica trick (Holzhey, Larsen, Wilczek '94):

1. For all integer $n > 1$, consider orbifold theory $\mathcal{C}^n/\mathbf{Z}_n$, and compute

$$I^{(n)}(x) = \frac{1}{n-1}\ln\frac{\langle\sigma_1(0)\sigma_{-1}(x)\sigma_1(1)\sigma_{-1}(\infty)\rangle}{\langle\sigma_1(0)\sigma_{-1}(x)\rangle\langle\sigma_1(1)\sigma_{-1}(\infty)\rangle}$$

   (mutual Rényi information) where $\sigma_{1,-1}$ are twist operators
2. Analytically continue to non-integer n and take limit $n \to 1$:

$$I(x) = \lim_{n\to 1} I^{(n)}(x)$$

4-point function of twist operators in $\mathcal{C}^n/\mathbf{Z}_n$ is related to partition function of $\mathcal{C}$ on genus-$(n-1)$ Riemann surface, whose complex structure depends on x. In holographic theories, this has a first-order phase transition (à la Hawking-Page) at $x = 1/2$ for all n (since $x = 1/2$ is fixed point of mapping-class group). Supports existence of phase transition in $I(x)$ at $x = 1/2$.

Second approach: first expand $I^{(n)}(x)$ in x, then take limit $n \to 1$. Using OPE,

$$\frac{\langle\sigma_1(0)\sigma_{-1}(x)\sigma_1(1)\sigma_{-1}(\infty)\rangle}{\langle\sigma_1(0)\sigma_{-1}(x)\rangle\langle\sigma_1(1)\sigma_{-1}(\infty)\rangle} = \sum_{\mathcal{O}_m\in\mathcal{C}^n/\mathbf{Z}_n} c^{\sigma}{}_{\sigma m}c^{m}{}_{\sigma\sigma}x^{2d_m}$$
$$= 1 + \frac{(n^2-1)^2 c}{144n^3}x^2 + \mathcal{O}(x^3)$$

(from identity & stress tensor)

$$\Rightarrow \quad I^{(n)}(x) = \frac{(n+1)^2(n-1)c}{144n^3}x^2 + \mathcal{O}(x^3)$$

$$\Rightarrow \quad I(x) = \lim_{n\to 1} I^{(n)}(x) = \mathcal{O}(x^3)$$

To go to higher order in x, use conformal blocks of $\mathcal{C}^n/\mathbf{Z}_n$. At each order expand in $1/c$:

$$I^{(n)}(x) = \frac{(n-1)(n+1)^2}{144n^3}\left(x^2 + x^3 + \frac{1309n^4 - 2n^2 - 11}{1440n^4}x^4 + \frac{589n^4 - 2n^2 - 11}{720n^4}x^5 + \mathcal{O}(x^6)\right)c + \mathcal{O}(c^0)$$

Order-c part of $I(x)$ vanishes, as predicted by RT

4 Monogamy of mutual information

What do we expect for $I(A:BC)$ versus $I(A:B) + I(A:C)$? Define

$$I_3(A:B:C) = I(A:B) + I(A:C) - I(A:BC)$$
$$= S(A) + S(B) + S(C) - S(AB) - S(BC) - S(AC) + S(ABC)$$

Can go either way. Two examples:

$$\rho(ABC) = \frac{1}{2}\left(|000\rangle\langle 000| + |111\rangle\langle 111|\right), \qquad I_3 = \ln 2$$

$$\rho(ABC) = \frac{1}{4}\left(|000\rangle\langle 000| + |011\rangle\langle 011| + |101\rangle\langle 101| + |110\rangle\langle 110|\right), \qquad I_3 = -\ln 2$$

Can go either way also in QFTs (Casini & Huerta '08)

Hayden, Headrick, Maloney '11: RT formula implies, for all A, B, C,

$$I_3(A:B:C) \leq 0$$

(generalization of SSA proof)

This implies an infinite set of inequalities generally obeyed by entanglement entropy recently discovered by Cadney, Linden, Winter '04, '11—another check of RT formula.

What does this special property of holographic theories mean?

Crucial difference between classical correlation & entanglement: entanglement is not shareable. For example, in state

$$|ABC\rangle = \frac{1}{2^{1/2}}\left(|000\rangle + |111\rangle\right)$$

there is no A–B or A–C entanglement. (Basis for quantum cryptography)

Quantities that measure only entanglement (not classical correlation), such as the entanglement of formation and distillable entanglement, obey the monogamy relation

$$f(A:BC) \geq f(A:B) + f(A:C) \tag{1}$$

The fact that in holographic theories the mutual information obeys the monogamy relation suggests that the correlations are dominated by entanglement