Matthew Headrick: Statement of Recent Work

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There have been two major themes to my work over the past few years. In this Statement I will concentrate on one of them, statistical mechanics and quantum information in holographic theories, because that is the area where my current research is focused. For completeness, however, let me first briefly summarize my work in the other theme, elliptic numerical relativity and its applications to string theory. Making use of recent advances in differential and algebraic geometry, particularly in the study of Ricci flow and of spectral representations of Calabi-Yau metrics, my collaborators and I created and applied efficient numerical methods for finding static and Euclidean solutions to the Einstein equation, including higher-dimensional black holes [1], toric manifolds [2, 3], and Calabi-Yaus [4].

My work on the statistical mechanics of holographic theories has focused on problems concerning the deconfinement transition, represented in the gravity dual as a Hawking-Page transition [5]. In the paper [6], I studied the behavior of the Polyakov loop, a key order parameter for deconfinement. Specifically, I showed how to describe in gravity microcanonical ensembles in which the Polyakov loop is fixed; using this tool, I discovered a novel phase transition as a function of the chemical potential for the Polyakov loop, generalizing the deconfinement/Hawking-Page transition. In the paper [7], Ebrahim and I considered a field theory undergoing a dynamical deconfinement transition following an external injection of energy, which is described holographically by a gravitational collapse and formation of a black hole, and studied the time dependence of statistical fluctuations such as Brownian motion of a quark. Statistical fluctuations in a holographic theory are represented in the bulk as quantum effects such as Hawking radiation [8, 9]; however, since the standard methods for computing Hawking radiation yield only the late-time radiation, we had to invent a new method that gave the full time dependence of these fluctuations. Surprisingly, we found that certain types of fluctuations become thermal instantly following the injection of energy.

Recently my attention has turned to the subject of entanglement entropies in holographic theories [10, 11, 12]. Let me start with some context before describing my work. In the last decade, entanglement entropies and related information-theoretic quantities have proven to be a valuable addition to the field theorist's toolkit, providing information that cannot be obtained from traditional computables like expectation values and scattering amplitudes. For example, entanglement entropies of spatial regions and mutual informations between them provide a measure of correlations that is independent of the choice of any particular observable, and are capable of detecting non-local effects like topological order that are invisible to any local observable. Unfortunately, these quantities are notoriously difficult to calculate, even in free field theories. In holographic theories, however, Ryu and Takayanagi (RT) [13, 14] conjectured a remarkably simple formula for the entanglement entropy of an arbitrary region in terms of the area of a certain minimal surface in the bulk. This formula has since been refined, generalized, and extensively tested by myself and others, and the study of entanglement entropies in holographic theories is now a rapidly-developing subject. Aside from their usefulness as probes of strongly-coupled field theories (for example in holographic models of condensed-matter systems), the study of these quantities goes to fundamental questions about gravity and holography, and also raises interesting new questions in quantum information theory itself.

My first paper on this subject, with Takayanagi, proved that the RT formula satisfies the strong-subadditivity (SSA) inequality [10], a fundamental physical property of entropy and a cornerstone of quantum information theory. Our theorem, akin to the area theorem showing that the Bekenstein-Hawking formula is consistent with the second law of thermodynamics, stands as one of the most stringent tests of the RT formula. Our proof was based on a simple geometrical argument, which is quite striking given the technical difficulty of the general proof of SSA, and provides another example of how gravity manages to encode deep physics in simple geometry.

More recently, Hayden, Maloney, and I generalized the holographic proof of SSA to show that the mutual information in fact obeys a stronger inequality, the monogamy relation [12]. Unlike SSA, this is not a general property of quantum systems (although, as we showed, it does imply a recently-discovered infinite set of general inequalities [15, 16], providing further evidence for the RT formula), but rather is a special property of holographic theories. In fact, it suggests a link to quantum cryptography, where monogamy relations play a central role, and hints that the correlations in holographic theories are dominated by quantum entanglement rather than classical correlations.

In another recent paper [11], I observed that the RT formula makes a striking prediction for the behavior of the mutual information between separated regions as a function of their separation (keeping their shapes and sizes fixed); for small separations the mutual information is non-zero, but at a certain finite separation it undergoes a first-order transition, dropping to zero (at leading order in $1/N$). Compared to the typical behavior of non-holographic theories, this behavior is so exotic that it made some (including Takayanagi!) doubt the validity of the RT formula for disconnected regions, so it was important to check this prediction from first principles. Focusing on the two-dimensional case, with the two regions being intervals, I was able to confirm both the existence of the transition and the vanishing of the mutual information at large separations. To do this I employed the replica trick, computing the required entanglement Rényi entropies using both holographic and conformal field theory techniques. As an intriguing by-product, I found that the mutual Rényi informations display a very curious behavior: while they also undergo a first-order transition, they do not vanish when the mutual information does. Naively this is impossible—the vanishing of the mutual information implies that the joint density matrix factorizes, hence all the Rényi informations also vanish. It is allowed here because the mutual information does not vanish exact but only at leading order in $1/N$. But it shows that there is some subtlety—worth investigating further—in how the $1/N$ expansion applies to the joint density matrix.

While these results about entanglement entropies have given us valuable information about holographic theories, in some sense they have also added to the mystery of holography. We don't yet know why the RT formula is correct (a putative proof by Fursaev [17] was incorrect, as I showed [11]), why entanglement entropies in holographic theories have the special properties described above, or what these facts are telling us at a fundamental level about quantum gravity. In the absence of a derivation of the RT formula, it is also difficult to go beyond it; for example, we don't know even in principle how to calculate $1/N$ corrections to the entanglement entropy. The papers [10, 11, 12] had in common a certain surprising observation, that I believe points the way towards addressing these fundamental questions. A certain amount is known (despite the absence of a derivation) about how the RT formula gets modified when the bulk action has higher-derivative corrections (i.e. slightly away from the strong-coupling limit of the field theory, but still at infinite N). My collaborators and I found that the features we had discovered—the simple proof of SSA, the monogamy, and the phase transition and vanishing of the mutual information but not mutual Rényi informations—all continue to hold in the presence of such corrections. In fact, we found evidence that they hold in generic large-N theories, at arbitrary coupling, independent of holography. (My calculations of Rényi entropies strongly suggest that the full entanglement spectrum, for any region, is actually completely theory-independent at large N in two dimensions [11].) Thus it seems that the key to understanding entanglement entropies in holographic theories is not strong coupling but large-N. This important issue is a principal focus of my current work.

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