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# Our strategy...

**Matthew Headrick**

Abstract: ...for taking over the world using numerical Calabi-Yau metrics

## Contents

LAB: -TOC-



## 1. General strategy

Let $X$ be an $n$-dimensional Calabi-Yau manifold with a fixed complex structure. Let $\Omega$ be the holomorphic $(n,0)$ form on $X$, and denote the corresponding volume form by $\mu$:

$$\mu \equiv (-i)^n \Omega \wedge \bar{\Omega}. \tag{1.1}$$

Let $M$ be a Kähler class on $X$. For a given representative $J$ of $M$, define the following functions on $X$:

$$v_J \equiv \frac{J^n}{n!\,\mu} = \frac{\det g_{i\bar{\jmath}}}{|\Omega_{1\ldots n}|^2}\,, \qquad f_J \equiv \ln v_J\,, \tag{1.2}$$

where $g_{i\bar{\jmath}}$ is the metric corresponding to $J$. The Ricci tensor of $g_{i\bar{\jmath}}$ is given by the matrix of second derivatives of $f_J$:

$$R_{i\bar{\jmath}}(J) = -\partial_i \partial_{\bar{\jmath}} f_J. \tag{1.3}$$

Yau's theorem states that $M$ contains a unique Ricci-flat representative, i.e. one satisfying the Monge-Ampère equation

$$v_J = \text{constant}. \tag{1.4}$$

Our task is to find a numerical approximation to this representative.

Our general strategy will contain two more or less independent ingredients:

1. An energy functional $H[J]$ on $M$ that is minimized on the Ricci-flat metric.
2. A way to numerically represent metrics in $M$.

We will describe these in turn, then discuss the details of how to evaluate $H[J]$ numerically.

## 2. Energy functional and optimization algorithm

The first ingredient in our strategy for calculating a numerical approximation to the Ricci-flat representative of $M$ is an energy functional that has a (unique) global minimum on the Ricci-flat representative. Here we would like to propose two such functionals, both of which have the additional nice property that they have no other critical points. The first is simply the standard deviation of $v_J$:

$$H_1[J] \equiv \int_X \mu \left( v_J - \frac{V_M}{V_\mu} \right)^2 = \int_X \mu \, v_J^2 - \frac{V_M^2}{V_\mu}, \tag{2.1}$$

where

$$V_M = \int_X \mu \, v_J \,, \qquad V_\mu = \int_X \mu \tag{2.2}$$

are the volumes of $X$ with respect to the metric and the volume form $d\mu$ respectively (the first depends on the Kähler class—not the representative–and the second on the normalization of $\Omega$). The second functional uses gradient information:

$$H_2[J] \equiv \int_X \mu \, g^{\bar{j}i} \, \partial_i f_J \, \partial_{\bar{j}} f_J = -\frac{1}{2} \int_X \mu \, R_J, \tag{2.3}$$

where $R_J = -\nabla_J^2 f_J$ is the Ricci scalar of $g_{i\bar{j}}$. Both $H_1$ and $H_2$ have the following properties:

1. They are non-negative.

2. They vanish on the Ricci-flat representative, and only there.

3. They have no other critical points.

The first two are obvious from the definitions, and the third follows from a short calculation in each case. Both functionals can serve as figures of merit in trying to approximate the Ricci-flat metric, and it remains to be seen whether one is superior to the other for some particular application. $H_1$ is certainly substantially simpler to compute than $H_2$. For all I know there may be yet other functionals with all of these properties. Whichever one we choose we will call it $H$.

Working numerically, we will be able to store only a finite amount of information about the metric, so in effect we will be working on a finite-dimensional submanifold $M' \subset M$. The task we set ourselves is to find the minimum of $H$ on $M'$. $H$ is positive on $M'$ (unless $M'$ happens to go through the Ricci-flat point in $M$, which generically it won't). Furthermore, for reasonable choices of $M'$ I think we can safely assume that $H$ has no critical points other than its minimum. These two properties should be kept in mind when choosing an optimization algorithm. A third property that should be kept in mind is that $H$ is likely to be very expensive to evaluate at each point $J$ in $M'$. Finally, in the case of $H_2$ its gradient is likely to be prohibitely complicated.

## 3. Spectral representation of the metric

The method we use to represent the metric is borrowed from Donaldson, who in turn borrowed it from Yau. We assume that $X$ is embedded in $\mathbf{C}P^N$, which can be done for any Calabi-Yau. More specifically, we consider the case $N = n + 1$, with $X$ defined by a homogeneous polynomial $P$ of degree $N + 1$ in the homogeneous coordinates of $\mathbf{C}P^N$. For $N = 1, 2, 3, 4$, this yields the manifolds $S^0$ in $\mathbf{C}P^1$, $T^2$ in $\mathbf{C}P^2$, K3 in $\mathbf{C}P^3$, and the quintic in $\mathbf{C}P^4$. The generalizations to complete intersections, and to algebraic Calabi-Yaus in weighted projective spaces, are presumably straightforward. Finally, we will assume that the Kähler class $M$ of interest is the one induced by the Fubini-Study class in $\mathbf{C}P^N$. In the case of K3 these assumptions restrict us to a 39-dimensional subspace of the 58-dimensional moduli space, but in the cases of $T^2$ and the quintic it so happens that they present no restriction at all.

Let $z^a$ $(a = 1, \ldots, N + 1)$ be homogeneous coordinates on $\mathbf{C}P^N$. On the patch $O_{(c)}$ defined by the condition $z^c \neq 0$, define $u^a_{(c)} \equiv z^a/z^c$. Letting the index $\alpha$ run from 1 to $N + 1$ skipping $c$, the $u^\alpha_{(c)}$ are inhomogeneous coordinates on $O_{(c)}$, while $u^c_{(c)} = 1$. Given a positive-definite Hermitian form $G_{a\bar{b}}$ on $\mathbf{C}^{N+1}$, the associated Fubini-Study metric on $\mathbf{C}P^N$ has the following Kähler potential on the patch $O_{(c)}$:

$$K^{\mathrm{FS}}_{(c)} \equiv \ln\left(\frac{G_{a\bar{b}} z^a \bar{z}^{\bar{b}}}{|z^c|^2}\right) = \ln\left(G_{a\bar{b}} u^a_{(c)} \bar{u}^{\bar{b}}_{(c)}\right), \tag{3.1}$$

and this is also its value on $X$ for the induced metric. Note that the Kähler class doesn't depend on $G_{a\bar{b}}$.

For any representative of $M$ with Kähler potential $K$, the difference $K - K^{\mathrm{FS}}$ is a globally defined function on $X$. Rather than representing this function locally, we wish to represent it spectrally, in other words expand it in a basis of eigenfunctions of an operator such as the Laplacian. Unfortunately, the Laplacian on $X$ (say, with respect to the induced Fubini-Study metric) is complicated, and we don't know its eigenfunctions analytically. Instead, we use as our basis of functions the eigenfunctions of the Laplacian on $\mathbf{C}P^N$ (with respect to the Fubini-Study metric), which have the following extremely simple form. Let $\rho^I(z)$ be a basis of homogeneous degree-$k$ polynomials in the homogeneous coordinates $z^a$.[1] The index $I$ runs from 1 to $N_k \equiv (N+k)!/N!k!$. The first $k$ eigenspaces of the Laplacian (counting the constant function as the zeroth eigenspace) are spanned by the functions

$$\frac{\rho^I(z)\bar{\rho}^{\bar{J}}(\bar{z})}{(G_{a\bar{b}} z^a \bar{z}^{\bar{b}})^k}. \tag{3.2}$$

These $N_k^2$ functions are linearly independent on $\mathbf{C}P^N$, but not when restricted to $X$. The linear combinations of the $\rho^I$ that vanish on $X$ are simply the polynomials of the form $gP$, where $g$ is any degree $k - (N + 1)$ homogeneous polynomial. There are therefore $N_{k-(N+1)}$

[1] These are referred to as $Z^A$ in Donaldson's paper and in our paper.

of them (if $k > N$), and the space of polynomials that aren't zero on $X$ is $N'_k$ dimensional, where

$$N'_k = \begin{Bmatrix} N_k, & k \le N \\ N_k - N_{k-(N+1)}, & k > N \end{Bmatrix} = \begin{Bmatrix} 2, & N=1 \\ 3k, & N=2 \\ 2(k^2+1), & N=3 \\ \frac{5}{6}k(k^2+5), & N=4 \end{Bmatrix} \sim k^n. \tag{3.3}$$

Let $p^A$ be a basis for this quotient space.

It turns out to be more convenient to decompose $e^{k(K-K^{\rm FS})}$, rather than $K - K^{\rm FS}$, in the basis of functions (3.2). So we write

$$\begin{aligned} K_{(c)} &= K^{\rm FS}_{(c)} + \frac{1}{k}\ln\left(\frac{h_{A\bar{B}}p^A(z)\bar{p}^{\bar{B}}(\bar{z})}{(G_{a\bar{b}}z^a\bar{z}^{\bar{b}})^k}\right) \\ &= \frac{1}{k}\ln\left(\frac{h_{A\bar{B}}p^A(z)\bar{p}^{\bar{B}}(\bar{z})}{|z^c|^{2k}}\right) \\ &= \frac{1}{k}\ln\left(h_{A\bar{B}}p^A(u_{(c)})\bar{p}^{\bar{B}}(\bar{u}_{(c)})\right), \end{aligned} \tag{3.4}$$

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where $h_{A\bar{B}}$ is a positive definite $N'_k \times N'_k$ Hermitian matrix. There is a redundancy, since for any positive real number $\lambda$, $h_{A\bar{B}}$ and $\lambda h_{A\bar{B}}$ give the the same metric. Kähler potentials of the form (3.4) define the subspace $M'$ of the Kähler class $M$ in which we will attempt to minimize the energy functional $H[J]$. The (real) dimensionality of $M'$ is

$$M_k \equiv N'^2_k - 1 \sim k^{2n}, \tag{3.5}$$

so we can roughly think of $k$ as the number of Fourier modes in each direction.

### 3.1 Symmetric polynomials

Unfortunately, in the cases of K3 ($n = 2$) and the quintic ($n = 3$), $M_k$ is unmanageably large for reasonable values of $k$. It therefore seems that we will be restricted to (or at least we should begin with) manifolds with a high degree of discrete symmetry. This would allow us to restrict the space of polynomials to those which are invariant under the symmetry, vastly cutting down the parameter space to be explored.

For concreteness, we will focus on the Fermat case,

$$P = \sum_a (z^a)^{N+1}, \tag{3.6}$$

which has the largest symmetry group $\Gamma$. $\Gamma$ is the product of three symmetry subgroups: the permutations of the $z^a$; multiplying each $z^a$ independently by an $(N+1)$ root of unity, modulo multiplying them all by the same root of unity; and complex conjugation. The total order of $\Gamma$ is

$$2(N+1)^N(N+1)! = \begin{cases} 108, & N=2 \\ 3072, & N=3 \\ 150{,}000, & N=4 \end{cases}. \tag{3.7}$$

A basis for the space of Γ-invariant homogeneous degree $(k,k)$ polynomials can be constructed as follows. Let $E_{Ia}$ be a $2\times(N+1)$ matrix of non-negative integers, satisfying the constraints

$$\sum_a E_{1a} = \sum_a E_{2a} = k\,, \qquad E_{1a} - E_{2a} \equiv 0 \bmod N+1\,. \tag{3.8}$$

Then the polynomial

$$p_E = \prod_a (z^a)^{E_{1a}} \prod_a (\bar{z}^a)^{E_{2a}} + \text{permutations}\,, \tag{3.9}$$

where the permutations include both permutations of the $z^a$ and complex conjugation, is homogeneous of degree $(k,k)$ and Γ-invariant. Furthermore, the polynomials $p_E$ for all matrices $E$ satisfying (3.8) span this space of polynomials. Because of the permutations in (3.9), $p_E = p_{P_L E P_R}$, where $P_L$ and $P_R$ are $2\times 2$ and $(N+1)\times(N+1)$ permutation matrices respectively.

For $k \geq N+1$, the polynomials proportional to $P$ and $\bar{P}$ must be quotiented out. Hence we must construct a basis for the degree $(k-(N+1),k)$ and $(k,k-(N+1))$ polynomials, multiply them by $P$ and $\bar{P}$ respectively, write the products as linear combinations of the basis $(k,k)$ polynomials described above, and remove the appropriate basis elements in order to have a basis for the quotient space. This will leave us with a basis

$$\mathcal{P}^l = c^l_{I\bar{J}} \bar{\rho}^{\bar{J}} \rho^I\,, \tag{3.10}$$

where each $\mathcal{P}^l$ is invariant under the full symmetry group, and they are linearly independent on the $P=0$ surface.

We will now consider the more general polynomial

$$P = \sum_a (z^a)^{N+1} - (N+1)\psi \prod_a (z^a)\,, \tag{3.11}$$

and we will take $\psi$ real. This has a slightly smaller symmetry group than the Fermat polynomial. We still have the permutations and the complex conjugation. Now, however, multiplying each $z^a$ by an $(N+1)$ root of unity $e^{2\pi i n_a/(N+1)}$ is a symmetry only if $\sum_a n_a \equiv 0 \bmod (N+1)$. This constraint means that the order of the group is now

$$2(N+1)^{N-1}(N+1)! = \begin{cases} 36, & N=2 \\ 768, & N=3 \\ 30{,}000, & N=4 \end{cases}\,. \tag{3.12}$$

This means that there are now somewhat more invariant polynomials. As above, a basis of such polynomials is given by polynomials of the form (3.9), where $E$ is again a $2\times(N+1)$ matrix of non-negative integers. Now, however, the conditions (3.8) on $E$ are relaxed to become

$$\sum_a E_{1a} = \sum_a E_{2a} = k\,, \qquad E_{1a} - E_{2a} \equiv j \bmod N+1 \qquad (j \text{ constant})\,. \tag{3.13}$$

The rest of the construction is as before.

## 4. Evaluation of the energy functional

### 4.1 Algebraic approach

In each patch $O_{(c)}$ of $\mathbf{C}P^N$, define the function $P_{(c)} \equiv P(u_{(c)}) = (z^c)^{-(N+1)}P(z)$ (on the overlaps we have $P_{(c')} = (u^c_{(c')})^{N+1}P_{(c)}$). Locally, a coordinate system on $X$ is provided by any subset of $n$ out of the $N$ coordinates $u^\alpha_{(c)}$ on $\mathbf{C}P^N$: pick a $\delta \neq c$ and let the coordinates on $X$ be $u^i_{(c)}$ $(i \neq c, \delta)$, with $u^\delta_{(c)}$ being a function of them defined implicitly by the equation $P_{(c)} = 0$. We will refer to this coordinate system as $O_{(c,\delta)}$. From now on we fix $c$ and suppress it in the notation, writing $P$ for $P_{(c)}$, etc. Keep in mind that the indices $\alpha, \beta$ run from 1 to $N+1$ skipping $c$, while $i, j$ also skip $\delta$.

Of course, for any set of values of $u^i$, the polynomial equation $P = 0$ has $N+1$ solutions for $u^\delta$, leading to a complicated structure of patches. Therefore we advocate a purely algebraic approach to the evaluation of the energy functional, which avoids such issues, as follows: (1) The integral is evaluated by a "zeroth-order" method, simply by summing the values of the integrand over a selection of points on $X$ distributed (possibly at random) according to the measure $\mu$—no trapezoid or higher order approximation to the integrand is constructed. Note that the measure $\mu$ is independent of $h_{A\bar{B}}$, so the points may be chosen at the outset and then stored as $h_{A\bar{B}}$ is varied in search of the minimum. (2) Analytic expressions for the integrand in terms of $h_{A\bar{B}}$ and $P$ will be given; again, the derivatives involved need not be approximated by fitting a polynomial through neighboring points. The idea is to avoid needing to know which points on $X$ lie on the same sheet of $P$.

A crucial question is, how many points on $X$ do we need in order to get a good approximation to the integral? More precisely, how many points on $X$ do we need in order for the optimum value of $h_{A\bar{B}}$ not to depend on the choice of points? Obviously, we will need at least $M_k$, the number of variables we're optimizing over (components of $h_{A\bar{B}}$), but we will probably need much more.

### 4.2 The measure

On the patch $O_{(c)}$ in $\mathbf{C}P^N$, define $Q_\alpha$ as the gradient of $P$: $Q_\alpha \equiv \partial P/\partial u^\alpha \equiv \hat{\partial}_\alpha P$, where the hat indicates that the derivative is taken in $\mathbf{C}P^N$ rather than on $X$, i.e. at fixed $u^\delta$ (if $\alpha \neq \delta$) rather than fixed $P$. The holomorphic $(n,0)$ form on $X$ is given in the coordinate system $O_{(c,\delta)}$ by

$$\Omega = Q_\delta^{-1} \prod_i du^i. \tag{4.1}$$

Therefore the volume form is

$$\mu = (-i)^n |Q_\delta|^{-2} \prod_i du^i \wedge \prod_j d\bar{u}^{\bar{\jmath}}. \tag{4.2}$$

This equals the coordinate volume form on $O_{(c)}$ times a delta function localized at $P = 0$:

$$\mu = (-i)^n \delta^2(P) \prod_\alpha du^\alpha \wedge \prod_\beta d\bar{u}^{\bar{\beta}}. \tag{4.3}$$

This expression shows that $\mu$ is independent of the choice of $\delta$ (the fact that it is also independent of $c$ is a short exercise).

Equation (4.3) can be used as the starting point for an algorithm that finds a set of points on $X$ randomly distributed according to the measure $\mu$. The idea is to spread the delta function out a bit: pick points randomly in $O_{(c)}$ according to the coordinate measure, keep those for which $|P|$ is less than a fixed small parameter $\epsilon$, and project each of those onto $X$. The projection can be done in various ways, of which the most natural is perhaps an orthogonal projection with respect to some metric on $\mathbf{C}P^N$, such as the Fubini-Study metric. The error due to spreading out the delta function is of order $\epsilon^2$. However, the smaller $\epsilon$ the slower the algorithm, because you need to try roughly $1/\epsilon^2$ random points in $O_{(c)}$ in order to find one that satisfies the criterion $|P| < \epsilon$. Note that this is an up-front cost—the points will be chosen at the outset and then stored as we calculate the energy functional at different values of $h_{A\bar{B}}$ in search of the minimum. Therefore we can afford a relatively slow algorithm.

Each patch $O_{(c)}$ covers all of $\mathbf{C}P^N$ except a set of measure zero (a copy of $\mathbf{C}P^{N-1}$). However, since its coordinate volume is infinite, we can't pick random values of the coordinates. Instead, we need to choose smaller patches $O'_{(c)} \subset O_{(c)}$ of finite coordinate volume, that overlap on sets of measure zero and together make up the whole $\mathbf{C}P^N$. The simplest choice, which also has the virtue of treating all the patches $O_{(c)}$ democratically, is:

$$O'_{(c)} \equiv \{u \in O_{(c)} : \forall \alpha,\, |u^\alpha| \leq 1\}. \tag{4.4}$$

Hence coordinate-wise each $O'_{(c)}$ is $D^N$, where $D$ is the unit disk. Note that with this choice (and only with this choice) the tube $|P| < \epsilon$ is continuous as it crosses different patches.

The algorithm is as follows. In each patch $O'_{(c)}$, do the following $N_{\text{trial}}$ times:

1. Pick a point $u_0$ at random with respect to the coordinate measure.
2. If $|P(u_0)| > \epsilon$, reject the point and go back to step 1.
3. For $u_0$, calculate the vector $w^\alpha = \bar{Q}_{\bar{\beta}} \delta^{\bar{\beta}\alpha} - \bar{Q}_{\bar{c}} u_0^\alpha$, where $Q_c \equiv (\partial P(z)/\partial z^c)_{z^a = u^a}$. This is (approximately) in the direction of the gradient of $P$, conjugated by the Fubini-Study metric, where we have arbitrarily chosen $G_{a\bar{b}} = \delta_{a\bar{b}}$.[2]
4. With $u = u_0 + mw$, solve $P = 0$ for the single complex variable $m$ using Newton-Raphson or whatever, starting at $m = 0$. (The solution may lie outside the coordinate patch $D^N$. It doesn't matter—keep the point anyway.)
5. Calculate and store the values of the following functions, which will be needed to calculate the value of the integrand: $Q_\alpha$, the monomials $p^A$, and their gradients $q^A_\alpha \equiv \hat{\partial}_\alpha p^A$.[3]

[2] We have $g^{\bar{\beta}\alpha} = e^{K_{\text{FS}}}(\delta^{\bar{\beta}\alpha} + \bar{u}^{\bar{\beta}} u^\alpha)$. The conjugated gradient is $\bar{Q}_{\bar{\beta}} \hat{g}^{\bar{\beta}\alpha} \propto Q_{\bar{\beta}} \delta^{\bar{\beta}\alpha} + \bar{Q}_{\bar{\beta}} \bar{u}^{\bar{\beta}} u^\alpha$. Since $P$ is homogeneous of degree $N+1$, we have $Q_\alpha u^\alpha = (N+1)P - Q_c$, and since we are near the $P = 0$ surface, we drop the first term.

[3] For the integrand of $H_2$, we will also need the matrices of second derivatives $Q_{\alpha\gamma} \equiv \hat{\partial}_\alpha \hat{\partial}_\gamma P$ and $q^A_{\alpha\gamma} \equiv \hat{\partial}_\alpha \hat{\partial}_\gamma p^A$.

It's important to use the same number of *trial* points—rather than try to keep the same number of retained points—in each patch. (Of course, if we're doing a symmetric Calabi-Yau, then all the patches $O_{(c)}$ are identical.)

### 4.3 Computation of $H_1$

We first deal with the calculation of $H_1$, then of $H_2$. It will be seen that $H_1$ is far simpler to calculate, so we should begin by tring to minimize that.

To begin we need the metric $g_{i\bar{\jmath}}$. The Kähler potential is the same as in $\mathbf{C}P^N$, so we have

$$g_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} K. \tag{4.5}$$

These partial derivatives, which are on $X$ of course, are related to those on $\mathbf{C}P^N$ by

$$\partial_i = \hat{\partial}_i - \frac{Q_i}{Q_\delta}\hat{\partial}_\delta. \tag{4.6}$$

With the help of a few matrix identities, we find the following expressions for the determinant and inverse of $g_{i\bar{\jmath}}$ in terms of the metric $\hat{g}_{\alpha\bar{\beta}} = \hat{\partial}_\alpha \hat{\partial}_{\bar{\beta}} K$ on $\mathbf{C}P^N$:

$$\det g_{i\bar{\jmath}} = \frac{|Q|^2}{|Q_\delta|^2} \det \hat{g}_{\alpha\bar{\beta}}, \tag{4.7}$$

where

$$|Q|^2 \equiv \hat{g}^{\bar{\beta}\alpha} Q_\alpha \bar{Q}_{\bar{\beta}}. \tag{4.8}$$

Using (4.2) we find

$$v = |Q|^2 \det \hat{g}_{\alpha\bar{\beta}}. \tag{4.9}$$

It's gratifying to see that this expression is manifestly independent of $\delta$ (the independence from $c$ is another short exercise).

It remains to calculate $\hat{g}_{\alpha\bar{\beta}}$. Recall that we have

$$K = \frac{1}{k}\ln\psi, \qquad \psi \equiv h_{A\bar{B}} p^A \bar{p}^{\bar{B}}\,. \tag{4.10}$$

Then we have

$$k\hat{g}_{\alpha\bar{\beta}} = \frac{1}{\psi}\psi_{\alpha\bar{\beta}} - \frac{1}{\psi^2}\psi_\alpha \bar{\psi}_{\bar{\beta}}, \tag{4.11}$$

where

$$\psi_\alpha \equiv \hat{\partial}_\alpha \psi = h_{A\bar{B}} q_\alpha^A \bar{p}^{\bar{B}} \qquad \psi_{\alpha\bar{\beta}} \equiv \hat{\partial}_\alpha \hat{\partial}_{\bar{\beta}} \psi = h_{A\bar{B}} q_\alpha^A \bar{q}_{\bar{\beta}}^{\bar{B}}. \tag{4.12}$$

There is a cute way to calculate $v$ without having to calculating $\hat{g}_{\alpha\bar{\beta}}$ as an intermediate step. We assume that $Q_\alpha$, $p^A$, and $q_\alpha^A$ have been previously calculated and stored. The idea is to work with $N+1$ dimensional vectors and matrices. So define $q_c^A \equiv p^A$ and $Q_c \equiv 0$. Then calculate the $(N+1)\times(N+1)$ matrix

$$\Psi_{a\bar{b}} \equiv q_a^A\, h_{A\bar{B}}\, \bar{q}_{\bar{b}}^{\bar{B}} \tag{4.13}$$

and its inverse $\Psi^{\bar{b}a}$. Then it's an easy matter to compute

$$v = k^{-N+1}\psi^{-N} \det \Psi \, \bar{Q}_{\bar{b}} \Psi^{\bar{b}a} Q_a. \tag{4.14}$$

(Recall $\psi = \Psi_{c\bar{c}}$.) The standard deviation of $v$ over the sample of points can be computed by standard methods. If the minimization algorithm requires gradient information, then we have

$$\delta v = v D^{\bar{B}A} \delta h_{A\bar{B}} \,, \tag{4.15}$$

where

$$D^{\bar{B}A} = -N\psi^{-1}\bar{p}^{\bar{B}}p^A + \bar{q}_{\bar{b}}^{\bar{B}} \Psi^{\bar{b}a} q_a^A - (\bar{Q}_{\bar{b}}\Psi^{\bar{b}a}Q_a)^{-1} \bar{q}_{\bar{b}}^{\bar{B}} \Psi^{\bar{b}a} Q_a \bar{Q}_{\bar{d}} \Psi^{\bar{d}e} q_e^A \,. \tag{4.16}$$

It's also worth writing down Donaldson's "balancing" map on the matrix $h$:

$$T : h \mapsto \left( N_k' \left\langle \frac{\bar{p}^{\bar{B}} p^A}{p^C h_{C\bar{D}} \bar{p}^{\bar{D}}} \right\rangle \right)^{-1} , \tag{4.17}$$

where the brackets mean the average with respect to the measure $\mu$ (or over the set of points chosen according to that measure). He proves that, starting from any $h$, this map converges to a fixed point, which yields the so-called "balanced" metric.

If we are working with the basis of symmetric polynomials $\mathcal{P}^l$ (3.10), then $h_{A\bar{B}}$ is replaced with a vector $h_l$, and (4.13), (4.15), and (4.16) become

$$\Psi_{a\bar{b}} = h_l \mathcal{Q}^l_{a\bar{b}} \,, \tag{4.18}$$

$$\delta v = v D^l \delta h_l \,, \qquad D^l = -N\psi^{-1}\mathcal{P}^l + \Psi^{\bar{b}a}\mathcal{Q}^l_{a\bar{b}} - (\bar{Q}_{\bar{b}}\Psi^{\bar{b}a}Q_a)^{-1} \bar{Q}_{\bar{d}} \Psi^{\bar{d}e} \mathcal{Q}^l_{e\bar{b}} \Psi^{\bar{b}a} Q_a \,, \tag{4.19}$$

where

$$q_c^I = \rho^I \,, \qquad q_\alpha^I = \hat{\partial}_\alpha \rho^I \,, \tag{4.20}$$

$$\mathcal{Q}^l_{a\bar{b}} = c^l_{I\bar{J}} \bar{q}_{\bar{b}}^{\bar{J}} q_a^I \,. \tag{4.21}$$

### 4.4 Computation of $H_2$

Of the two expressions for $H_2$ given in (2.3), the first, namely $\int_X \mu \, g^{\bar{j}i} \partial_i f \partial_{\bar{j}} f$, is more amenable to numerical analysis, for two reasons (1) it's slightly simpler, involving only first derivatives of $f$, and (2) the integrand is positive definite, so the integral should converge faster and be less prone to numerical errors. Here we compute that integrand $I \equiv g^{\bar{j}i} \partial_i f \partial_{\bar{j}} f$.

First, we calculate the inverse metric $g_{i\bar{j}}$, finding

$$g^{\bar{j}i} = \tilde{g}^{\bar{j}i}, \tag{4.22}$$

where we've defined

$$Q^{\bar{\beta}} \equiv \hat{g}^{\bar{\beta}\alpha} Q_\alpha, \qquad \tilde{g}^{\bar{\beta}\alpha} \equiv \hat{g}^{\bar{\beta}\alpha} - \frac{Q^{\bar{\beta}} \bar{Q}^\alpha}{|Q|^2}. \tag{4.23}$$

Using (4.6) and (4.22), we obtain a $\delta$-free expression for the integrand:

$$I = g^{\bar{\jmath}i}\partial_i f \partial_{\bar{\jmath}} f = \tilde{g}^{\bar{\beta}\alpha}\hat{\partial}_\alpha f \hat{\partial}_{\bar{\beta}} f. \tag{4.24}$$

In case it's useful for the future, we also note a similar expression for the Ricci scalar:

$$\frac{1}{2}R = -g^{\bar{\jmath}i}\partial_i\partial_{\bar{\jmath}} f = -\tilde{g}^{\bar{\beta}\alpha}\hat{\partial}_\alpha\hat{\partial}_{\bar{\beta}} f. \tag{4.25}$$

The gradient $\hat{\partial}_\alpha f$ is given by

$$\hat{\partial}_\alpha f = \hat{g}_{\alpha\gamma\bar{\beta}}\tilde{g}^{\bar{\beta}\gamma} + \frac{Q_{\alpha\gamma}\bar{Q}^\gamma}{|Q|^2}, \tag{4.26}$$

where

$$\hat{g}_{\alpha\gamma\bar{\beta}} \equiv \hat{\partial}_\alpha \hat{g}_{\gamma\bar{\beta}} = \hat{\partial}_\alpha\hat{\partial}_\gamma\hat{\partial}_{\bar{\beta}} K = \frac{1}{k}\left(\frac{1}{\psi}\psi_{\alpha\gamma\bar{\beta}} - \frac{1}{\psi^2}\left(\psi_\alpha\psi_{\gamma\bar{\beta}} + \psi_\gamma\psi_{\alpha\bar{\beta}} + \psi_{\bar{\beta}}\psi_{\alpha\gamma}\right) + \frac{2}{\psi^3}\psi_\alpha\psi_\gamma\psi_{\bar{\beta}}\right), \tag{4.27}$$

and

$$\psi_{\alpha\gamma} \equiv \hat{\partial}_\alpha\hat{\partial}_\gamma\psi = h_{A\bar{B}}q^A_{\alpha\gamma}\bar{p}^{\bar{B}} \qquad \psi_{\alpha\gamma\bar{\beta}} \equiv \hat{\partial}_\alpha\hat{\partial}_\gamma\hat{\partial}_{\bar{\beta}}\psi = h_{A\bar{B}}q^A_{\alpha\gamma}\bar{q}^{\bar{B}}_{\bar{\beta}}. \tag{4.28}$$

Let's sum up the algorithm for computing the integrand. We assume that $Q_\alpha$, $Q_{\alpha\gamma}$, $p^A$, $q^A_\alpha$, and $q^A_{\alpha\gamma}$ are known, having been computed and stored at the outset. We list the quantities to compute, in the order they should be computed, repeating all the necessary formulas for handy reference:

1.

$$\psi = h_{A\bar{B}}p^A\bar{p}^{\bar{B}} \tag{4.29}$$
$$\psi_\alpha = h_{A\bar{B}}q^A_\alpha\bar{p}^{\bar{B}} \tag{4.30}$$
$$\psi_{\alpha\gamma} = h_{A\bar{B}}q^A_{\alpha\gamma}\bar{p}^{\bar{B}} \tag{4.31}$$
$$\psi_{\alpha\bar{\beta}} = h_{A\bar{B}}q^A_\alpha\bar{q}^{\bar{B}}_{\bar{\beta}} \tag{4.32}$$
$$\psi_{\alpha\gamma\bar{\beta}} = h_{A\bar{B}}q^A_{\alpha\gamma}\bar{q}^{\bar{B}}_{\bar{\beta}} \tag{4.33}$$

2.

$$\hat{g}_{\alpha\bar{\beta}} = \frac{1}{k}\left(\frac{1}{\psi}\psi_{\alpha\bar{\beta}} - \frac{1}{\psi^2}\psi_\alpha\psi_{\bar{\beta}}\right) \tag{4.34}$$
$$\hat{g}_{\alpha\gamma\bar{\beta}} = \frac{1}{k}\left(\frac{1}{\psi}\psi_{\alpha\gamma\bar{\beta}} - \frac{1}{\psi^2}\left(\psi_\alpha\psi_{\gamma\bar{\beta}} + \psi_\gamma\psi_{\alpha\bar{\beta}} + \psi_{\bar{\beta}}\psi_{\alpha\gamma}\right) + \frac{2}{\psi^3}\psi_\alpha\psi_\gamma\psi_{\bar{\beta}}\right) \tag{4.35}$$

3.

$$\hat{g}^{\bar{\beta}\alpha} \qquad \text{(inverse of } \hat{g}_{\alpha\bar{\beta}}\text{)} \tag{4.36}$$

4.

$$Q^{\bar{\beta}} = \hat{g}^{\bar{\beta}\alpha}Q_\alpha \tag{4.37}$$

5.
$$|Q|^2 = Q^{\bar{\beta}} \bar{Q}_{\bar{\beta}} \tag{4.38}$$

6.
$$\tilde{g}^{\bar{\beta}\alpha} = \hat{g}^{\bar{\beta}\alpha} - \frac{Q^{\bar{\beta}} \bar{Q}^{\alpha}}{|Q|^2} \tag{4.39}$$

7.
$$\hat{\partial}_\alpha f = \hat{g}_{\alpha\gamma\bar{\beta}} \tilde{g}^{\bar{\beta}\gamma} + \frac{Q_{\alpha\gamma} \bar{Q}^{\gamma}}{|Q|^2} \tag{4.40}$$

8.
$$I = \tilde{g}^{\bar{\beta}\alpha} \hat{\partial}_\alpha f \hat{\partial}_{\bar{\beta}} f \tag{4.41}$$

Eight easy steps!

It's clear from these expressions that calculating the gradient of $H_2$ with respect to $h_{A\bar{B}}$ would be exceedingly complicated (and take significantly longer than $M_k$ evaluations of $H$). We would therefore need to use an optimization algorithm that does not depend on gradient information.

### 4.5 Normalized notation

As indicated in subsection 4.2, the integrations (but not the calculation of $v$) are performed with respect to a measure $\tilde{\mu}$ that approximates $\mu$. It is useful to normalize the various integrals, so we define

$$\langle \cdot \rangle = \frac{1}{V_{\tilde{\mu}}} \int_X \tilde{\mu} \cdot , \qquad \eta = \frac{v}{\langle v \rangle} , \qquad E[J] = \left\langle (\eta - 1)^2 \right\rangle \approx \frac{1}{V_{\tilde{\mu}} \langle v \rangle^2} H_1[J] . \tag{4.42}$$