Quantum Entanglement, Classical Gravity, and Convex Programming:
New Connections

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0 Introduction

This talk is about *holographic entanglement entropy*

Active area of research, ~ 10 years old

Has revealed unsuspected connections between quantum mechanics and gravity

Brings together several subjects from physics and geometry:

- General relativity
- Quantum information theory
- Quantum field theory
- Geometric measure theory
- Convex programming

Many of these connections are not yet fully explored or understood

I will explain the necessary background and try to give a flavor of the subject

Contents

1 Entanglement entropy

Associated to a given quantum system is a Hilbert space $\mathcal{H}$

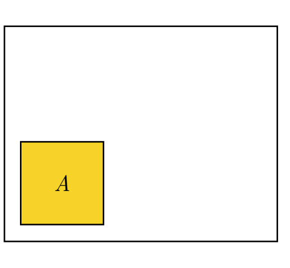
A *state* is described by an operator ρ obeying $\rho = \rho^\dagger$, $\operatorname{Tr}\rho = 1$, $\rho \geq 0$

If ρ is a projector, $\rho = |\psi\rangle\langle\psi|$, then the state is *pure* (definite); otherwise *mixed* (uncertain)

Mixedness can be quantified in many ways; one good way is *von Neumann entropy*:

$$S(\rho) = -\operatorname{Tr}\rho\log\rho$$

If we divide system into subsystems A, A^c, Hilbert space factorizes: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$



Given state ρ on full system, but observer who only has access to subsystem A (can only measure observables of the form $\mathcal{O}_A \otimes I_{A^c}$), *effective state* is

$$\rho_A := \operatorname{Tr}_{\mathcal{H}_{A^c}}\rho$$

Even when full system is in pure state, it may not be a product:

$$\rho = |\psi\rangle\langle\psi|\,, \qquad |\psi\rangle = \sum_i \lambda_i |i\rangle_A \otimes |i\rangle_{A^c} \qquad \text{(Schmidt decomposition)}$$

($|i\rangle_{A,A^c}$ are orthonormal states on $\mathcal{H}_{A,A^c}$) This is *entanglement*. As a result, ρ_A is mixed:

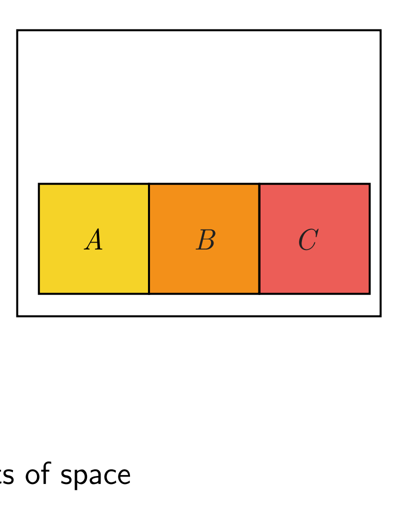
$$\rho_A = \sum_i |\lambda_i|^2 |i\rangle_A\langle i|_A$$

Quantify amount of entanglement by entropy $S_A := S(\rho_A) = -\sum_i |\lambda_i|^2 \log|\lambda_i|^2$

In general, ρ_A may be mixed due to (1) entanglement with A^c; (2) full system being mixed; (3) a combination

Nonetheless, $S_A := S(\rho_A)$ is generally called *entanglement entropy* (EE)
("subsystem entropy" would be a better name)

For multiple subsystems $A, B, \ldots$, we have EEs S_A, S_B, S_{AB}, etc.
These obey many important properties, including

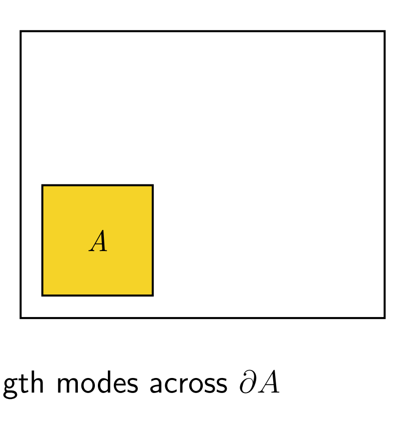


- Subadditivity: $S_{AB} \leq S_A + S_B$
- Araki-Lieb: $S_{AB} \geq |S_A - S_B|$
- Strong subadditivity (Lieb-Ruskai '73): $S_{AB} + S_{BC} \geq S_B + S_{ABC}$

2 EE in quantum field theories

The Hilbert space $\mathcal{H}$ of a QFT is (roughly) a tensor product over all points of space
(since local operators commute at spacelike separation)

For any choice of spatial region A, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$



S_A depends on:

- parameters of theory
- state
- region A

S_A is ultraviolet divergent
Leading divergent term $\propto \operatorname{area}(\partial A)$ due to entanglement of short-wavelength modes across ∂A

First calculation of an EE in QFT by Bombelli-Koul-Lee-Sorkin '86

Since 2003, has become established as powerful tool for studying QFTs and many-body condensed-matter systems

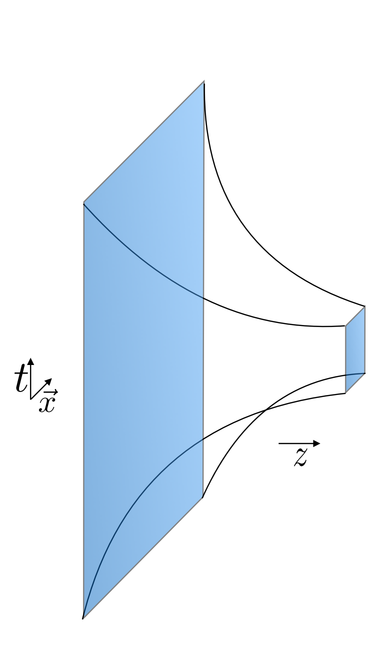
Useful for diagnosing and understanding:

- quantum criticality
- topological order
- renormalization-group flows
- many-body localization
- quantum quenches
- energy conditions
- much more...

In practice, EEs are usually very difficult to compute (even in free theories)

However, there exists a class of QFTs where S_A is given in terms of a classical geometry problem

3 Holographic QFTs



Certain QFTs with a large number of strongly-interacting fields admit a *dual description* in terms of *general relativity* with cosmological constant $\Lambda < 0$, coupled to certain matter fields, in 1 higher dimension

Different states of the QFT (pure or mixed) are represented by different solutions to the Einstein-matter equations

All of these solutions are asymptotically anti-de Sitter (AdS) spacetime:

$$ds^2 = \frac{R^2}{z^2}(-dt^2 + dz^2 + d\vec{x}^2)\,, \qquad z > 0$$

The (conformal) boundary at $z = 0$ is Minkowski space, parametrized by $t, \vec{x}$; this is where the QFT "lives"

These *holographic dualities* are extremely useful for studying many aspects of strongly-interacting QFTs

I will just discuss EEs

4 Holographic EEs

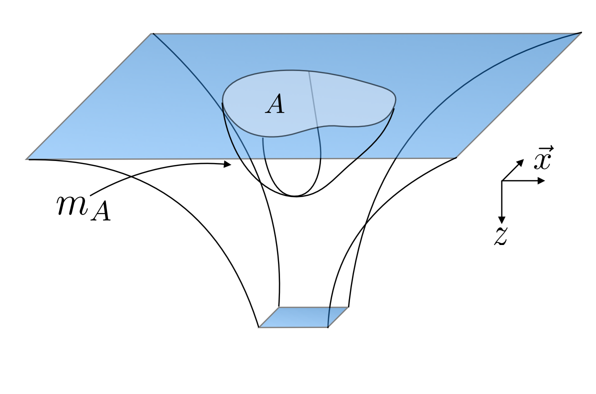
First, consider static (t-independent) solution

Fixed-t slice is an asymptotically hyperbolic Riemannian manifold

Let A be a region on its conformal boundary

Ryu-Takayanagi '06:

$$\begin{aligned} S_A &= \operatorname{area}(m_A) \\ m_A &= \text{minimal-area surface homologous to } A \end{aligned}$$



EE is given by solution to Plateaux problem!

Generalizes Bekenstein-Hawking black-hole entropy formula

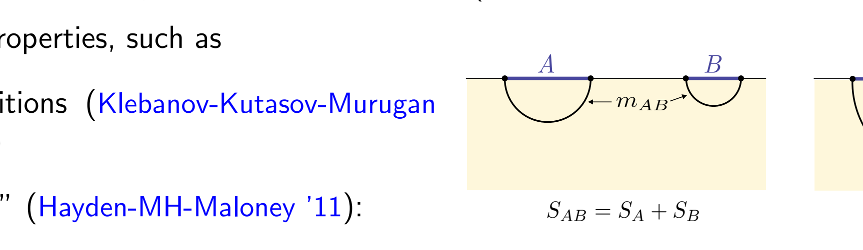
Area is divergent; leading term $\propto \operatorname{area}(\partial A)$

RT formula is very useful calculationally

Has led to many general insights into structure of entanglement in QFTs

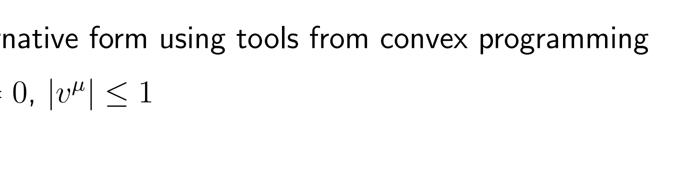
Also shows how quantum information theory is encoded in the geometry of spacetime

For example, RT leads to simple proof of strong subadditivity (MH-Takayanagi '07):

$S_{AB} + S_{BC} =$  $= S_B + S_{ABC}$

In fact, RT obeys all known general properties of EEs (Hayden-MH-Maloney '11; MH '13)

Also has special properties, such as



$S_{AB} = S_A + S_B$ $\qquad$ $S_{AB} < S_A + S_B$

- phase transitions (Klebanov-Kutasov-Murugan '07, MH '10)
- "monogamy" (Hayden-MH-Maloney '11):

$$S_{AB} + S_{BC} + S_{AC} \geq S_A + S_B + S_C + S_{ABC}$$

5 Max flow-min cut

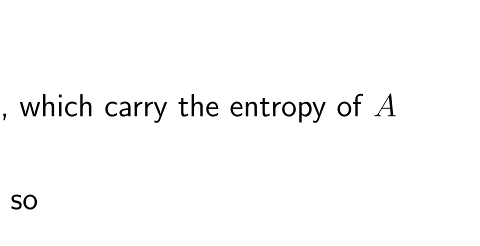
The Plateaux problem can be put into a useful alternative form using tools from convex programming

Define a *flow* as a vector field v^μ satisfying $\nabla_\mu v^\mu = 0$, $|v^\mu| \leq 1$

For any surface $m \sim A$

$$\int_A n_\mu v^\mu = \int_m n_\mu v^\mu \leq \operatorname{area}(m) \qquad (n_\mu = \text{unit normal})$$

Theorem (max flow-min cut): Maximizing on one side and minimizing on the other, this bound is saturated:



$$\max_v \int_A n_\mu v^\mu = \operatorname{area}(m_A)$$

(Originates in network theory; adapted to Riemannian setting by Federer '74, Strang '83, Nozawa '90)

Proven by turning min cut problem into convex program by (1) passing to real homology (convex relaxation), (2) applying strong duality principle of convex programming (introducing Lagrange multipliers to enforce constraints, then solving for original variables to obtain convex program for Lagrange multipliers)

We can thus write (Freedman-MH '16)

$$S_A = \max_v \int_A n_\mu v^\mu$$

This can be given a physical interpretation in terms of "bit threads", which carry the entropy of A

Can also give a "dual" proof of strong subadditivity:
Let $\hat{v}^\mu$ be a flow that maximizes flux through B and through ABC, so

$$S_B = \int_B n_\mu \hat{v}^\mu\,, \qquad S_{ABC} = \int_{ABC} n_\mu \hat{v}^\mu$$

Also have

$$S_{AB} = \max_v \int_{AB} n_\mu v^\mu \geq \int_{AB} n_\mu \hat{v}^\mu\,, \qquad S_{BC} = \max_v \int_{BC} n_\mu v^\mu \geq \int_{BC} n_\mu \hat{v}^\mu$$

Together,

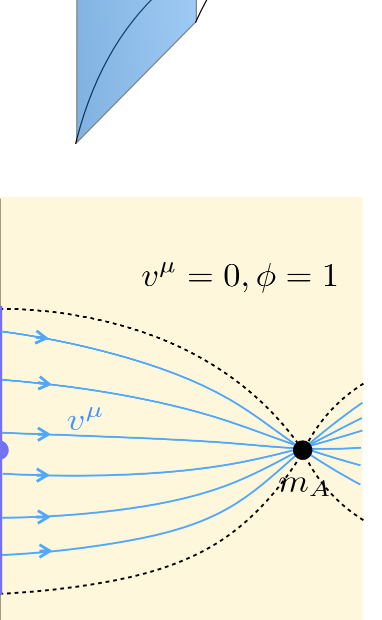
$$S_{AB} + S_{BC} \geq \int_{AB} n_\mu \hat{v}^\mu + \int_{BC} n_\mu \hat{v}^\mu = \int_B n_\mu \hat{v}^\mu + \int_{ABC} n_\mu \hat{v}^\mu = S_B + S_{ABC}$$

6 Time-dependent case

In the time-dependent case, we cannot restrict a priori to a time-slice

Have to deal with full Lorentzian spacetime, making life much more complicated

Instead of RT, have Hubeny-Rangamani-Takayanagi '07 formula:



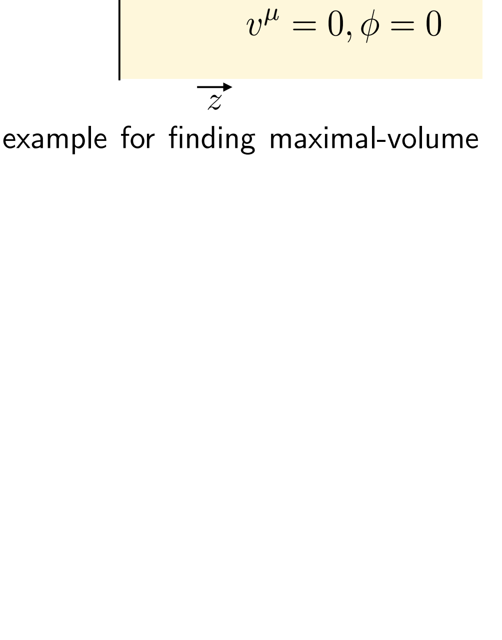
$$\begin{aligned} S_A &= \operatorname{area}(m_A) \\ m_A &= \text{minimal-area extremal spacelike codimension-2 surface} \sim A \end{aligned}$$

More difficult to prove theorems about than RT
Alternative "maximin" formulation by Wall '12 is useful:

$$S_A = \max_{\substack{\text{Cauchy}\\ \text{slice } C}} \min_{\substack{m \subset C\\ m \sim A}} \operatorname{area}(m)$$

We can now apply convex relaxation and duality to obtain a Lorentzian version of max flow-min cut (MH-Hubeny, to appear):

$$S_A = \max_v \int_{D(A)} n_\mu v^\mu$$



where $D(A) =$ causal domain of A, v^μ obeys

$$\nabla_\mu v^\mu = 0\,, \qquad \partial_\mu\phi \pm v^\mu \in J^+(0)\,, \qquad \phi \to \begin{cases} 0\,, & t \to -\infty \\ 1\,, & t \to +\infty \end{cases}$$

Result is that $v^\mu = 0$ in causal future and past of m_A

Flow-based proof of strong subadditivity goes through

Can dualize again to obtain minimizing convex program for S_A

One can prove other Lorentzian max flow-min cut theorems, for example for finding maximal-volume slices

7 Conclusion

There are deep and surprising connections among these subjects:

- Entanglement
- Quantum field theory
- General relativity
- Plateaux problem
- Convex programming

The physics underlying these connections remains somewhat mysterious

Hopefully, the math involved will help us to unravel it