Quantum Entanglement and the Geometry of Spacetime

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Black Hole Entropy

Bekenstein, Hawking ’74:

\[ S = \frac{k_B c^3 a}{4 G_N \hbar} = k_B \frac{a}{4 l_P^2} \]

\( G_N \rightarrow \) gravity

\( \hbar \rightarrow \) quantum mechanics

\( k_B \rightarrow \) statistical mechanics

area of event horizon

Planck length

\( \sim 10^{-33} \text{ cm} \)

Mysteries:

What are the “atoms” of the black hole?

Why is \( S \propto a \)?
General relativity:
- Gravity is a manifestation of the curvature of spacetime
- Geometry of spacetime (metric $g_{\mu\nu}$) is dynamical

Einstein equation: 
$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu} - \Lambda g_{\mu\nu}$$

We know of many quantum theories of gravity (from string theory, . . . ).
At long distances (compared to Planck length), they reduce to GR.
They have various
- numbers of dimensions
- types of matter fields
- values of $\Lambda$

Unfortunately, we don’t understand them well enough to directly answer the above questions.
Suppose we have a quantum theory of gravity in $d + 1$ dimensions ($d = 2, 3, \ldots$).

Let spacetime geometry fluctuate, fixing boundary conditions at infinity.

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Closed quantum system.

Simplest solution to Einstein equation is *anti-de Sitter (AdS) spacetime*.

No matter ($T_{\mu \nu} = 0$).

Space is hyperbolic (Lobachevsky) space.

Boundary is infinitely far away, with infinite potential wall.

Light can reach boundary (and reflect back) in finite time.

Holographic DUALITIES
Holographic Dualities

Maldacena ’97:
Quantum gravity in $d + 1$ dimensions with AdS boundary conditions
= $d$ dimensional ordinary quantum field theory (without gravity).
QFT “lives on the boundary”.
Map between the two theories is non-local.

QFT has a large number of strongly interacting fields:

$$N = \left(\frac{R}{l_P}\right)^{d-1} = \frac{R^{d-1}}{G_N \hbar} \gg 1$$
## Holographic Dualities

<table>
<thead>
<tr>
<th>Quantum gravity</th>
<th>Quantum field theory</th>
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<tbody>
<tr>
<td>( \frac{R^{d-1}}{G_N \hbar} )</td>
<td>( N )</td>
</tr>
<tr>
<td>( \hbar \to 0 )</td>
<td>( N \to \infty )</td>
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<tr>
<td>classical limit</td>
<td>thermodynamic limit</td>
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<tr>
<td>general relativity</td>
<td>macroscopic (collective) description</td>
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<tr>
<td>empty AdS</td>
<td>vacuum</td>
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<tr>
<td>[ S = 0 ]</td>
<td></td>
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<tr>
<td>thermal gas of particles in AdS</td>
<td>[ S = \mathcal{O}(1) ]</td>
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<td>confined thermal state</td>
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<tr>
<td>black hole</td>
<td>[ S = \frac{a}{4G_N \hbar} = \mathcal{O}(N) ]</td>
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<tr>
<td>deconfined plasma</td>
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</tbody>
</table>
This helps to understand black hole entropy.

But mysteries remain.

Nothing special happens at a black hole horizon.

What about other surfaces? Can their areas represent entropies?

Are there entropies that are intrinsic to a system --- not thermal?
**Entanglement Entropy**

Classical mechanics:
- definite state $\rightarrow$ certain outcome for any measurement

Quantum mechanics:
- definite state $\rightarrow$ uncertain outcomes for some measurements

Example: $|\uparrow\rangle$
- measurement of $S_z$ definitely gives $+\frac{1}{2}\hbar$
- measurement of $S_x$ gives $+\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$ with equal probability

When only certain kinds of measurements are allowed, a definite (pure) state will **effectively** be indefinite (mixed).

Suppose a system has two parts, but we can only measure one.

Spin singlet state: $|AB\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right)$ $S_{AB} = 0$

To see that this is a pure state (superposition, not mixture, of $|\uparrow\rangle|\downarrow\rangle$ and $|\downarrow\rangle|\uparrow\rangle$) requires access to both $A$ and $B$.

For an observer who only sees $A$, effective state is mixed:

$$\rho_A = \frac{1}{2} \left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right)$$ $S_A = k_B \ln 2$

In classical mechanics, if whole is in a definite state then each part is also: $S_A \leq S_{AB}$
In quantum field theories, spatial regions are highly entangled.

New way of thinking about QFTs.
Entanglement entropy $S_A$ is a function of the state and the region $A$.

Reveals a lot about the theory:
- quantum criticality
- topological order
- renormalization-group monotones

Unfortunately, difficult to compute, even in simple theories.
Standard method is *replica trick*:
1. using path integral, compute *Rényi entropies* 
   \[ S_A(n) = \frac{1}{1-n} \ln \text{tr} \rho_A^n \quad (n = 2, 3, \ldots) \]
2. extrapolate to $n = 1$ 
   \[ \lim_{n \to 1} S_A(n) = -\text{tr}(\rho_A \ln \rho_A) = S_A^{\Delta S_n(r)} \]

MH, Lawrence, Roberts ’12:
Showed that entanglement entropy is invariant under bosonization in 1+1 dimensions.

Agón, MH, Jafferis, Kasko ’13:
Calculated $S_A$ for disk of radius $r$ in 2+1 dimensional electromagnetism.
Black hole = thermal state

Maldacena ’01:  
2 black holes joined by Einstein-Rosen bridge  
= 2 entangled QFTs 

\[ S_A = \frac{a}{4G_N \hbar} \]

Ryu, Takayanagi ’06 proposed that, in general, 

\[ S_A = \frac{a}{4G_N \hbar} \text{ area of minimal surface between } A \text{ and } B \]

“Entanglement is the fabric of spacetime”

Simple & beautiful . . . widely applied . . . but is it right?
Quantum information theory is built into spacetime geometry.
Holographic entanglement also has a special property: “monogamy of mutual information”.

**Holographic Entanglement Entropy**

MH, Takayanagi ’07; Hayden, MH, Maloney ’11; MH ’13:
Holographic formula obeys all general properties of entanglement entropies.

Examples:

If full system is pure then $S_A = S_B$

Otherwise,

$S_A \neq S_B$

Strong subadditivity:

$$S_{AB} + S_{BC} \geq S_{ABC} + S_B$$

Examples:

$\geq$

$=$

$\geq$

$\geq$

$=$

$= S_{ABC} + S_B$
MH ’10:
- Explained how to apply replica trick to holographic theories.
- Debunked previous “derivation” of holographic formula.
- Showed that holographic formula predicts phase transition for separated regions.
- Confirmed using Euclidean quantum gravity & orbifold CFT techniques.

Lewkowycz, Maldacena ’13: General “derivation” of holographic formula.

\[
S_{AB} = S_A + S_B
\]

correlations mediated by \(O(1)\) confined degrees of freedom

\[
S_{AB} < S_A + S_B
\]

correlations mediated by \(O(N)\) elementary degrees of freedom
What about *time*?

Original (Ryu-Takayanagi) holographic formula assumes state is *static*.

**Hubeny, Rangamani, Takayanagi ’07:** For non-static states, replace *minimal* surface in bulk space with *extremal* surface in bulk spacetime.

Simple & beautiful . . . widely applied . . . but is it right?

**Callan, He, MH ’12:** Obeys strong subadditivity in examples.

Extremal surface goes behind horizons!

**MH, Hubeny, Lawrence, Rangamani ’14:** Nonetheless obeys causality. Implies that QFT state is encoded by (part of) spacetime behind horizon.

**MH, Myers, Wien ’14:** Proved that area of a *general* surface (not just extremal) is given by *differential entropy* in QFT.
Entanglement entropy in holographic theories:
• Enormous progress in recent years.
• Still many mysteries.
• Suggests a deep and general connection between entanglement and the geometry of spacetime.

(How) does spacetime itself emerge from quantum mechanics?

Stay tuned . . .