A New Perspective on Holographic Entanglement

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1 How should one think about the minimal surface?

In semiclassical gravity, surface areas are related to entropies

Bekenstein-Hawking ['74]:

For black hole,

\[ S = \frac{1}{4G_N} \text{area(horizon)} \]

Why?

Possible answer:

Microstate bits “live” on horizon, 1 bit / 4 Planck areas

Ryu-Takayanagi ['06]: For region in holographic field theory (classical Einstein gravity, static state)

\[ S(A) = \frac{1}{4G_N} \text{area}(m(A)) \]

\[ m(A) = \text{bulk minimal surface homologous to } A \]

Do microstate bits of \( A \) “live” on \( m(A) \)?

Unlike horizon, \( m(A) \) is not a special place; by choosing \( A \), we can put \( m(A) \) almost anywhere
Puzzles:

- Under continuous changes in boundary region, minimal surface can jump

  Example: Union of separated regions $A, B$

\[
m(AB) = m(A) \cup m(B)
\]

- Information-theoretic quantities are given by differences of areas of surfaces passing through different parts of bulk:

  Conditional entropy: 
  \[
  H(A|B) = S(AB) - S(B)
  \]

  Mutual information: 
  \[
  I(A : B) = S(A) + S(B) - S(AB)
  \]

  Conditional mutual information: 
  \[
  I(A : B|C) = S(AB) + S(BC) - S(ABC) - S(C)
  \]
\[ H(A|B) = S(AB) - S(B) \quad I(A : B) = S(A) + S(B) - S(AB) \]

Information-theoretic meaning (heuristically):

Classical:
\[ H(A|B) = \# \text{ of (independent) bits belonging purely to } A \]
\[ I(A : B) = \# \text{ shared with } B \]

Quantum: Entangled (Bell) pair contributes 2 to \( I(A : B) \), -1 to \( H(A|B) \); can lead to \( H(A|B) < 0 \)

\[ -\frac{1}{\sqrt{2}} (|1\rangle + |0\rangle) \]

\[ I(A : B|C) = S(AB) + S(BC) - S(ABC) - S(C) = \text{correlation between } A \text{ & } B, \text{ conditioned on } C \]

What do differences between areas of surfaces, passing through different parts of bulk, have to do with these measures of information?
• RT obeys strong subadditivity [Headrick-Takayanagi ’07]

\[ I(A : BC) \geq I(A : C) \]

What does proof (by cutting & gluing minimal surfaces) have to do with information-theoretic meaning of SSA (monotonicity of correlations)?

To try to answer these questions, I will present a new formulation of RT

• Does not refer to minimal surfaces (demoted to a calculational device)

• Suggests a new way to think about the holographic principle & about the connection between spacetime geometry and information
2 Reformulation of RT

Consider a Riemannian manifold with boundary

Flow: vector field $v$ obeying $\nabla \cdot v = 0$, $|v| \leq 1$

Think of flow as a set of oriented threads (flow lines) beginning & ending on boundary, transverse density $= |v| \leq 1$

Let $A$ be a subset of boundary
Max flow-min cut theorem (originally on graphs; Riemannian version: [Federer '74, Strang '83, Nozawa '90]):

$$\max_v \int_A v = \min_{m \sim A} \text{area}(m)$$

Note:

- Max flow is highly non-unique (except on $m(A)$, where $v = $ unit normal)
Let $v(A)$ denote any max flow

- Finding max flow is a convex optimization problem
RT version 2.0:

\[
S(A) = \max_v \int_A v \quad (4G_N = 1)
\]

\[
= \max \text{ # of threads beginning on } A
\]

Threads can end on \(A^c\) or horizon

Each thread has cross section of 4 Planck areas & is identified with 1 (independent) bit of \(A\)

Automatically incorporates homology & global minimization conditions of RT

Threads are “floppy”: lots of freedom to move them around in bulk & move where they attach to \(A\)

Also lots of room near boundary to add extra threads that begin & end on \(A\) (don’t contribute to \(S(A)\))

Role of minimal surface: bottleneck, where threads are maximally packed, hence counted by area

Holographic principle: entropy \(\propto\) area because bits are carried by one-dimensional objects

Bekenstein-Hawking:
3 Threads & information

Now we address conceptual puzzles with RT raised before

First: even when $m(A)$ jumps, $v(A)$ changes continuously with $A$

Next, consider two regions $A$, $B$

We can maximize flux through $A$ or $B$, not in general both
But we can always maximize through $A$ and $AB$ (nesting property)
Call such a flow $v(A, B)$

Example 1: $S(A) = S(B) = 2$, $S(AB) = 3 \Rightarrow I(A : B) = 1$, $H(A|B) = 1$

Lesson 1:

• Threads that are stuck on $A$ represent bits unique to $A$

• Threads that can be moved between $A$ & $B$ represent correlated pairs of bits
Example 2: \( S(A) = S(B) = 2, \ S(AB) = 1 \Rightarrow I(A : B) = 3, \ H(A|B) = -1 \Rightarrow \text{entanglement!} \)

One thread leaving \( A \) must go to \( B \), and vice versa

Lesson 2:

- Threads that connect \( A \) & \( B \) (switching orientation) represent entangled pairs of bits

Apply lessons to single region:

- freedom to move beginning points around reflects correlations within \( A \)
- freedom to add threads that begin & end on \( A \) reflects entanglement within \( A \)
In equations:

**Conditional entropy:**

\[ H(A|B) = S(AB) - S(B) \]
\[ = \int_{AB} v(AB) - \int_B v(B) \]
\[ = \int_{AB} v(B, A) - \int_B v(B, A) \]
\[ = \int_A v(B, A) \]
\[ = \text{min flux on } A \text{ (maximizing on } AB) \]

**Mutual information:**

\[ I(A : B) = S(A) - H(A|B) \]
\[ = \int_A v(A, B) - \int_A v(B, A) \]
\[ = \text{max} - \text{min flux on } A \text{ (maximizing on } AB) \]
\[ = \text{flux movable between } A \text{ and } B \text{ (maximizing on } AB) \]

Max flow can be defined even when flux is infinite: flow that cannot be augmented

Regulator-free definition of mutual information:

\[ I(A : B) = \int_A (v(A, B) - v(B, A)) \]
Conditional mutual information:

\[
I(A : B|C) = H(A|C) - H(A|BC)
\]

\[
= \int_A v(C, A, B) - \int_A v(C, B, A)
\]

\[
= \text{max} - \text{min} \text{ flux on } A \text{ (maximizing on } C \& ABC)
\]

\[
= \text{flux movable between } A \& B \text{ (maximizing on } C \& ABC)
\]

\[
= (\text{flux movable between } A \& BC) - (\text{movable between } A \& C)
\]

\[
= I(A : BC) - I(A : C)
\]

Strong subadditivity \((I(A : B|C) \geq 0)\) is clear

In each case, clear connection to information-theoretic meaning of quantity/property

Open problem: Use flows to prove “monogamy of mutual information” property of holographic EEs

[Hayden-Headrick-Maloney '12]:

\[
I(A : BC) \geq I(A : B) + I(A : C)
\]

(and generalizations to more parties [Bao et al '15])

Flow-based proof may illuminate the information-theoretic meaning of these inequalities
4 Extensions

4.1 Emergent geometry

Metric \(\nabla \cdot \mathbf{v} = 0, |\mathbf{v}| \leq 1\)

Set of thread configurations \(\rightarrow\) metric:

Given set \(\{w\}\) of closed \((d - 1)\)-forms, find \(g_{\mu\nu}\) with pointwise smallest \(\text{det}\) such that \(|w| \leq 1 \ \forall w\)

4.2 Quantum corrections

Faulkner-Lewkowycz-Maldacena [13]:

Quantum (order \(G_{N}^0\)) correction to RT is from entanglement of bulk fields

May be reproduced by allowing threads to jump from one point to another (or tunnel through microscopic wormholes, à la ER = EPR [Maldacena-Susskind ’13])
4.3 Covariant bit threads

With Veronika Hubeny (to appear)

Hubeny-Rangamani-Takayanagi ['07] covariant entanglement entropy formula:

\[ S(A) = \text{area}(\text{min}(A)) \]

\( m(A) = \) minimal extremal surface homologous to \( A \)

Need generalization of max flow-min cut theorem to Lorentzian setting

Define a flow as a vector field \( v \) (in full Lorentzian spacetime) obeying

- \( \nabla \cdot v = 0 \)
- no flux into or out of singularities
- integrated norm bound: \( \forall \) timelike curve \( C \),

\[ \int_C ds |v_\perp| \leq 1 \quad (v_\perp = \text{projection of } v \text{ orthogonal to } C) \]

Any observer counts, over her lifetime, total of at most 1 thread / 4 Planck areas

(Can be expressed as local constraint)
Theorem (assuming NEC, using results of Wall ['12] & Headrick-Hubeny-Lawrence-Rangamani ['14]):

\[
\max_v \int_{D(A)} v = \text{area}(m(A)) \quad D(A) = \text{boundary causal domain of } A
\]

Linearizes problem of finding extremal surface area

HRT version 2.0:

\[
S(A) = \max_v \int_{D(A)} v
\]

To maximize flux, threads seek out \(m(A)\), automatically confining themselves to entanglement wedge.

Threads can lie on common Cauchy slice (equivalent to Wall’s ['12] maximin by standard max flow-min cut) or spread out in time.