

# Inflation Determination under a Taylor Rule: Consequences of Endogenous Capital Accumulation

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## Abstract

Dupor's paper "Investment and Interest Rate Policy" [Journal of Economic Theory, 2001] concludes that adding investment to a continuous-time sticky price model reverses standard results on the stabilization property of interest rate policy rules: with endogenous capital, passive monetary policy ensures determinacy while active policy leads to indeterminacy. In this paper I analyze the determinacy of equilibrium in a discrete-time version of Dupor's model. The determinacy results obtained contrast with those of Dupor, even when the length of the discrete period is made arbitrarily short. I find that the continuous-time limit proposed by Dupor does not correctly approximate the behavior of a discrete-time model with arbitrarily short periods. An important lesson is that before using the continuous-time approach, one needs to verify that the continuous-time limit is well-behaved. This warning may be of importance for macroeconomic modeling in other settings as well.

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# 1 Introduction

Most existing research on interest rate policy thus far has been in optimization-based models without endogenous capital accumulation. One well established result is that active interest rate feedback rules of the form advocated by Taylor imply equilibrium determinacy: Stability of the economy can be achieved by monetary authority raising interest rate instrument more than one-for-one with increases in inflation. On the other hand, passive policies lead to indeterminacy. Thus a central policy recommendation is that the central bank should conduct an active monetary policy. Dupor (2001) makes one of the first attempts to incorporate endogenous investment into interest rate policy analysis. The author proposes a continuous time perfect foresight model and concludes that by simply appending endogenous capital to a benchmark monopolistic competition sticky price model, previous results on determinacy are reserved: Only passive monetary policy ensures determinacy while active Taylor rule leads to indeterminacy<sup>1</sup>.

A natural question to ask is how reliable are Dupor's results? Does this mean Taylor's principle, proposed as a characterization of recent U.S. monetary policy and supported both empirically and theoretically by previous studies, is actually incorrect? Realizing the importance of investment, both as a significant and a highly volatile component of GDP, it is crucial to understand whether the introduction of investment to a model has indeed a major reverse impact on stabilization property of monetary policies. Such an investigation also has its practical importance for central bank policies.

To answer this question, this paper presents a rational expectations discrete time version of Dupor's model and I find that when period length is reasonably long, introducing endogenous investment does not reverse previous findings: passive monetary policy leads to indeterminacy while active policy results in determinacy. On the other hand, when period length is made small, to achieve determinacy much more aggressive policy is needed than in previous models without investment. In the extreme case, when length of period goes to zero, equilibrium is always indeterminate for the given policy. In other words, in the limit, a stabilizing policy should be such that the interest rate responds at an infinite speed to tiny changes in inflation. This is not a completely useless result for it originates from some underlying assumptions of

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<sup>1</sup>Other models with investment have been studied, see Casares and McCallum (2000), Chari, Kehoe and McGrattan (2000), Hairault and Portier (1993), Kimball(1995), King and Watson (1996), King and Wolman (1996) and Yun (1996). All of these papers deal with intertemporal general-equilibrium sticky price models with investment dynamics. But they do not analyze the determinacy question and none of them assume that monetary policy is described by an interest rate rule. Woodford (2003) also studies endogenous capital but it differs from literature mentioned above in that it introduces the adjustment costs at the level of the individual firm rather than for the aggregate capital stock with free mobility of capital among firms.

the model that become extreme as period is made short. But still, it disagrees with Dupor's conclusion that passive policy achieves determinacy.

It is puzzling that these conflicting results arise from two extremely similar models except for the choice of a discrete or continuous approach. To understand the problem, further effort is made to identify the sources underlying Dupor's results. Instead of just making period length short, I explicitly derive the continuous time limit by making a specific transformation. There are important findings. First, the resulting continuous time limit is identical to Dupor's model. Second, such a transformation I have used in getting the continuous time limit is not valid because I show that it fails an important condition which guarantees that, after the transformation, the continuous time limit still well-mimics the dynamic properties of the discrete time model with arbitrarily small period length. Therefore, the sort of continuous time limit proposed by Dupor should not be used as the limiting approximation to the behavior of the discrete time model and is not appropriate for policy evaluation. One message from this study is: macroeconomists must be careful when using continuous time limit to approximate the behavior of a model with discrete time periods of arbitrarily small length. Under some circumstances, a continuous time approximation may be misleading.

The rest of the paper is organized as follows. In section 2, Dupor's continuous perfect foresight model is briefly summarized. Then in section 3, a discrete rational expectations version of Dupor's model is developed, dynamic properties of equilibria under Taylor type interest rate rules are studied. Section 4 studies how the discrete time model and its determinacy issue depend on the length of period. Section 5 is devoted to discussion of the continuous time limit of the discrete model and it is compared with Dupor's model. Section 6 concludes.

## 2 The Continuous-time Model with Investment

I begin by reviewing and summarizing Dupor's work in terms of model setup, methodology and major findings<sup>2</sup>. This allows me to derive the discrete version of this model accordingly in the next section and it also serves as the reference model for the subsequent comparison and discussion in section 4.

The economy is assumed to be characterized by a large number of household-firm units, each of which makes all of household and firm decisions: as a household, it consumes all of the differentiated products; as an owner of inputs, it supplies labor and capital; as a firm, it is specialized in the production of one good. In a symmetric equilibrium, given the same

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<sup>2</sup>Note in order to facilitate model comparison in section 4, some notations in Dupor's paper are changed to match the discrete model developed latter.

initial conditions, each agent faces identical decision problem and chooses identical functions for consumption, capital, asset holdings and prices. The preference of such an infinitely lived representative household is represented by

$$\int_0^{\infty} e^{-\theta t} \left[ \log C + \log m - H - \frac{\gamma}{2} \left( \frac{\dot{p}}{p} - \pi^* \right)^2 \right] dt \quad (1)$$

where  $c, m, H$  denote consumption, real money balances and labor supply.  $p$  is the price that the household-firm unit charges for its product. Flow utility is additively separable in  $c$  and  $m$ , disutility in labor is linear in  $H$  and price stickiness is introduced by the quadratic adjustment cost specification with  $\pi^*$  being the steady state inflation rate.

The production sector is that of standard monopolistic competition. An individual firm's instantaneous production function is given by a constant return to scale technology

$$y = k^\alpha h^{1-\alpha} \quad (2)$$

where  $k$  and  $h$  are its demands for capital and labor. In equilibrium, output produced equals the demand derived using Dixit-Stiglitz preferences over differentiated goods

$$y = Y d\left(\frac{p}{P}\right) \quad (3)$$

where  $d(\cdot)$  has the standard properties.

Household's holding of capital  $K$ , follows the law of motion

$$\dot{K} = I - \delta K \quad (4)$$

where  $I$  denotes the flow investment and  $\delta$  is the flow depreciation rate of capital.

Let  $b$  be the value of real government bond, the only non-capital wealth of the household in the perfect foresight model. The instant budget constraint of the household-firm unit expressed in real terms is then obtained by combining household and firm budget constraints into one single equation

$$\dot{b} = (i - \pi)b - im + \frac{p}{P}y - \omega h + \omega H - \rho k + \rho K - I - C - \tau \quad (5)$$

where  $i$  is the nominal interest rate,  $\pi$  denotes the rate of inflation.  $\omega$  and  $\rho$  are real labor wage and real rental price for capital respectively, which are the same faced by all firms<sup>3</sup>. The first

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<sup>3</sup>Homogeneous factor markets are implicitly assumed for both labor and capital since all producers face the same wage rate and capital rental price. Thus the household supplies of both labor and capital services can be employed in production of any good. Actually the assumption greatly reduces the degree of strategic complementarity between the pricing decisions of different producers. This in turn undercuts the central mechanism that results in sluggish adjustment of the overall price level. In this sense, the assumption of segmented factor markets is more appealing. For detailed discussion on this issue, see Woodford (2003). But the simplified assumption is still useful in an early-stage investigation of investment in monetary policy analysis and I will keep it latter in the discrete model as well.

term on the right hand side is the real interest earnings on government bond. It is obtained by using Fisher parity  $i = r + \pi$  where  $r$  is the real interest rate.

To complete the model, the following simple interest rate feedback rule is considered

$$i = \psi(\pi) \quad (6)$$

where  $\psi(\cdot)$  is an increasing function.

The household-firm's problem is then to choose sequences for  $c, I, b, m, H, h, K, k, p$  and  $y$  to maximize (1) subject to (2) - (5).

Imposing first-order necessary conditions along with feedback rule (6), market clearing condition, equilibrium symmetry  $P = p, Y = y, H = h, K = k$  and the assumption of Ricardian fiscal policy, the following system of structural equations of the economy is obtained

$$\begin{aligned} \text{consumption Euler:} \quad \dot{c} &= c(\psi(\pi) - \theta - \pi) \\ \text{no arbitrage:} \quad \rho &= \psi(\pi) - \pi + \delta \\ \text{capital/labor relation:} \quad \beta\rho k &= \alpha HC \\ \text{aggregate supply:} \quad \dot{\pi} &= \theta(\pi - \pi^*) - g(K, C, Y) \\ \text{capital accumulation:} \quad \dot{K} &= I - \delta K \\ \text{market clearing:} \quad Y &= I + C \end{aligned} \quad (7)$$

Note that because capital is also a variable input in addition to labor, there are two important features in this model which are absent when capital is treated fixed: First, a static no arbitrage condition between holding capital and riskless bond must be satisfied in equilibrium. Second, the aggregate supply curve relates inflation to not only output but also capital and consumption.

Next step consists of manipulating the above equations to eliminate  $\rho, Y$  and  $I$  which yields a system of three differential equations in  $\pi, C$  and  $K$

$$\dot{\pi} = F(\pi, C, K); \quad \dot{C} = G(\pi, C, K); \quad \dot{K} = sH(\pi, C, K)$$

Linearizing the three equations around the unique steady state gives

$$\begin{bmatrix} \dot{\pi} \\ \dot{C} \\ \dot{K} \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & 0 \\ g_1 & 0 & 0 \\ h_1 & h_2 & h_3 \end{bmatrix} \begin{bmatrix} \hat{\pi} \\ \hat{C} \\ \hat{K} \end{bmatrix} \quad (8)$$

where  $\hat{x} = x - x^*$ , and  $f_i, g_i, h_i$  are derivatives evaluated at steady state values  $(\pi^*, C^*, K^*)$ . Examining (8), the  $\dot{\pi}$  and  $\dot{C}$  equations are independent of  $K$ . Thus the dynamics of  $(\pi, C)$  are completely characterized by

$$\hat{A} = \begin{bmatrix} f_1 & f_2 \\ g_1 & 0 \end{bmatrix}$$

Since capital stock is the only pre-determined variable in the system and one eigenvalue  $h_3$  is strictly positive, determinacy requires one and only one eigenvalue of  $\hat{A}$  be negative. Otherwise indeterminacy is implied. Stability property of interest rate rules are then characterized by Dupor in the following Theorems which completely reverse previous results without endogenous capital:

**Theorem 1:** *If interest rate policy is passive, there exists a unique perfect foresight equilibrium in which  $(\pi, C, K)$  converge asymptotically to the steady state  $(\pi^*, C^*, K^*)$ .*

**Theorem 2:** *Under an active monetary policy:*

1. *If  $f_1 > 0$  and  $K_0 \neq K^*$ , no PFE exist in which  $(\pi, C, K)$  converge asymptotically to the steady state.*
2. *If  $f_1 < 0$ , a continuum of PFE exist in which  $(\pi, C, K)$  converge asymptotically to the steady state.*

### 3 A Discrete-time Model with Investment

One major advantage of using continuous time model is that it often delivers neat analytical results, while dealing with its discrete variant may sometimes involve cumbersome calculation. However the kind of limiting approximation provided by continuous time model may have its own limitations. In this section I build up a discrete time version of Dupor's model. As the first step of my investigation, the role of period length is not considered in this section. My main results are consistent with previous findings from models without endogenous capital, that is, passive Taylor rule can not lead to determinacy.

#### 3.1 Why Discrete-time model?

I take most of the assumptions used in Dupor, for example, additively separable utility function, single homogeneous market for either labor or capital services, constant return to scale technology, monopolistic competition among individual producers, contemporaneous Taylor rule, etc. The objective is to see with minimum modifications—simply by moving from continuous time to discrete time—whether there are differences in terms of model prediction on determinacy.

There are reasons to be suspicious of the continuous time approach. First, in continuous time, the assumption of a purely contemporaneous Taylor rule is much less appealing than in discrete time. Looking at contemporaneous rule in discrete time is one way to get some idea about why it has bad property in continuous time. Second, in continuous time (and assuming

a purely contemporaneous Taylor rule) there is no distinction between a rule that responds to inflation which has just occurred and a rule which responds to expected inflation from now on. While in discrete time (without endogenous capital), it is known that one gets qualitatively very different results for these two classes of rules, no matter how short the periods are. All these suggest it might be dangerous to use continuous time approach under some circumstances.

In addition to the discrete approach instead of the continuous approach, there are other two departures from Dupor's original model. But as I argue below they would not have impact on dynamic property of the model I consider here. First, the price dynamics are derived explicitly from producers' optimization with staggered price-setting, a discrete variant of a model proposed first by Calvo (1983). Such a model results in price dynamics that are qualitatively identical to those implied by a model with a quadratic cost of price adjustment as is assumed in Dupor<sup>4</sup>. This modification does not matter for determinacy. But it provides better micro-foundation for firm behavior. Thus in my model presented below, households and firms are modeled separately: households make decisions on consumption and investment while firms are responsible for production including individual price setting<sup>5</sup>. Second, my rational expectations model takes the assumption of complete financial markets for the nominal risky assets. This is mainly for convenience and it would allow me to compare my results not only directly to those of Dupor but also to those of some earlier interest rate policy analysis using discrete time sticky price models but without endogenous capital<sup>6</sup>.

In the following subsection, I first look at the households' optimization problem. The resulting first order necessary conditions together with the interest rate rule describe the aggregate demand side of the model. Then I turn to the producers' optimization and derive the corresponding aggregate supply curve. Dynamics of the whole economy are thus fully characterized by combining equilibrium conditions from both sides. In the second subsection, log-linearized equilibrium conditions enable me to study determinacy property of interest rate rules. Results are summarized.

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<sup>4</sup>For a detailed discussion on this issue, see Rotemberg (1987).

<sup>5</sup>The choice of who make investment decisions, households or firms, when they are modeled separately is arbitrary. In his original paper, although Dupor treats the household as the single decision maker who makes all decisions, in fact production and investment are two activities that can be completely decoupled in his model. In other words, they could be carried out inside the same firms, or by different firms, or by households as opposed to firms. Thus it does not matter whether households or firms choose investment.

<sup>6</sup>For theoretical work that do not model investment, see Benhabib, Schmitt-Grohe and Uribe (2001a and 2001b) for continuous time models and Rotemberg (1982), Woodford (2003) for discrete time models. For empirical papers that do not model investment, see Rotemberg and Woodford (1998, 1999).

### 3.2 The Model and Equilibrium Conditions

Assuming identical initial capital and non-capital wealth together with complete financial markets and homogeneous factor prices, all households face the same problem and make the same consumption, investment and factor supply decisions. Thus the household sector of the economy can be characterized by a representative household. As both a consumer and an owner of factor inputs, it seeks to maximize<sup>7</sup>

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t (\log C_t + \log m_t - H_t) \right\} \quad (9)$$

where  $\beta$  is a discount factor,  $m_t$  is the real money balances held at the end of period  $t$ ,  $H_t$  is the quantity of labor supply that can be used to produce any differentiated good and  $C_t$  is an Dixit-Stiglitz consumption index which is defined as

$$C_t \equiv \left[ \int_0^1 c_t(z)^{\frac{\theta-1}{\theta}} dz \right]^{\frac{\theta}{\theta-1}} \quad (10)$$

with  $\theta > 1$  and that the corresponding price index is

$$P_t \equiv \left[ \int_0^1 p_t(z)^{1-\theta} dz \right]^{\frac{1}{1-\theta}} \quad (11)$$

The household's period budget constraint takes the form

$$P_t C_t + m_t P_t \frac{i_t}{1+i_t} + E_t(Q_{t,t+1} W_{t+1}) \leq W_t + (\omega_t H_t + \rho_t K_t + profit_t - I_t P_t - T_t) \quad (12)$$

where  $P_t C_t$  is nominal consumption expenditure,  $W_t$  denotes beginning-of-period nominal financial wealth (including both money and all other assets held at the end of previous period),  $T_t$  represents nominal tax collection by the government,  $i_t$  is the riskless one-period nominal interest rate,  $m_t P_t i_t / (1 + i_t)$  is then the opportunity cost of holding wealth in liquid form. the term on the right hand side within the bracket is the household's after tax net income,  $Q_{t,t+1}$  is the stochastic nominal pricing kernel which is uniquely defined because of the existence of complete markets. Thus  $E_t(Q_{t,t+1} W_{t+1})$  represents the expected present value of tomorrow's financial wealth conditional on the state of the world at date  $t$ . Slightly different from Dupor's real flow constraint (5), the constraint (12) is expressed in nominal terms; future wealth is

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<sup>7</sup>One can also put a vector of exogenous disturbances in the household utility. But it will not affect the determinacy calculations of the sort I am interested in. In the log-linear approximation, such a disturbance term is simply an exogenous additive term that does not affect the coefficients on the endogenous variables and so does not affect determinacy.

stochastic and  $profit_t$ <sup>8</sup> is taken as given for the household because this is separately determined in the producers' optimization problem.

Finally, capital stock evolves according to

$$K_{t+1} = (1 - \delta)K_t + I_t$$

where  $K_t$  represents the beginning-of-period capital stock at date  $t$  while  $I_t$  is the amount of investment during date  $t$ .

The household's optimization problem is then to choose processes  $C_t, m_t, H_t, K_{t+1}, I_t, W_t$  for all dates  $t \geq 0$  to maximize (9) subject to (12) (with equality held in equilibrium) and capital evolution. Also, no ponzi-game condition must be imposed as well. This is essentially a standard concave optimization problem subject to sequences of constraints. The resulting first order conditions along with market clearing are as follows<sup>9</sup>.

$$\frac{C_t}{m_t} = \frac{i_t}{1 + i_t} \quad (13)$$

$$\frac{1}{1 + i_t} = \beta E_t \left[ \frac{1}{\Pi_{t+1}} \frac{C_t}{C_{t+1}} \right] \quad (14)$$

$$C_t = \tilde{\omega}_t \quad (15)$$

$$1 = \beta E_t \left[ \frac{C_t}{C_{t+1}} (\tilde{\rho}_{t+1} + 1 - \delta) \right] \quad (16)$$

$$K_{t+1} = (1 - \delta)K_t + I_t \quad (17)$$

$$C_t + I_t = Y_t \quad (18)$$

where  $\tilde{\omega}, \tilde{\rho}$  refer to real wage rate and real capital rental price, respectively. The intratemporal relation represented in equation (13) is the LM equation of this system<sup>10</sup>. Equation (14) is the consumption Euler. It is a Fisher-type relation and it is obtained by taking conditional expectation of the first order condition on nominal pricing kernel

$$Q_{t,t+1} = \beta \frac{C_t}{C_{t+1}} \frac{P_t}{P_{t+1}} \quad (19)$$

Equation (15) emerges from household's optimal labor supply decision. And (16) is an intertemporal no-arbitrage condition between real capital rent and real pricing kernel where the

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<sup>8</sup>For the typical household  $profit_t = \int_0^1 profit_t(z) dz$ . That is it is the sum over its profits from all differentiated goods.

<sup>9</sup>I study Ricardian monetary-fiscal policies in the model so that the flow budget constraint (12) can be ignored in the subsequent analysis.

<sup>10</sup>As noted by Dupor, this LM equation can be safely ignored in the following analysis since in this model, utility is additively separable in real money balances. In the presence of an interest rate rule which is the focus of the exercise, this equilibrium condition plays no role in determining inflation, output, capital and interest rate

intertemporal marginal rate of substitution of the household plays the role of the real interest discount factor. Market clearing identity is given by equation (18).

Following Dupor, I study the simplest possible interest rate feedback which takes the form

$$i_t = \phi(\Pi_t) \quad (20)$$

Thus the aggregate demand (AD) block of the model consists of equations (14), (15), (16), (17),(18) and monetary policy rule (20).

As is standard, latter study on dynamics relies on local approximations to the model's structural equations. Around the steady state<sup>11</sup>, log-linearization of the AD block gives

$$\hat{i}_t = E_t \pi_{t+1} + E_t \hat{C}_{t+1} - \hat{C}_t \quad (21)$$

$$E_t \hat{C}_{t+1} - \hat{C}_t = [1 - \beta(1 - \delta)] E_t \hat{\rho}_{t+1} \quad (22)$$

$$\hat{K}_{t+1} = (1 - \delta) \hat{K}_t + \delta \hat{I}_t \quad (23)$$

$$\hat{Y}_t = s_C \hat{C}_t + s_I \hat{I}_t \quad (24)$$

$$\hat{i}_t = \phi_\pi \pi_t \quad (25)$$

where in (24),  $s_C$  and  $s_I$  are shares of  $C$  and  $I$  in aggregate output evaluated at steady state values, and in (25),  $\phi_\pi$  represents the elasticity of  $\phi(\cdot)$  with respect to  $\pi$  evaluated at the steady state.

I now turn to my optimizing model of staggered price-setting and aggregate supply<sup>12</sup>. The period profit of an individual producer of good  $z$  is defined by

$$profit_t(z) = p_t(z)y_t(z) - [\omega_t h_t(z) + \rho_t k_t(z)] \quad (26)$$

with the overall demand of this good satisfying<sup>13</sup>

$$y_t(z) = Y_t \left( \frac{p_t(z)}{P_t} \right)^{-\theta} \quad (27)$$

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<sup>11</sup>The unique steady state of this model is solved in Appendix A.1.

<sup>12</sup>This part is largely based on the model in Woodford (2003) in which labor is the only variable input and labor market is segmented. Here I extend that model by introducing capital into production, but also simplify the analysis by assuming homogeneous factor markets.

<sup>13</sup>This individual demand curve in the presence of investment is derived as follows: Given the  $C$  index defined in (10), the optimal allocation of the household's consumption over the differentiated goods is given by  $c_t(z) = C_t \left( \frac{p_t(z)}{P_t} \right)^{-\theta}$ . On the other hand,  $I_t$  that determines the evolution of the capital stock in (17) is also a Dixit-Stiglitz aggregate of the expenditures  $i_t(z)$  on individual goods, with the same  $\theta$  as for consumption goods. In this case, the optimal choice of  $i_t(z)/I_t$  depends on the relative price in exactly the same way as in the case of consumption demand. This then leads to the demand curve (27).

The production technology is of constant return to scale

$$y_t(z) = k_t(z)^\alpha h_t(z)^{1-\alpha} \quad (28)$$

Assuming monopolistic competition, an individual producer regards itself as unable to affect aggregate variables  $Y_t$  and  $P_t$  and it takes input prices as given as well.

In equilibrium, individual firms' factor demands  $h(z)$  and  $k(z)$  are related to the household supplies of labor and capital by

$$H_t = \int_0^1 h_t(z) dz; \quad K_t = \int_0^1 k_t(z) dz$$

Staggered price-setting is modeled following Calvo (1983). There are two types of producers. Type I producers (with a fraction of  $(1 - \gamma)$ ) are assumed to be able to change prices at  $t$  and new prices are in effect immediately. In a symmetric equilibrium in which each of type I producers faces the same decision problem, they choose the same price, that is,  $p_t(z) = p_t^*$ . Type II producers (with a remaining proportion of  $\gamma$ ) are constrained to charge the aggregate price of last time period  $P_{t-1}$ . Since firms are identical within types, the aggregate price index in (11) becomes

$$P_t = \left[ (1 - \gamma) p_t^{*1-\theta} + \gamma P_{t-1}^{1-\theta} \right]^{\frac{1}{1-\theta}} \quad (29)$$

It follows that to determine the evolution of this index, in addition to its initial value, one only needs to know the evolution of  $p^*$  each period, that is, price-setting of type I producers.

A representative type I firm's optimization problem is to set its new price  $p(z)$  each period along with its choice of factor demands  $h(z)$  and  $k(z)$  to maximize the discounted sum of its nominal profits

$$E_t \sum_{T=t}^{\infty} \gamma^{T-t} Q_{t,T} [profit_T(z)]$$

where profit in each period is defined by (26), (27) and (28) jointly. The factor  $\gamma$  can be understood as the probability that the price set beginning in period  $t$  is still in effect in period  $T \geq t$ .  $Q_{t,T}$  is the stochastic nominal pricing kernel between date  $t$  and  $T$  as in households' problem.

It is convenient in this case to decompose it into a two-step optimization problem in which  $h(z)$ ,  $k(z)$  and  $p(z)$  are chosen sequentially. In the first step, the producer chooses optimal factor demands for a given level of output by solving a static cost-minimization problem ; then in the second step, the producer determines its optimal price using the results from the first step.

The first step cost minimization subject to technology (28) yields

$$\frac{k_t(z)}{h_t(z)} = \frac{\alpha}{1 - \alpha} \frac{\omega_t}{\rho_t} \quad (30)$$

or, expressed in  $y_t(z)$  and  $k_t(z)$ , the above can be equivalently written as

$$\frac{k_t(z)}{y_t(z)} = \left[ \frac{\alpha}{1-\alpha} \frac{\omega_t}{\rho_t} \right]^{1-\alpha} \quad (31)$$

Also, real marginal cost function is given by

$$s_t(z) = \left[ \frac{\tilde{\omega}_t}{1-\alpha} \right]^{1-\alpha} \left[ \frac{\tilde{\rho}_t}{\alpha} \right]^\alpha \quad (32)$$

where  $\tilde{\omega}$  and  $\tilde{\rho}$  are real wage and real capital rent.

Note that because of homogeneous factor markets, producers face the same factor prices regardless of their types<sup>14</sup>, which in turn implies

$$\frac{k_t(z)}{y_t(z)} = \frac{K_t}{Y_t}$$

for each good  $z$  in equilibrium. For the same reason the real marginal cost is identical among all producers and it can be further written as a function of aggregate variables  $K, Y$  and  $C$

$$s_t(z) = \left[ \frac{\tilde{\omega}_t}{1-\alpha} \right]^{1-\alpha} \left[ \frac{\tilde{\rho}_t}{\alpha} \right]^\alpha = \frac{1}{1-\alpha} \tilde{\omega}_t \left[ \frac{1-\alpha}{\alpha} \frac{\tilde{\rho}_t}{\tilde{\omega}_t} \right]^\alpha = \frac{1}{1-\alpha} C_t \left[ \frac{Y_t}{K_t} \right]^{\frac{\alpha}{1-\alpha}} \quad (33)$$

The last equality follows from substitution of household first order condition (15) for  $\tilde{\omega}$  and cost minimization result (31) for  $\tilde{\rho}/\tilde{\omega}$ .

In the second step, with results just derived, an individual type I producer seeks to maximize

$$E_t \sum_{T=t}^{\infty} \gamma^{T-t} Q_{t,T} [ p_t(z) y_T(z) - (P_T s_T) y_T(z) ] \quad (34)$$

with  $y$  satisfying the demand function (27). Alternatively, the problem can be transformed into an unconstrained optimization by substituting the demand curve into the objective<sup>15</sup>

$$E_t \sum_{T=t}^{\infty} \gamma^{T-t} Q_{t,T} Y_T P_T^\theta \left[ p_t(z)^{1-\theta} - (P_T s_T) p_t(z)^{-\theta} \right] \quad (35)$$

Thus the unique optimal price of Type I producers in equilibrium must satisfy the first order condition

$$E_t \sum_{T=t}^{\infty} \gamma^{T-t} Q_{t,T} Y_T P_T^\theta \left[ \frac{p_t^*}{P_T} - \mu s_T \right] = 0 \quad (36)$$

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<sup>14</sup>Note that for type II firms the restriction imposed on their ability to change their product prices does not prevent them from setting their factor demands in the optimal ratio each period.

<sup>15</sup>(34) is a well-behaved objective function. To see this, note the expression inside the bracket in (34) is strictly concave in  $p_t(z)^{-\theta}$  which in turn is a monotonic function of  $p_t(z)$ . Thus the whole expression has a unique maximum achieved at the price satisfying the first order condition.

where  $\mu = \frac{\theta}{\theta-1} > 1$  is the producer's steady state markup. Furthermore, using household first order condition (19), the nominal pricing kernel  $Q_{t,T}$  in the above can be replaced to obtain a first order condition solely in  $(Y_T, C_T, K_T)$  and price ratio.

$$E_t \sum_{T=t}^{\infty} (\gamma\beta)^{T-t} u_c(C_T) Y_T P_T^\theta \left[ \frac{P_t^*}{P_T} - \mu s_T \right] = 0 \quad (37)$$

The “aggregate supply block” of the model then consists of the above condition for type I price-setting, the expression of the common real marginal cost(33) and the evolution of price index (29).

In order to reduce the three equations to a single aggregate supply relation, one can log-linearize these conditions around a steady state in which  $Y_t = \bar{Y}$ ,  $P_t = P_{t-1}$ ,  $p_t^* = P_t$ ,

$$E_t \sum_{T=t}^{\infty} (\gamma\beta)^{T-t} \left[ \hat{p}_t^* - \hat{s}_T - \sum_{s=t+1}^T \pi_s \right] = 0 \quad (38)$$

$$\hat{s}_t = \hat{C}_t + \frac{\alpha}{1-\alpha} \left[ \hat{Y}_t - \hat{K}_t \right] \quad (39)$$

$$\pi_t = \frac{1-\gamma}{\gamma} \hat{p}_t^* \quad (40)$$

where  $\hat{p}_t^* = \log \frac{P_t^*}{P_t}$  and the other variables measure the percentage deviations from their steady state values. Note expression (38) is equivalent to

$$\left[ \frac{1}{1-\gamma\beta} \right] \hat{p}_t^* = E_t \sum_{T=t}^{\infty} (\gamma\beta)^{T-t} \hat{s}_T + \left[ \frac{1}{1-\gamma\beta} \right] E_t \sum_{T=t}^{\infty} (\gamma\beta)^{T-t} \pi_T$$

Multiplying both sides by  $(1-\gamma\beta)$ , I quasi-difference the above to yield

$$\hat{p}_t^* = \gamma\beta E_t \pi_{t+1} + (1-\gamma\beta) \hat{s}_t + \gamma\beta E_t \hat{p}_{t+1}^*$$

Then using (40) to substitute out  $\hat{p}_t^*$  and  $\hat{p}_{t+1}^*$ , I get a difference equation for inflation

$$\pi_t = \beta E_t \pi_{t+1} + \frac{(1-\gamma)(1-\gamma\beta)}{\gamma} \hat{s}_t \quad (41)$$

The above equation has the same form as the New Keynesian Phillips curve derived from Calvo-pricing models without endogenous capital<sup>16</sup>. The only difference as a result of the introduction of capital is that the real marginal cost is now not a function solely of the level of output. Instead, it depends upon  $Y$  as well as  $K$  and  $C$ , just as in Dupor's continuous time model. Using (39) to substitute out  $\hat{s}_t$  I obtain a single AS curve

$$\pi_t = \beta E_t \pi_{t+1} + \xi_c \hat{C}_t + \xi_y \hat{Y}_t - \xi_k \hat{K}_t \quad (42)$$

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<sup>16</sup>See Woodford (2003) for details.

where  $\xi_c = \frac{(1-\gamma)(1-\gamma\beta)}{\gamma} > 0$ ;  $\xi_y = \xi_c \frac{\alpha}{1-\alpha}$ . This is a direct discrete time analog of the AS curve of (7) in Dupor's model and it can be shown these two AS equations have identical implications on price dynamics in the presence of investment.

Finally, combining the equilibrium conditions from both households and firms optimization problems, dynamics of the economy are then fully characterized by the set of eight structural equations (14), (15), (16), (17), (18), (20), (31) and (42) which contains eight unknowns, namely,  $i, \tilde{\omega}, \tilde{\rho}, \pi, C, Y, K, I$ .

To make the system tractable, I substitute out  $(i, \tilde{\omega}, \tilde{\rho}, Y, I)$  and reduce it to the following set of three difference equations. This forms the basis of subsequent analysis on local dynamics.

$$\begin{aligned} E_t \pi_{t+2} &= f_1 E_t \pi_{t+1} - (\phi_\pi f_2) \pi_t - f_3 \hat{C}_t \\ E_t \hat{C}_{t+1} &= \hat{C}_t + \phi_\pi \pi_t - E_t \pi_{t+1} \\ \hat{K}_{t+1} &= -h_1 E_t \pi_{t+1} + h_2 \pi_t - h_3 \hat{C}_t + h_4 \hat{K}_t \end{aligned} \quad (43)$$

where

$$\begin{aligned} f_1 &= \frac{\xi_y \gamma_c + \xi_c \gamma_y + \gamma_y}{\beta \gamma_y} > 1 \\ f_2 &= \frac{\xi_y \gamma_c + \xi_c \gamma_y}{\beta \gamma_y} > 0 \\ f_3 &= \frac{\xi_y \gamma_c + \xi_c \gamma_y - \xi_y}{\beta \gamma_y} > 0 \\ h_1 &= \frac{\beta \delta}{s_I \xi_y} > 0 \\ h_2 &= \frac{\delta}{s_I \xi_y} > 0 \\ h_3 &= \frac{\delta}{s_I} \left( \frac{\xi_c}{\xi_y} + s_c \right) > 0 \\ h_4 &= 1 - \delta + \frac{\delta}{s_I} > 1 \\ \gamma_c &= \beta(1 - \delta) \\ \gamma_y &= \frac{1 - \beta(1 - \delta)}{1 - \alpha} \end{aligned}$$

Note that  $f_1 > f_2 > f_3 > 0$ . There are a couple of points worth noting. First the evolution path of inflation does not depend upon  $K$  even when capital is endogenous. This can be traced back to the assumption of constant return to scale production. One can show that with a more general form of production function  $y = k^\alpha h^\beta$  with  $\alpha + \beta$  not equal to 1 inflation function becomes dependent upon capital. Second, inflation equation is a second order difference equation which implies the expected change in inflation also plays a role in the evolution of inflation. A detailed derivation of the above three equations is contained in Appendix A.2.

### 3.3 Determinacy analysis

I first write the three equations (43) in the first order vector form  $E_t X_{t+1} = AX_t$  by adjoining the identity  $E_t \pi_{t+1} = E_t \pi_{t+1}$  as an auxiliary equation. The elements of vector  $X_t$  are  $E_t \pi_{t+1}$ ,  $\pi_t$ ,  $\hat{C}_t$ ,  $\hat{K}_t$ .

$$\begin{bmatrix} E_t \pi_{t+2} \\ E_t \pi_{t+1} \\ E_t \hat{C}_{t+1} \\ \hat{K}_{t+1} \end{bmatrix} = \begin{bmatrix} f_1 & -\phi_\pi f_2 & -f_3 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & \phi_\pi & 1 & 0 \\ -h_1 & h_2 & -h_3 & h_4 \end{bmatrix} \begin{bmatrix} E_t \pi_{t+1} \\ \pi_t \\ \hat{C}_t \\ \hat{K}_t \end{bmatrix} \quad (44)$$

Denote the square matrix on the right hand side by  $A$ . In the above system, note that in vector  $X_t$  only element  $K$  is a pre-determined state variable. Rational expectations equilibrium (REE) is determinate if and only if  $A$  has exactly one eigenvalue inside unit circle and three outside. Since dynamics of  $\pi$  and  $C$  are independent of capital stock and one eigenvalue of matrix  $A$  is given by  $h_4 > 1$ , stability of inflation and consumption is completely determined by the following  $3 \times 3$  submatrix

$$\hat{A} = \begin{bmatrix} f_1 & -\phi_\pi f_2 & -f_3 \\ 1 & 0 & 0 \\ -1 & \phi_\pi & 1 \end{bmatrix}$$

Determinacy then requires one and only one eigenvalue of  $\hat{A}$  is less than one. I wish to examine whether this is true and in particular, how it depends upon parameter  $\phi_\pi$ . It turns out that my findings do not support those of Dupor.

**Proposition 1:**

1. Under a passive interest rate rule ( $\phi_\pi < 1$ ) REE is locally indeterminate.
2. Under an active interest rate rule ( $\phi_\pi > 1$ ), REE may be locally determinate or indeterminate.

*Proof of part(1):* See Appendix A.3.

Next consider part(2). Although part (1) of my results has shown that active policy is necessary for a REE to be determinate, this condition alone does not guarantee determinacy. I can show that not all values of  $\phi_\pi > 1$  implies determinacy. Consider a specific parameterization  $(\alpha, \beta, \gamma, \delta) = (0.85, 0.90, 0.85, 0.00)$  with active policy  $\phi_\pi = 1.5$

$$\hat{A} = \begin{bmatrix} 1.510 & -0.597 & -0.007 \\ 1 & 0 & 0 \\ -1 & 1.5 & 1 \end{bmatrix}$$

It has eigenvalues  $(0.787 + 0.092i, 0.787 - 0.092i, 0.936)$ , all of which are within unit circle.

However, in the above experiment the parameter values are empirically irrelevant. It can be shown that for conventional values of model parameters, one always have determinacy for  $\phi_\pi > 1$ . To do so, 4 parameter values need to be specified: capital share  $\alpha$ , time discount  $\beta$ , capital depreciation rate  $\delta$  and fraction of firms unable to reset their prices in any period  $\gamma$ . Note that all of them except for the first one depend upon the length of period. I calibrate them on quarterly basis. Following RBC literature, I assume  $(\alpha, \beta, \gamma, \delta) = (1/3, 0.99, 2/3, 0.025)$ . With these parameter values, the proof of determinacy under an active rule is straightforward, see Appendix A.3 for details.

## 4 The Discrete-time Model Revisited: Period Length Matters

Now that results of the discrete time model stand in sharp contrast to those of Dupor, one question to ask is which features of my model and his model help explain the difference. A natural starting point would be to study how the discrete time model depends on period length. In this section, I revisit determinacy issue with a focus on the role of period length. I find that my model is very well-behaved for longer periods but poorly-behaved for very short periods.

How does the system depends upon the period length? First of all, the coefficients matrix in (44) can be written in period length  $\Delta$ . To see this, note all  $f_i$  and  $h_i$  are functions of  $\alpha, \beta, \gamma$  and  $\delta$ , where  $\alpha$  is the capital share,  $\beta$  is the period time discount,  $\gamma$  is the fraction of firms unable to change their prices each period and  $\delta$  is the period capital depreciation rate. Except for  $\alpha$ , these parameters are obviously related to  $\Delta$ ,

$$\beta = e^{-\theta\Delta}; \quad \gamma = e^{-\mu\Delta}; \quad \delta = \tilde{\delta}\Delta,$$

where  $\theta, \mu$  and  $\tilde{\delta}$  are constants that don't depend on period length. Thus, each element in the coefficient matrix can be expressed as a function of  $\Delta$ . Secondly, some endogenous variables like inflation and investment also depends upon  $\Delta$ .

$$\pi_t^d = \pi^c \Delta, \quad I_t = \tilde{I}_t \Delta$$

where  $\pi_t^d = \log(P_t/P_{t-1})$  is the inflation defined in discrete time,  $\pi^c = \dot{P}/P = \frac{d \log P}{dt}$  is the rate of inflation defined in the continuous time model and  $\tilde{I}_t$  is the investment flow in continuous time. Therefore, an alternative representation of (44) in variables  $(\hat{\pi}_t^c, \pi_t^c, \hat{C}_t, \hat{K}_t)$  is as follows:

$$\begin{bmatrix} E_t \hat{\pi}_{t+1}^c \\ E_t \pi_{t+1}^c \\ E_t \hat{C}_{t+1} \\ \hat{K}_{t+1} \end{bmatrix} = \begin{bmatrix} f_1 - 1 & \frac{f_1 - 1 - \phi_\pi f_2}{\Delta} & -\frac{f_3}{\Delta^2} & 0 \\ \Delta & 1 & 0 & 0 \\ -\Delta^2 & (\phi_\pi - 1)\Delta & 1 & 0 \\ -h_1 \Delta^2 & (h_2 - h_1)\Delta & -h_3 & h_4 \end{bmatrix} \begin{bmatrix} E_t \hat{\pi}_t^c \\ \pi_t^c \\ \hat{C}_t \\ \hat{K}_t \end{bmatrix} \quad (45)$$

Derivation of this representation is included in Appendix B.1. This is an equivalent representation of the original model (44). To see this, denote the above coefficient matrix by  $M$ . Due to the linearity of the transformation, matrix  $M$  and matrix  $A$  in (44) have identical eigenvalues and hence identical stability property for any given  $\Delta$ . On the other hand, rewriting the system this way helps me explicitly study the role of  $\Delta$ , which was impossible in the previous section when period length is not taken into account. Determinacy analysis on (45) gives the following result.

**Proposition 2:** *All results in Proposition 1 remain. Moreover, as  $\Delta$  goes to 0, no finite value of  $\phi_\pi$  leads to determinate equilibrium.*

*Proof:* see Appendix B.2.

The above results are illustrated in Figure 1 in appendix<sup>17</sup>. It is seen that the discrete time model is very well-behaved as long as  $\Delta$  is not too small. For example, when  $\Delta$  is 0.25, i.e., on quarterly basis, any value of  $\phi_\pi$  a bit above 1 leads to determinacy. This is a useful result, and not what Dupor's conclusions would suggest. But the performance of the model deteriorates as  $\Delta$  is made small. In the extreme case, as  $\Delta$  approaches 0, no finite values of  $\phi_\pi$  can induce a determinate equilibrium under the given policy. Further discussion on why the model does not behave well when periods are short is provided in the final section.

## 5 Limit of the Discrete-time Model and Model Comparisons

In the above section, I have shown that the discrete time model indeed does not have a determinate equilibrium as  $\Delta$  is made arbitrarily small. Nevertheless, there is still one point on which my model disagree with Dupor's conclusions. In his paper, the equilibrium is determinate when  $\phi_\pi$  is less than 1, which is not what I conclude about the discrete time model. Thus one still wants to challenge the correctness of his conclusions about Taylor rules with  $\phi_\pi < 1$ . Instead of just making periods small, I approach the problem by looking at its continuous time limit and see how it differs from Dupor's model. To facilitate the analysis, I introduce  $\widetilde{M} = (M - I)/\Delta$ ,

$$\widetilde{M} = \begin{bmatrix} \frac{f_1-2}{\Delta} & \frac{f_1-1-\phi_\pi f_2}{\Delta^2} & -\frac{f_3}{\Delta^3} & 0 \\ 1 & 0 & 0 & 0 \\ -\Delta & \phi_\pi - 1 & 0 & 0 \\ -h_1\Delta & h_2 - h_1 & -\frac{h_3}{\Delta} & \frac{h_4-1}{\Delta} \end{bmatrix}$$

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<sup>17</sup>Figure 1 is produced under conventional values of parameters. The specific parameterization is as follows:  $\theta = -4\log 0.99$ ,  $\mu = -4\log 2/3$ ,  $\tilde{\delta} = 4 \cdot 0.025$ . These values are chosen to match the conventional values on quarterly basis mentioned in previous section, namely  $(\alpha, \beta, \gamma, \delta) = (1/3, 0.99, 2/3, 0.025)$ .

and correspondingly the system can be represented by

$$E_t \dot{Z}_{t+1} = \widetilde{M} Z_t \quad (46)$$

This is just a pure change of variable, so for any given value of  $\Delta$ , eigenvalues of  $M$  and  $\widetilde{M}$  are related by

$$\lambda_M = 1 + \Delta \cdot \lambda_{\widetilde{M}} \quad (47)$$

where  $\lambda_M$  and  $\widetilde{M}$  are corresponding eigenvalues of the two matrices.

In the rest of the section, I first analyze some property of the four eigenvalues of  $\widetilde{M}$ . Then I introduce the procedure to derive the continuous time limit. I also provide a necessary and sufficient condition under which the continuous time limit derived does not distort the determinacy property of the discrete time model when periods are arbitrarily small. In other words, if and only if this condition is satisfied, can continuous time limit be a correct approximation of the behavior of the model with discrete periods of arbitrarily short length. There are two important findings. First, this condition is not satisfied by the model. Second, the continuous time limit derived from my discrete time model is identical to Dupor's model. This gives the key to explain why the model proposed by Dupor is misleading.

**Proposition 3:** *Consider  $\widetilde{M}$ . As  $\Delta$  goes to 0, one root of  $\widetilde{M}$  goes to  $-\infty$  and the other three roots are finite.*

*Proof:* see Appendix B.3.

In what follows, I describe a procedure that transforms the 4 equation system (46) to a 3 equation continuous time limit. For such a transformation to work, ie, to produce a correct continuous time approximation in the sense that it does not distort the discrete time system's stability property as periods are made arbitrarily small, the following condition has to be satisfied:

$$\lim_{\Delta \rightarrow 0} |1 + \lambda_1 \cdot \Delta| > 1 \quad (48)$$

where  $\lambda_1$  is the root of  $\widetilde{M}$  that goes to  $-\infty$  as  $\Delta$  goes to 0. Essentially, condition (48) means that the corresponding eigenvalue of  $M$ , the coefficient matrix in (45) must lie outside the unit circle in the limit. It will be clear immediately why this condition is important when I illustrate the transformation procedure below.

**Observation:** *If  $\lambda_1$  is positive, condition (48) is always true; If  $\lambda_1$  is negative, (48) may fail.*

To get a three-equation continuous time limit, I begin with matrix  $\widetilde{M}$ . Solving the corresponding left eigenvector associated with  $\lambda_1$  yields

$$e' = \left[ 1, \frac{m_2}{\lambda_1} + \frac{m_3}{\lambda_1^2}(\phi_\pi - 1), \frac{m_3}{\lambda_1}, 0 \right]$$

where  $m_2 = \frac{f_1 - 1 - \phi_\pi f_2}{\Delta^2}$  and  $m_3 = -\frac{f_3}{\Delta^3}$ . If and only if condition (48) is satisfied, in any bounded equilibrium it requires the following restriction holds at all time<sup>18</sup>.  $e'Z_t = 0$ . Partition  $e'$  and  $Z_t$  such that  $e' = [1, -f']$  and  $Z_t = [Z_{1t}, \bar{Z}'_t]$  where  $-f'$  is a  $1 \times 3$  vector,  $Z_{1t}$  is a scalar and  $\bar{Z}'_t$  is a  $3 \times 1$  vector. The restriction then implies at any time

$$Z_{1t} - f'\bar{Z}'_t = 0 \quad (49)$$

Partitioning the system (46) in a similar fashion gives

$$\begin{bmatrix} E_t \dot{Z}_{1,t+1} \\ E_t \dot{\bar{Z}}_{t+1} \end{bmatrix} = \begin{bmatrix} \widetilde{M}_{11} & \widetilde{M}_{12} \\ \widetilde{M}_{21} & \widetilde{M}_{22} \end{bmatrix} \begin{bmatrix} Z_{1t} \\ \bar{Z}_t \end{bmatrix} \quad (50)$$

where  $\widetilde{M}_{11}$  is  $1 \times 1$ ,  $\widetilde{M}_{12}$  is  $1 \times 3$ ,  $\widetilde{M}_{21}$  is  $3 \times 1$  and  $\widetilde{M}_{22}$  is  $3 \times 3$ . Focusing on the last three variables, i.e.,  $\bar{Z}_t$  and substitute out  $Z_{1t}$  using (49), I get the transformed system  $E_t \dot{\bar{Z}}_{t+1} = [\widetilde{M}_{22} + \widetilde{M}_{21}f'] \bar{Z}_t$ . Let  $\widetilde{N} = \widetilde{M}_{22} + \widetilde{M}_{21}f'$ , I have

$$E_t \dot{\bar{Z}}_{t+1} = \widetilde{N} \bar{Z}_t$$

By construction, such a transformation eliminates the first state variable in (50) and also the eigenvalue that goes to  $-\infty$ . Moreover, the eigenvalues of the resulting continuous time limit are exactly the other three roots of the original system (50). So the dynamic property of the continuous time limit is completely characterized by the three roots. This is a general statement no matter condition (48) holds or not.

However, condition (48) matters for determinacy issue. The key insight is as follows. The variable that is eliminated during the transformation is an inflation related variable, which is not predetermined. Thus systems before and after the transformation both requires one and only one stable root to achieve determinacy. If  $\lim_{\Delta \rightarrow 0} |1 + \Delta\lambda_1|$  is outside the unit circle, elimination of this unstable root does not make the number of stable roots different in the two systems. On the other hand, failure of this condition implies the transformation throws away a stable root, therefore  $\widetilde{M}$  and  $\widetilde{N}$  no longer have the same number of stable roots (in fact  $\widetilde{N}$

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<sup>18</sup>To see why I need condition (48), note that premultiplying  $E_t \dot{\bar{Z}}_{t+1} = \widetilde{M} \bar{Z}_t$  by  $e'$  gives  $e' E_t \dot{\bar{Z}}_{t+1} = \lambda_1 \cdot e' \bar{Z}_t$ , or equivalently,  $e' E_t \bar{Z}_{t+1} = (1 + \Delta\lambda_1) \cdot e' \bar{Z}_t$ . When condition (48) holds, the above equation implies  $e' \bar{Z}_t = 0$  in any bounded equilibrium.

has one less stable root than  $\widetilde{M}$ ). Under this situation, determinacy results may be different between the two systems.

Simple calculation yields

$$\widetilde{N} = \begin{bmatrix} -\frac{m_2}{\lambda_1} & -\frac{m_3}{\lambda_1} & 0 \\ \phi_\pi - 1 & 0 & 0 \\ H_2 - H_1 \frac{m_2}{\lambda_1} & H_3 - H_1 \frac{m_3}{\lambda_1} & H_4 \end{bmatrix}$$

where  $m_2$  and  $m_3$  are the same as defined before and  $H_1 = -\Delta \cdot h_1$ ,  $H_2 = h_2 - h_1$ ,  $H_3 = -\frac{h_3}{\Delta}$  and  $H_4 = \frac{h_4 - 1}{\Delta}$ .

First of all, by inspection, structure of  $\widetilde{N}$  resembles Dupor's system (8). Also note that  $\phi_\pi$  shows up in  $m_2$  but not  $m_3$  which also matches Dupor's system. Moreover, determinacy property of  $\widetilde{N}$  given the truth  $\lambda_1$  goes to  $-\infty$  also turns out to be the same as Dupor's conclusions.

**Proposition 4:** *Determinacy of system  $E_t \dot{Z}_{t+1} = \widetilde{N} \cdot \bar{Z}_t$  depends upon value of  $\phi_\pi$  as follows:  $\phi_\pi < 1$  leads to determinate equilibrium and  $\phi_\pi > 1$  leads to indeterminacy.*

*Proof:* see Appendix B.4.

All evidence suggests the resulting continuous time limit of the discrete time model is qualitatively identical to Dupor's model. The mystery outlined at the beginning of this section seems remain, that is, when period length is made arbitrarily small, it seems that passive policy has stabilizing ability. However, as I have noted earlier, when  $\lambda_1$  has a negative sign, condition (48) may or may not be true. Unfortunately, it turns out (48) is not satisfied in this case.

**Proposition 5:** *Consider  $\widetilde{M} \cdot \lim_{\Delta \rightarrow 0} |1 + \lambda_1 \cdot \Delta| < 1$*

*Proof:* see Appendix B.5.

Therefore it is not appropriate to use the sort of continuous time limit proposed by Dupor for determinacy analysis for stability property is altered during the transformation when a stable root is thrown away. The following proposition helps further understanding of the nature of the problem.

**Proposition 6:** *If  $\phi_\pi > 1$ ,  $\widetilde{M}$  has one unstable root and three stable roots. If  $\phi_\pi < 1$ ,  $\widetilde{M}$  has two unstable roots and two stable roots.*

*Proof:* see Appendix B.6.

Based on inherent connections between various systems I have examined so far, implication of the above proposition is illustrated below in Table 1. Recall  $\widetilde{M}$ 's eigenvalues is related to

that of  $M$  by (47), so they have the same determinacy result. On the other hand,  $\tilde{N}$  always has one less stable root than either  $M$  or  $\tilde{M}$ . That is, as  $\Delta$  approaches 0,  $\phi_\pi > 1$  or not leads to

Table 1: Comparison of Determinacy

		$M$	$\tilde{M}$	$\tilde{N}$	Agree?
$\phi_\pi > 1$	# unstable roots	1	1	1	agree
	# stable roots	3	3	2	
	determinacy	indeterminate	indeterminate	indeterminate	
$\phi_\pi < 1$	# unstable roots	2	2	2	disagree
	# stable roots	2	2	1	
	determinacy	indeterminate	indeterminate	<b>determinate</b>	

different numbers of stable roots in the discrete time model. I note that in either case there are more than one stable root, the conclusion that equilibrium is indeterminate does not depend on  $\phi_\pi$ . After the transformation, however, one stable root is lost in either case, which makes the total number of stable roots exactly one in one case and still more than one in the other, therefore determinacy results of the two cases differ dramatically. The continuous time limit which is essentially Dupor's model gets wrong conclusions on determinacy issue for it does not correctly approximate the discrete time model's behavior when periods are short.

## 6 Conclusions

The first main finding of this paper is that, in discrete time when length of periods is reasonably long, adding investment in a standard sticky price optimizing model does not change the stabilization property of interest rate rules. This conclusion contrasts to the finding of Dupor. My second finding is, performance of the discrete time model deteriorates as periods are made short. In the limit when period length is arbitrarily small, equilibrium is always indeterminate, no matter the policy is active or passive. This result still disagrees with Dupor's conclusions. Investigation aiming to identify sources for such contrasting results leads to the third finding, the kind of continuous time limit proposed by Dupor is misleading in the sense that it does not correctly approximate the behavior of the discrete time model of arbitrarily small periods. The generality of the third finding goes beyond the context of monetary policy analysis. In macroeconomics modeling, one should be aware of the possibility that continuous time approximation may distort the dynamic properties of a model with discrete periods of an arbitrarily

short length.

As to why the discrete time model is not well-behaved as length of period is made short, I note that some assumptions of the model are not realistic for small period length. It also indicates directions for future improvement. For example, in my model, as well as in Dupor's model, there are no adjustment costs of capital and the Taylor rule is interpreted as a purely contemporaneous relation between inflation and interest rate. As  $\Delta$  is made very short, decisions are made frequent enough by both the private sector and the central bank. The assumption of no any adjustment costs allows complete adjustment of the capital to its new desired level, no matter how short one period is. Smaller  $\Delta$  is, the faster is the adjustment, in the limit, as  $\Delta$  approaches 0, this implies an arbitrarily large rate of adjustment. Therefore adding adjustment costs is an important extension. For similar reason, the assumption that the central bank responds immediately to inflation becomes more and more unrealistic when  $\Delta$  is made small. From this point of view, it is more appealing to assume policy rules that involve some time-averaging of inflation, rather than instantaneous inflation, so that the time horizon over which inflation is averaged can be held fixed even when the period is made shorter. Whether modifications on these assumptions can improve my model (and Dupor's model) predictions for small time periods will be an interesting extension of the current exercise.

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## 8 Appendix: The Discrete-Time Model

### 8.1 Appendix A: When Period Length Is Not Taken into Account

#### 8.1.1 A.1: The Steady State

The steady state of the model is defined to be a set of constants  $(\bar{m}, \bar{C}, \bar{i}, \bar{H}, \bar{K}, \bar{I}, \bar{Y}, \bar{\omega}, \bar{\rho})$  that satisfies the following structural equations.

$$\begin{aligned}
 \frac{\bar{C}}{\bar{m}} &= \frac{\bar{i}}{1+\bar{i}} \\
 \frac{1}{1+\bar{i}} &= \beta \\
 \bar{C} &= \bar{\omega} \\
 \frac{1}{1+\bar{i}} &= \frac{1}{\bar{\rho}+1-\delta} \\
 \bar{K} &= (1-\delta)\bar{K} + \bar{I} \\
 \bar{C} + \bar{I} &= \bar{Y} \\
 \frac{1}{\mu} &= \frac{1}{1-\alpha} \bar{C} \left[ \frac{\bar{H}}{\bar{K}} \right]^\alpha \\
 \bar{Y} &= \bar{K}^\alpha \bar{H}^{1-\alpha} \\
 \frac{\bar{K}}{\bar{Y}} &= \left[ \frac{\alpha \bar{\omega}}{1-\alpha \bar{\rho}} \right]^{1-\alpha}
 \end{aligned}$$

Solving these equations jointly yields the unique steady state of the model,

$$\begin{aligned}
 \bar{i} &= \beta^{-1} - 1 \\
 \bar{\rho} &= \beta^{-1} - 1 - \delta \\
 \bar{\omega} &= \bar{C} = \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{\mu^{\frac{1}{1-\alpha}}(\beta^{-1}-1+\delta)^{\frac{\alpha}{1-\alpha}}} \\
 \bar{m} &= \frac{\bar{C}}{1-\beta} \\
 \bar{K} &= \bar{C} \left[ \frac{\mu(\beta^{-1}-1+\delta)}{\alpha} - \delta \right]^{-1} \\
 \bar{I} &= \delta \bar{K} \\
 \bar{Y} &= \frac{\mu(\beta^{-1}-1+\delta)}{\alpha} \bar{K} \\
 \bar{H} &= \frac{1-\alpha}{\alpha} (\beta^{-1}-1+\delta) \frac{\bar{K}}{\bar{C}}.
 \end{aligned}$$

### 8.1.2 A.2: Derivation of the Reduced Form

The structural equations used for solving the model include (15), (21), (22), (23), (24), (25), (42), (31) and (42). In what follows, I reduce the eight equations to a three variable system in  $\pi$ ,  $C$  and  $K$ .

Step1: Eliminate  $\hat{\rho}$ . First from cost minimization result (31) I have

$$\tilde{\rho}_t = \frac{\alpha}{1-\alpha} \tilde{\omega}_t \left[ \frac{Y_t}{K_t} \right]^{\frac{1}{1-\alpha}} = \frac{\alpha}{1-\alpha} C_t \left[ \frac{Y_t}{K_t} \right]^{\frac{1}{1-\alpha}}$$

It log-linearizes to  $\hat{\rho}_t = \hat{C}_t + \frac{1}{1-\alpha}(\hat{Y}_t - \hat{K}_t)$ . By forwarding the above once and substituting into (22) I obtain

$$\gamma_c E_t \hat{C}_{t+1} = \gamma_y (E_t \hat{Y}_{t+1} - \hat{K}_{t+1}) \quad (51)$$

where  $\gamma_c = \beta(1-\delta)$  and  $\gamma_y = \frac{1-\beta(1-\delta)}{1-\alpha}$ .

Step2: Eliminate  $\hat{i}$  and  $\hat{I}$ . This can be done by using (25)  $\hat{i}_t = \phi_\pi \pi_t$  and by rewriting (24) as  $\hat{I}_t = \frac{1}{s_I} \hat{Y}_t - \frac{s_c}{s_I} \hat{C}_t$ . Then the Euler (21) and capital accumulation (23) become

$$\phi_\pi \pi_t = E_t \pi_{t+1} + E_t \hat{C}_{t+1} - \hat{C}_t \quad (52)$$

$$\hat{K}_{t+1} = (1-\delta) \hat{K}_t + \frac{\delta}{s_I} \hat{Y}_t - \frac{\delta s_c}{s_I} \hat{C}_t \quad (53)$$

Step3: Eliminate  $\hat{Y}$ . Note that  $\hat{Y}_t$  and  $E_t \hat{Y}_{t+1}$  can be substitute out by first rewriting (42) as  $\hat{Y}_t = \frac{1}{\xi_y} \pi_t - \frac{\beta}{\xi_y} E_t \pi_{t+1} - \frac{\xi_c}{\xi_y} \hat{C}_t + \hat{K}_t$ . Then (51) and (53) can be written as

$$\left[ \gamma_c + \gamma_y \frac{\xi_c}{\xi_y} \right] E_t \hat{C}_{t+1} - \hat{C}_t = \frac{\gamma_y}{\xi_y} [E_t \pi_{t+1} - \beta E_t \pi_{t+2}] \quad (54)$$

$$\hat{K}_{t+1} = \left[ (1-\delta) + \frac{\delta}{s_I} \right] \hat{K}_t + \frac{\delta}{s_I \xi_y} (\pi_t - \beta E_t \pi_{t+1}) - \frac{\delta}{s_I} \left[ \frac{\xi_c}{\xi_y} + s_c \right] \hat{C}_t \quad (55)$$

Step4: Eliminate  $E_t \hat{C}_{t+1}$  in (54). Simply by writing (52) as  $E_t \hat{C}_{t+1} = \phi_\pi \pi_t - E_t \pi_{t+1} + \hat{C}_t$  and substituting the above into (54), I get the final reduced form for inflation. Together with (52) and (55) I have the following reduced form relation in equilibrium.

$$E_t \pi_{t+2} = \left[ \frac{\xi_y \gamma_c + \xi_c \gamma_y + \gamma_y}{\beta \gamma_y} \right] E_t \pi_{t+1} - \phi_\pi \left[ \frac{\xi_y \gamma_c + \xi_c \gamma_y}{\beta \gamma_y} \right] \pi_t - \left[ \frac{\xi_y \gamma_c + \xi_c \gamma_y - \xi_y}{\beta \gamma_y} \right] \hat{C}_t \quad (56)$$

$$E_t \hat{C}_{t+1} = \phi_\pi \pi_t - E_t \pi_{t+1} + \hat{C}_t \quad (57)$$

$$\hat{K}_{t+1} = - \left[ \frac{\beta \delta}{s_I \xi_y} \right] E_t \pi_{t+1} + \left[ \frac{\delta}{s_I \xi_y} \right] \pi_t - \left[ \frac{\delta}{s_I} \left( \frac{\xi_c}{\xi_y} + s_c \right) \right] \hat{C}_t + \left[ 1 - \delta + \frac{\delta}{s_I} \right] \hat{K}_t \quad (58)$$

### 8.1.3 A.3: Determinacy Analysis

To prove the determinacy results of the discrete time model I apply the following proposition due to Woodford (2003). It provides the necessary and sufficient conditions for exactly one root of a cubic equation to be inside the unit circle.

**Lemma:** *Let the characteristic equation of a  $3 \times 3$  matrix be written in the form  $p(\lambda) = \lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 = 0$ . Thus this equation has one root inside the unit circle and two roots outside if and only if*

- case I:

$$p(1) = 1 + A_2 + A_1 + A_0 < 0 \quad (59)$$

$$p(-1) = -1 + A_2 - A_1 + A_0 > 0 \quad (60)$$

- case II:

$$p(1) = 1 + A_2 + A_1 + A_0 > 0 \quad (61)$$

$$p(-1) = -1 + A_2 - A_1 + A_0 < 0 \quad (62)$$

$$A_0^2 - A_0A_2 + A_1 - 1 > 0 \quad (63)$$

- case III: (61) - (62) hold, and in addition

$$A_0^2 - A_0A_2 + A_1 - 1 < 0 \quad (64)$$

$$|A_2| > 3 \quad (65)$$

Proof of part (1) of Proposition 1: The matrix under study is

$$\hat{A} = \begin{bmatrix} f_1 & -\phi_\pi f_2 & -f_3 \\ 1 & 0 & 0 \\ -1 & \phi_\pi & 1 \end{bmatrix}$$

The characteristic equation of matrix  $\hat{A}$  is

$$p(\lambda) = \det(\hat{A} - \lambda I) = \lambda^3 - (1 + f_1)\lambda^2 + (f_1 - f_3 + \phi_\pi f_2)\lambda + \phi_\pi(f_3 - f_2)$$

Thus for this system

$$A_2 = -(1 + f_1) < 0$$

$$A_1 = f_1 - f_3 + \phi_\pi f_2$$

$$A_0 = \phi_\pi(f_3 - f_2) < 0$$

The last inequality holds because  $f_3 - f_2 = \frac{-\xi_y}{\beta\gamma_y} < 0$ . Suppose case I holds. This requires

$$p(1) = (\phi_\pi - 1)f_3 < 0$$

$$p(-1) = \phi_\pi(f_3 - 2f_2) - 2 - 2f_1 + f_3 > 0$$

or equivalently,  $\phi_\pi < 1$  and  $\phi_\pi < \frac{2+2f_1-f_3}{f_3-2f_2} < 0$  jointly. The last inequality holds because  $f_1 > f_2 > f_3 > 0$ . Then a negative  $\phi_\pi$  is required for determinacy, a contradiction to the assumption  $\phi_\pi > 0$ . On

the other hand, both of case II and case III require conditions (61) and (62) be satisfied simultaneously, that is,

$$\begin{aligned} p(1) &= (\phi_\pi - 1)f_3 > 0 \\ p(-1) &= \phi_\pi(f_3 - 2f_2) - 2 - 2f_1 + f_3 < 0 \end{aligned}$$

Solving jointly leads to  $\phi_\pi > 1$  (active policy). This completes the proof of part (1).

*Proof of part (2) of Proposition 1:* Consider an active Taylor rule  $\phi_\pi > 1$ . Thus (61) and (62) are satisfied as discussed in the proof of part (1). For my parameterization,

$$\begin{aligned} A_2 &= -3.77 < 0 \\ A_1 &= 2.66 + 1.76\phi_\pi \\ A_0 &= -1.65\phi_\pi \end{aligned}$$

Then the left hand side of (63) or (64) becomes

$$A_0^2 - A_0A_2 + A_1 - 1 = 2.71\phi_\pi^2 - 4.45\phi_\pi + 1.66$$

This a quadratic function in  $\phi_\pi$  with two distinct roots 1.07 and 0.57. But note that (65) is always satisfied regardless of the value of  $\phi_\pi$ . Thus (63) and (64) become mutually exclusive and exhaustive. So any  $\phi_\pi \in \mathcal{R}$  fits into either (63) or (64). Obviously this is also true for the subset of  $\phi_\pi$  satisfying (61) and (62), that is, with values greater than one. Thus, any value of  $\phi_\pi > 1$  satisfies conditions for either case II or case III and I conclude that active interest rate rule leads to determinacy for the calibrated parameter values.

## 8.2 Appendix B: When Period Length Is Taken into Account

### 8.2.1 B.1: Derivation of System (45)

Starting from reduced form (44), I want to rewrite the system in state variables  $(\dot{\pi}_t^c, \pi_t^c, \hat{C}_t, \hat{K}_t)$ , where  $\hat{C}_t, \hat{K}_t$  are the same variables as before and  $\pi_t^c$  is related to the discrete time inflation measure  $\pi_t^d$  by  $\pi_t^d = \Delta \cdot \pi_t^c$  and  $E_t \pi_{t+1}^d = \Delta \cdot \pi_t^c + \Delta^2 \cdot \dot{\pi}_t^c$ . To get the second equation, I need to use the identity  $E_t \pi_{t+1}^c = \pi_t^c + \Delta \cdot \dot{\pi}_t^c$ .

Inflation equation: Note that LHS =  $\Delta E_t \pi_{t+1}^c + \Delta^2 E_t \dot{\pi}_{t+1}^c$  and RHS =  $f_1 \Delta E_t \pi_{t+1}^c - \phi_\pi f_2 \Delta \pi_t^c - f_3 \hat{C}_t$ . Rearranging and collecting terms yield

$$E_t \dot{\pi}_{t+1}^c = (f_1 - 1) \dot{\pi}_t^c + \frac{f_1 - 1 - \phi_\pi f_2}{\Delta} \pi_t^c - \frac{f_3}{\Delta^2} \hat{C}_t$$

Consumption equation: Rewrite RHS in  $\pi_t^c$  and  $\dot{\pi}_t^c$  and collect terms,

$$E_t \hat{C}_{t+1} = \hat{C}_t + (\phi_\pi - 1) \Delta \pi_t^c - \Delta^2 \dot{\pi}_t^c$$

Capital equation: Applying the same procedure to substitute out  $E_t \pi_{t+1}^d$  and  $\pi_t^d$  and rearranging gives

$$\hat{K}_{t+1} = -h_1 \Delta^2 \dot{\pi}_t^c + (h_2 - h_1) \Delta \pi_t^c - h_3 \hat{C}_t + h_4 \hat{K}_t$$

Finally adding identity and stacking four equations together, I obtain (45).

### 8.2.2 B.2: Proof of Proposition 2

Either matrix  $M$  or  $A$  can be used to formulate the following proof. This is because due to the linearity of the transformation, they have identical eigenvalues and hence identical characteristic function. Recall in Appendix A.3 I have shown that  $\phi_\pi > 1$  is necessary to achieve determinacy. In addition, other conditions are needed, they are

$$\text{Case } i: \quad A_0^2 - A_0 A_2 + A_1 - 1 > 0$$

$$\text{Case } ii: \quad A_0^2 - A_0 A_2 + A_1 - 1 < 0 \quad \text{and} \quad |A_2| > 3$$

Plug in expressions for  $A_0$ ,  $A_1$  and  $A_2$  as recorded in Appendix A.3, the above conditions become

$$\text{Case } i: \quad a \phi_\pi^2 + b \phi_\pi + c > 0$$

$$\text{Case } ii: \quad a \phi_\pi^2 + b \phi_\pi + c < 0 \quad \text{and} \quad f_1 > 2$$

where  $a = (f_3 - f_2)^2$ ,  $b = f_3 + f_1 f_3 - f_1 f_2$  and  $c = f_1 - f_3 - 1$ . Next, I show that neither of the two sets of conditions hold, for any value of  $\phi_\pi$ . First, consider *Case ii*. Note that

$$f_1 = \frac{\xi_y \gamma_c}{\beta \gamma_y} + \frac{\xi_c + 1}{\beta} = \frac{\alpha(1-\delta)(1-\gamma)(1-\gamma\beta)}{\gamma[1-\beta(1-\delta)]} + \frac{(1-\gamma)(1-\gamma\beta) + \gamma}{\gamma\beta}$$

Because  $\lim_{\Delta \rightarrow 0} \alpha = \alpha$ ,  $\lim_{\Delta \rightarrow 0} \beta = 1$ ,  $\lim_{\Delta \rightarrow 0} \gamma = 1$  and  $\lim_{\Delta \rightarrow 0} \delta = 0$ , it is trivial to see  $\lim_{\Delta \rightarrow 0} f_1 = 1$ . Therefore, in the limit it is impossible to satisfy conditions listed in *Case ii*.

Next, consider *Case i*. I express  $a$ ,  $b$  and  $c$  in  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\delta$

$$\begin{aligned}
a &= \left[ \frac{\alpha(1-\gamma)(1-\gamma\beta)}{\gamma\beta[1-\beta(1-\delta)]} \right]^2 \\
b &= \frac{\alpha(1-\gamma)(1-\gamma\beta)(1-\delta)}{\gamma[1-\beta(1-\delta)]} - \frac{\alpha^2(1-\gamma)^2(1-\gamma\beta)^2(1-\delta)}{\gamma^2\beta[1-\beta(1-\delta)]^2} + \frac{(1-\gamma)(1-\gamma\beta)}{\beta\gamma} \\
&\quad - \frac{\alpha(1-\gamma)^2(1-\gamma\beta)^2}{\beta^2\gamma^2[1-\beta(1-\delta)]} - \frac{\alpha(1+\beta)(1-\gamma)(1-\gamma\beta)}{\gamma\beta^2[1-\beta(1-\delta)]} \\
c &= \frac{1-\beta}{\beta} + \frac{\alpha(1-\gamma)(1-\gamma\beta)}{\gamma\beta[1-\beta(1-\delta)]}
\end{aligned}$$

In the limit, I have  $\lim_{\Delta \rightarrow 0} a = \lim_{\Delta \rightarrow 0} b = \lim_{\Delta \rightarrow 0} c = 0$ . But they approach 0 at different rates. To see this, rescale  $a$ ,  $b$  and  $c$  by a common factor

$$\bar{d} = \frac{(1-\gamma)(1-\gamma\beta)}{\gamma\beta[1-\beta(1-\delta)]}$$

and denote the rescaled coefficients by  $\tilde{a} = a/\bar{d}$ ,  $\tilde{b} = b/\bar{d}$ ,  $\tilde{c} = c/\bar{d}$ . Then, in the limit  $\lim_{\Delta \rightarrow 0} \tilde{a} = 0$ ,  $\lim_{\Delta \rightarrow 0} \tilde{b} = -\alpha$ ,  $\lim_{\Delta \rightarrow 0} \tilde{c} = \alpha$ . This implies

$$\lim_{\Delta \rightarrow 0} [\tilde{a}\phi_\pi^2 + \tilde{b}\phi_\pi + \tilde{c}] = \alpha(1 - \phi_\pi)$$

Note the scaling factor is positive no matter how small  $\Delta$  is. Thus such a rescaling does not change the sign of  $\tilde{a}$ ,  $\tilde{b}$ ,  $\tilde{c}$ . So I get

$$\text{sign} [a\phi_\pi^2 + b\phi_\pi + c] = \text{sign} [\tilde{a}\phi_\pi^2 + \tilde{b}\phi_\pi + \tilde{c}]$$

Then for condition in *Case i* to hold,  $\phi_\pi < 1$  is needed. But this is a violation of the necessary condition for determinacy, namely,  $\phi_\pi > 1$ . In other words, there does not exist a finite  $\phi_\pi$  that can lead to determinacy.

### 8.2.3 B.3: Proof of Proposition 3

First I need to restate the coefficient by using  $\beta = e^{-\theta\Delta}$ ,  $\delta = \tilde{\delta} \cdot \Delta$ ;  $\gamma = e^{-\nu\Delta}$ . Substituting these relations into the expressions of coefficients, I get the following second order approximations.

$$\begin{aligned}
s_I &\doteq \frac{\alpha\tilde{\delta}}{\mu(\theta + \tilde{\delta})} < 1 \\
\gamma_c &\doteq 1 - (\theta + \tilde{\delta})\Delta + \theta\tilde{\delta}\Delta^2 \\
\gamma_y &\doteq \frac{1}{1-\alpha} [ (\theta + \tilde{\delta})\Delta - \theta(\theta + \tilde{\delta})\Delta^2 ] \\
\xi_c &\doteq (\theta\nu + \nu^2)\Delta^2 \\
\xi_y &\doteq \frac{\alpha}{1-\alpha}(\theta\nu + \nu^2)\Delta^2 \\
f_1 &\doteq 1 + \left[ \theta + \frac{\alpha(\theta\nu + \nu^2)}{\theta + \tilde{\delta}} \right] \Delta + \left[ (\theta\nu + \nu^2)\left(1 + \frac{\alpha(\theta - \tilde{\delta})}{\theta + \tilde{\delta}}\right) \right] \Delta^2 \equiv 1 + B_1\Delta + B_2\Delta^2 \\
f_2 &\doteq \left[ \frac{\alpha(\theta\nu + \nu^2)}{\theta + \tilde{\delta}} \right] \Delta + \left[ (\theta\nu + \nu^2)\left(1 + \frac{\alpha(\theta - \tilde{\delta})}{\theta + \tilde{\delta}}\right) - \frac{\theta^2}{2} \right] \Delta^2 \equiv C_1\Delta + C_2\Delta^2 \\
f_3 &\doteq \left[ (1-\alpha)(\theta\nu + \nu^2) - \frac{\theta^2}{2} \right] \Delta^2 \equiv D_1\Delta^2 \\
h_1 &\doteq \left[ \frac{(1-\alpha)\tilde{\delta}}{\alpha s_I(\theta\nu + \nu^2)} \right] \frac{1}{\Delta} - \left[ \frac{\theta(1-\alpha)\tilde{\delta}}{\alpha s_I(\theta\nu + \nu^2)} \right] \\
h_2 &\doteq \left[ \frac{(1-\alpha)\tilde{\delta}}{\alpha s_I(\theta\nu + \nu^2)} \right] \frac{1}{\Delta} \\
h_3 &\doteq -\frac{\tilde{\delta}}{s_I} \left( s_c + \frac{1-\alpha}{\alpha} \right) \Delta \\
h_4 &\doteq 1 + \left( \frac{1}{s_I} - 1 \right) \tilde{\delta}\Delta
\end{aligned}$$

$\tilde{M}$  is reproduced below for convenience.

$$\tilde{M} = \begin{bmatrix} \frac{f_1-2}{\Delta} & \frac{f_1-1-\phi_\pi f_2}{\Delta^2} & -\frac{f_3}{\Delta^3} & 0 \\ 1 & 0 & 0 & 0 \\ -\Delta & \phi_\pi - 1 & 0 & 0 \\ -h_1\Delta & h_2 - h_1 & -\frac{h_3}{\Delta} & \frac{h_4-1}{\Delta} \end{bmatrix}$$

Step1: One root  $\frac{h_4-1}{\Delta}$  is finite. This is true because

$$\lim_{\Delta \rightarrow 0} \frac{h_4-1}{\Delta} = \left[ \frac{1}{s_I} - 1 \right] \tilde{\delta} = > 0$$

It is an unstable root of  $\tilde{M}$ , moreover it is finite.

Step 2: There is one root goes to infinite as  $\Delta$  goes to 0. To see this, given Step 1, dynamic property of

$\widetilde{M}$  is determined by the following submatrix

$$\widehat{M} = \begin{bmatrix} \frac{f_1-2}{\Delta} & \frac{f_1-1-\phi_\pi f_2}{\Delta^2} & -\frac{f_3}{\Delta^3} \\ 1 & 0 & 0 \\ -\Delta & \phi_\pi - 1 & 0 \end{bmatrix}$$

whose characteristic function is

$$p(\lambda) = \lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 = 0$$

where

$$\begin{aligned} A_2 &= -\frac{f_1-2}{\Delta} \\ A_1 &= -\frac{f_3+f_1-1-\phi_\pi f_2}{\Delta^2} \\ A_0 &= \frac{f_3}{\Delta^3}(\phi_\pi - 1) \end{aligned}$$

Using second order approximations of  $f_1$ ,  $f_2$  and  $f_3$ ,  $A_2$ ,  $A_1$  and  $A_0$  can be approximated as

$$\begin{aligned} A_2 &= \frac{1-B_1\Delta}{\Delta} \\ A_1 &= -\frac{(\phi_\pi C_1 - B_1) + (\phi_\pi C_2 - D_1 - B_2)\Delta}{\Delta} \\ A_0 &= \frac{D_1}{\Delta}(\phi_\pi - 1) \end{aligned}$$

Then  $\lim_{\Delta \rightarrow 0} A_i \rightarrow \infty$  for  $i = 0, 1, 2$  and they go to  $\infty$  at the same rate. Suppose one eigenvalue is  $\infty$  in the limit, then  $\exists k(\Delta)$  s.t.  $|\lim_{\Delta \rightarrow 0} k| < \infty$ ,  $|\lim_{\Delta \rightarrow 0} k \cdot A_i| < \infty \quad \forall i$ . In my case, choose  $k(\Delta) = \Delta$ , it is trivial to see

$$\lim_{\Delta \rightarrow 0} \Delta = 0, \quad \lim_{\Delta \rightarrow 0} \Delta \cdot A_2 = 1, \quad \lim_{\Delta \rightarrow 0} \Delta \cdot A_1 = \phi_\pi C_1 - B_1, \quad \lim_{\Delta \rightarrow 0} \Delta \cdot A_0 = D_1(\phi_\pi - 1)$$

All these limits are finite. Therefore, I conclude that one root of  $\widetilde{M}$  must go to  $\infty$  in the limit.

Step 3: The other two roots of  $\widehat{M}$  are both finite. To derive this result, first denote the root that goes to  $\infty$  by  $\lambda_1$  and the other two roots by  $\lambda_2$  and  $\lambda_3$  respectively. I multiply the characteristic function of  $\widehat{M}$  by  $k(\Delta) = \Delta$ ,

$$\Delta\lambda^3 + \Delta A_2\lambda^2 + \Delta A_1\lambda + \Delta A_0 = 0$$

According to the limit of  $\Delta$ ,  $\Delta A_2$ ,  $\Delta A_1$  and  $\Delta A_0$ , the above equality implies in the limit that

$$\lambda^2 + \frac{A_1}{A_2}\lambda + \frac{A_0}{A_2} = 0$$

By doing so,  $\lambda_1$  is eliminated, and  $\lambda_2$  and  $\lambda_3$  are the two roots of the above equation. It is then straightforward to show

$$\begin{aligned} -\lim_{\Delta \rightarrow 0} (\lambda_2 + \lambda_3) &= \lim_{\Delta \rightarrow 0} \frac{A_1}{A_2} = \phi_\pi C_1 - B_1 < \infty \\ \lim_{\Delta \rightarrow 0} (\lambda_2 \lambda_3) &= \lim_{\Delta \rightarrow 0} \frac{A_0}{A_2} = D_1(\phi_\pi - 1) < \infty \end{aligned}$$

These two ratios are finite if and only if both  $\lambda_2$  and  $\lambda_3$  are finite.

Step 4: The sign of  $\lambda_1$  is negative. To see this,

$$\text{sign}(\lambda_1) = -\text{sign}(A_2) = -\text{sign}(\Delta A_2) = -1$$

The first equality holds because  $A_2 = -\sum \lambda_i$  and as  $\Delta \rightarrow 0$ ,  $\lambda_1$  is the only dominating term. This completes the proof.

### 8.2.4 B.4: Proof of Proposition 4

Since  $H_4 = (h_4 - 1)/\Delta > 0$  is one eigenvalue of  $\tilde{N}$ , I only need to examine the submatrix

$$\hat{\tilde{N}} = \begin{bmatrix} -\frac{m_2}{\lambda_1} & -\frac{m_3}{\lambda_1} \\ \phi_\pi - 1 & 0 \end{bmatrix}$$

whose determinant is

$$\det(\hat{\tilde{N}}) = \frac{m_3}{\lambda_1}(\phi_\pi - 1)$$

Determinacy requires  $\det(\hat{\tilde{N}}) < 0$ . Given  $m_3 = -\frac{f_3}{\Delta^3} < 0$  and  $\lambda_1 < 0$ , this is true if and only if  $\phi_\pi < 1$ . Therefore, for the system characterized by  $\tilde{N}$ ,  $\phi_\pi < 1$  gives determinacy while  $\phi_\pi > 1$  leads to indeterminacy. This is exactly what Dupor concludes.

### 8.2.5 B.5: Proof of Proposition 5

Recall (i)  $\Delta(\lambda_1 + \lambda_2 + \lambda_3) = -\Delta \cdot A_2 \rightarrow -1$  and (ii)  $\lambda_2, \lambda_3$  are finite. These two results imply

$$\lim_{\Delta \rightarrow 0} |1 + \Delta\lambda_1| = \lim_{\Delta \rightarrow 0} |1 - \Delta A_2| = 0$$

That is, in the limit  $(1 + \Delta\lambda_1)$  stays within the unit circle.

### 8.2.6 B.6: Proof of Proposition 6

I follow all notations in Appendix A.3 and Appendix B.2. First, I derive the limits of several quantities. As  $\Delta$  goes to 0, I have

$$\begin{aligned} p(1) &= (1 - \lambda_1)(1 - \lambda_2)(1 - \lambda_3) = \phi_\pi - 1 \\ p(-1) &= (-1 - \lambda_1)(-1 - \lambda_2)(-1 - \lambda_3) = -4 \\ \text{sign}[(\lambda_1\lambda_2 - 1)(\lambda_1\lambda_3 - 1)(\lambda_2\lambda_3 - 1)] &= \text{sign}(1 - \phi_\pi) \end{aligned}$$

The last equality holds because  $\text{sign}[(\lambda_1\lambda_2 - 1)(\lambda_1\lambda_3 - 1)(\lambda_2\lambda_3 - 1)] = \text{sign}[\tilde{a}\phi_\pi^2 + \tilde{b}\phi_\pi + \tilde{c}]$  and the limit of  $\tilde{a}\phi_\pi^2 + \tilde{b}\phi_\pi + \tilde{c}$  is  $\alpha(1 - \phi_\pi)$ .

Case 1:  $\phi_\pi > 1$ . In this case, I will show all three roots are stable. First suppose all roots are real.  $p(-1) < 0$  and  $p(1) > 0$  imply there are odd number of roots within the unit circle. Hence all of them have to be stable. Otherwise I get determinacy if only one root is stable, but I have already shown in Appendix B.2 that determinacy is impossible as  $\Delta$  goes to 0. Next consider one root  $\lambda_3$  is real and the other two,  $\lambda_1$  and  $\lambda_2$  are complex pair then

$$(\lambda_1\lambda_2 - 1)(\lambda_1\lambda_3 - 1)(\lambda_2\lambda_3 - 1) < 0$$

This implies  $|\lambda_1| = |\lambda_2| < 1$  because  $(\lambda_1\lambda_3 - 1)$  and  $(\lambda_2\lambda_3 - 1)$  are complex conjugates. Also, from

$$\begin{aligned} p(1) &= (1 - \lambda_1)(1 - \lambda_2)(1 - \lambda_3) > 0 \\ p(-1) &= (-1 - \lambda_1)(-1 - \lambda_2)(-1 - \lambda_3) < 0 \end{aligned}$$

I can derive  $-1 < \lambda_3 < 1$ , i.e., it is also a stable root. Thus, once again three roots are all within the unit circle.

Case 2:  $\phi_\pi < 1$ . For this case, two roots are stable and one root is unstable. To see this, suppose first all roots are real. In addition to  $p(-1) < 0$  and  $p(1) < 0$ , (i)  $p(\lambda) > 0$  holds for all large enough positive  $\lambda$ , and (ii)  $p(\lambda) < 0$  for all large enough negative  $\lambda$ , where  $\lambda^3$  term dominates the others. Then there must be two roots within the unit circle and one outside. Now suppose one root  $\lambda_3$  is real and  $\lambda_1, \lambda_2$  are complex conjugate roots. As before,  $p(1) < 0$  implies  $\lambda_3 > 1$ . On the other hand,

$$(\lambda_1\lambda_2 - 1)(\lambda_1\lambda_3 - 1)(\lambda_2\lambda_3 - 1) > 0$$

which implies  $|\lambda_1| = |\lambda_2| > 1$  because the second and third terms are complex conjugates. So all three roots must be outside the unit circle. But then this is a contradiction to the result I have shown in Appendix B.5 that one root is within the unit circle in the limit. So when  $\phi_\pi < 1$ , it is impossible to have two complex roots. Combining the two cases together, the proof is completed.

