Lecture 3: Probability, diagnostic testing and Bayes

Supplementary Reading: Pagano/Gauvreau; Chapter 6

Probability

Frequentist viewpoint: Large-sample limit of a proportion, i.e. a proportion of a defined set of outcomes

Bayesian viewpoint: The degree to which a person (or community) believes that a proposition is true, subject to available information.

Ultimately, both definitions relate to *information*

Definitions

experiment - any procedure that can be repeated (in principle, if not logistically) an arbitrarily large number of times and has a well defined set of possible outcomes

sample space - set of all possible outcomes

event - any set of outcomes of interest

probability - (of an event) relative frequency of this set of outcomes of interest

Conceptually (from a frequentist point of view): if an experiment is repeated n times under essentially the same conditions and if the event A occurs m times, then as n gets large the ratio m/n approaches a fixed limit p_0 that is the probability of A; $p_0 = \mathrm{P}(A) \approx m/n$. This is called the *law of large numbers*.

Example: Coin toss

Consider the experiment consisting of flipping a coin three times. The sample space is

$S = \{$ HHH, HHT, HTH, THH, HTT, THT, TTH, TTT$\}$

Event A = {Majority of coins show heads}

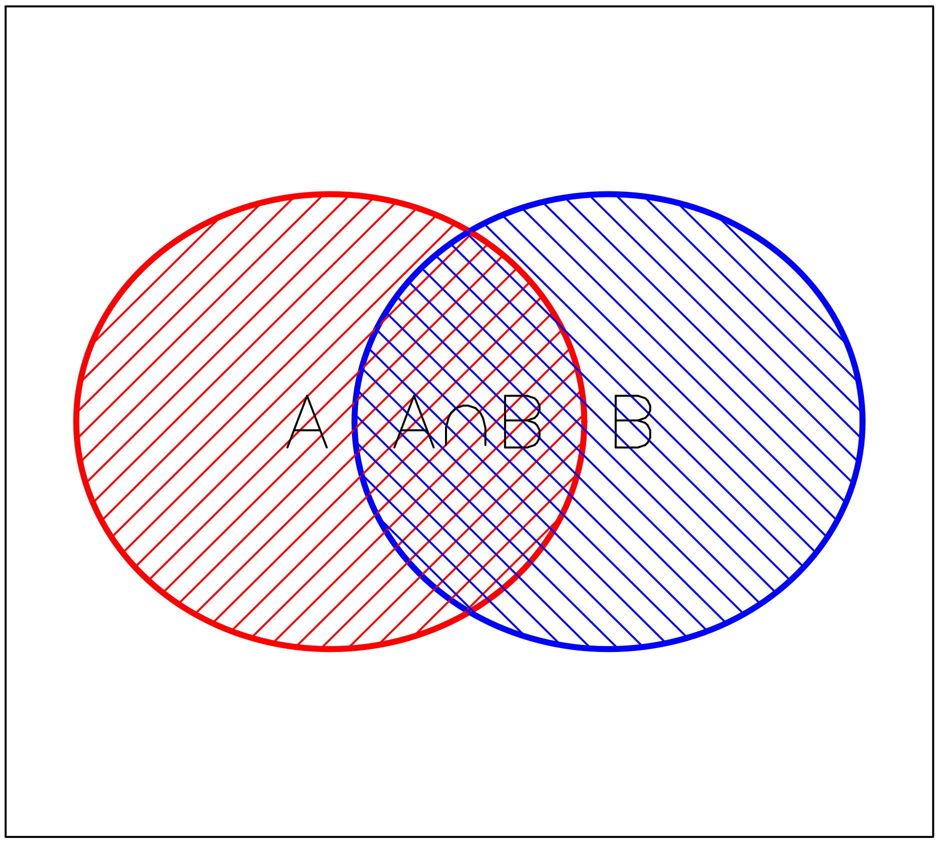
$A = \{$HHH, HHT, HTH, THH$\}$

Basic properties of probabilities

1. The probability of an event, $\mathrm{P}(E)$, always satisfies $0 \leq \mathrm{P}(E) \leq 1$
2. In outcomes A and B are two events that cannot both happen at the same time, then A and B are **mutually exclusive**: $\mathrm{P}(A \text{ or } B) = \mathrm{P}(A) + \mathrm{P}(B)$

More definitions/notation

1. $A \cup B$ is the event that either A or B occurs or they both occur - **union**
2. $A \cap B$ is the event that both A and B occur simultaneously - **intersection**
3. A^c (or $\bar{A}$) is the event that A does not occur - **complement**; $\mathrm{P}(A^c) = 1 - \mathrm{P}(A)$
4. Note that $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$. Why?


A
A∩B
B

Operations on events

Let events

A = {30 year old woman lives to see her 70^{th} birthday}
B = {her 30 year old husband lives to see his 70^{th} birthday}

Intersection of events A and B:
A ∩ B = { both live to 70 }

Union of events A and B:
A ∪ B = {the woman lives to 70 or the man lives to 70, or both live to 70}

Complement of event : A^c = {30 year old woman does not live to 70}

Example :

Let A = set of integers x for which $x^2 + 2x = 8$
Let B = set of integers x for which $x^2 + x = 6$
Let C = set of integers x for which $x^2 - 6x = 5$

A = {-4,2}, B ={-3,2}, and C = {1,5}

A ∩ B = {2}
A ∪ B = {-4,-3,2}

A ∩ C = ∅ (A and C are *mutually exclusive*)

Definitions/notation (con't)

Two events A and B are **independent** if $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$

If two events are **dependent**, then $\mathrm{P}(A \cap B) \neq \mathrm{P}(A) \times \mathrm{P}(B)$

Example: Consider all possible diastolic blood pressure (DBP) measurements from a mother and her first-born child.

Let A = mothers DBP $\geq$ 95 and B = first-born child's DBP $\geq$ 80

Suppose P(A $\cap$ B) = 0.05; P(A) = 0.1; and P(B) = 0.2

Then, P(A $\cap$ B) = 0.05 > P(A) $\times$ P(B)

So A and B are dependent

A^c (or $\bar{A}$) is the event that A does not occur - **complement**; $\mathrm{P}(A^c)$ = 1 - $\mathrm{P}(A)$

In general, if A_1, ..., A_k are independent events, then $\mathrm{P}(A_1 \cap ... \cap A_k) = \mathrm{P}(A_1) \times ... \times \mathrm{P}(A_k)$ - **multiplicative law of probability**

Properties of probabilities

$$P(A^c) = 1 - P(A)$$

$$P(A \cup A^c) = 1$$

$$P(A \cap A^c) = 0$$

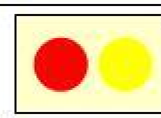
Examples:

P(a child is not male) = P(a child is female)

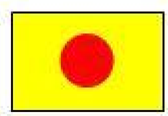
P(a child is male or female) = 1

P(a child is male and female) = 0

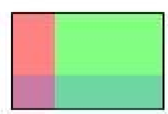
$$P(A \cup B) = P(A) + P(B) \quad \text{if } A \cap B = \emptyset$$



$$1 = P(A \cup A^c) = P(A) + P(A^c)$$



$$P(A \cap B) = P(A)P(B)$$
iff A and B are independent



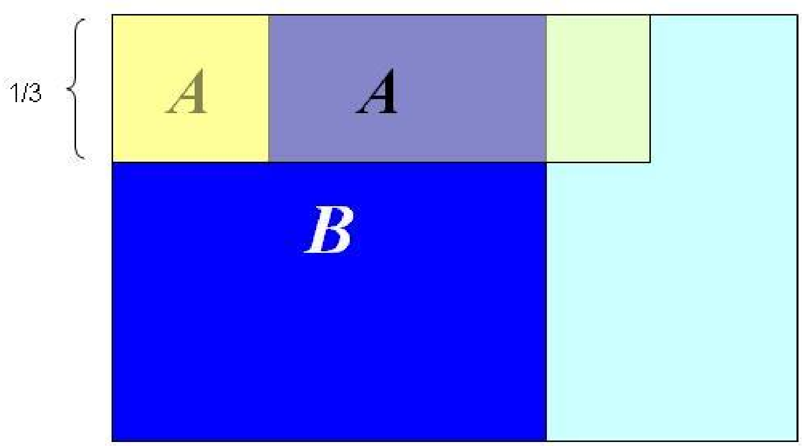
Conditional probability

Say we are interested in finding the probability a person will survive for the next five years given that she/he has already reached the age of 60.

Interested in the probability that one event will occur given another has already taken place, i.e., the conditional probability denoted $\mathrm{P}(A|B)$

Definition: $\mathrm{P}(A|B) = \mathrm{P}(A \cap B)/\mathrm{P}(B)$

> If A and B are independent events, then $\mathrm{P}(A|B) = \mathrm{P}(A) = \mathrm{P}(A|B^c)$
>
> If two events A and B are dependent, then $\mathrm{P}(A|B) \neq \mathrm{P}(A) \neq \mathrm{P}(A|B^c)$



Example: Compute the probability of the event B|A where

A = a person in the US who lives to 60 yrs

B = a person in the US who lives to 65 yrs

→ B|A = a 60 yr old person in the US will live to age 65

Data: P(A) = 0.85; P(B) = 0.79

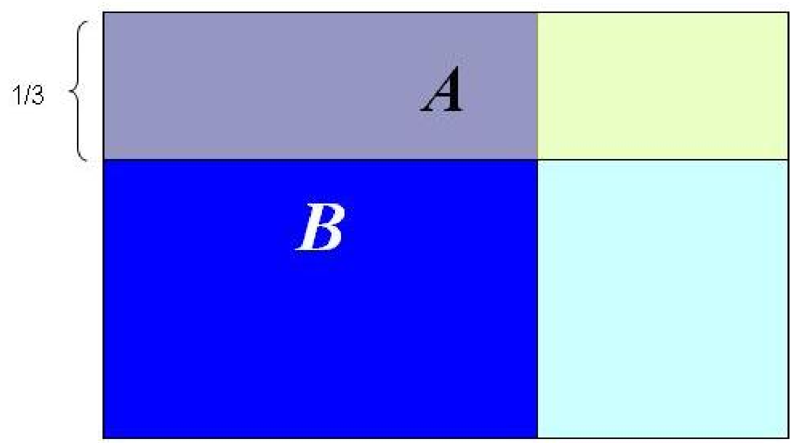
Exercise: Show $\mathrm{P}(B) = \mathrm{P}(B|A) \times \mathrm{P}(A) + \mathrm{P}(B|A^c) \times \mathrm{P}(A^c)$

Definitions/notation (con't)

Independent events: Two events are said to be independent if $\mathrm{P}(A|B) = \mathrm{P}(A)$

Intuitively, the knowledge that B has occurred does not affect the probability that A will occur.

Note that $\mathrm{P}(A \cap B) = \mathrm{P}(A|B)\mathrm{P}(B) = \mathrm{P}(A)\mathrm{P}(B)$ – multiplicative rule from before.



Example: Diagnostic testing

Notation:

D = has disease

D^c = does not have disease

T = test positive

T^c = test negative

Goal: To determine the probability that individual has the disease of interest, given they tested positive ($\rightarrow$ P(D|T))

More notation:

$\mathrm{P}(D)$ = **prevalence** of the disease

$\mathrm{P}(T|D)$ = **sensitivity** of the test

$\mathrm{P}(T|D^c)$ = probability of a false positive test

$\mathrm{P}(T^c|D)$ = probability of a false negative test

$\mathrm{P}(T^c|D^c)$ = **specificity** of the test

Ideally, for a diagnostic test to be effective at predicting "disease" (or a procedure to be effective at predicting an "event"), we want both sensitivity and specificity to be high. There are usually tradeoffs.

Bayes' Rule

Let T = symptom (positive test) and D = disease

$$P(D|T) = \frac{P(T|D) \times P(D)}{P(T|D) \times P(D) + P(T|D^c) \times P(D^c)}$$

$$P(D|T) = \frac{\text{sensitivity} \times \text{prevalence}}{\text{sensitivity} \times \text{prevalence} + (1 - \text{specificity}) \times (1 - \text{prevalence})}$$

$P(D|T)$ – **predictive value of a positive test**

$P(D^c|T^c)$ – **predictive value of a negative test**

Example: X-Ray screening for TB

	Tuberculosis		
X-Ray	No	Yes	Total
Negative	1739	8	1747
Positive	51	22	73
Total	1790	30	1820

Question: What is P($D|T$) – What is the probability someone has TB given they had a positive X-Ray?

Prevalence of TB: P(D) =9.3 in 100,000

$$
\begin{aligned}
P(D) &= 0.000093 \ \ \text{(given)} \\
P(D^c) &= 1 - P(D) = 0.999907 \\
P(T|D) &= \\
P(T^c|D) &= \\
P(T|D^c) &= \\
P(T^c|D^c) &=
\end{aligned}
$$

So,

$$P(D|T) = \rule{6cm}{0.4pt}$$

Bayes

Revising previous beliefs based on new "evidence" (Bayesian concept)

$$\begin{aligned} P(D) &= 0.000093 \ \text{ prior probability} \\ P(D|T) &= 0.00239 \ \text{ posterior probability} \end{aligned}$$

P(D|T) is updated probability of TB based on new "evidence"

Consider the ratio of the prior and posterior probabilities: $\frac{0.00239}{0.000093} = 25.7$. In essence, updating our knowledge has reduced our degree of uncertainty about the prevalence of TB by roughly 26 fold.

Example: Accuracy of home pregnancy test kits

Background

2 kinds of pregnancy tests (blood and urine); test detects levels of hCG (human chorionic gonadotropic)– a hormone secreted by at different times during pregnancy and builds up over time; hCG generally doubles every 2 days during the first few weeks of pregnancy

Home tests: most sensitive 20 miu/ml of hCG in urine (250 miu/ml)

$A_1 = \{\text{woman pregnant}\}$

$A_2 = \{\text{woman NOT pregnant}\}$

$T = \{\text{positive test result}\}$

Sensitivity = 80%

Specificity = 68%

Prevalence of pregnancy = 60%

1. What is the probability that a woman with a positive home pregnancy test result is actually pregnant?
2. What is the probability that a woman with a negative home pregnancy test result is NOT pregnant?