EXPLICIT EQUIVALENCES BETWEEN CAT(0) HYPERBOLIC TYPE GEODESICS

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ABSTRACT. We prove an explicit equivalence between various hyperbolic type properties for quasi-geodesics in CAT(0) spaces. Specifically, we prove that for \( X \) a CAT(0) space and \( \gamma \subset X \) a quasi-geodesic, the following four statements are equivalent and moreover the quantifiers in the equivalences are explicit: (i) \( \gamma \) is S-Slim, (ii) \( \gamma \) is M(K,L)–Morse, (iii) \( \gamma \) is (b,c)–contracting, and (iv) \( \gamma \) is C–strongly contracting. In particular, this explicit equivalence proves that for \( f \) a \((K,L)\) quasi-isometry between CAT(0) spaces, and \( \gamma \) a C–strongly contracting \((K',L')\)–quasi-geodesic, then \( f(\gamma) \) is a \(C'(C, K, L, K', L')\)–strongly contracting quasi-geodesic. This result is necessary for a key technical point with regard to Charney’s contracting boundary for CAT(0) spaces.

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1. INTRODUCTION AND OVERVIEW

In the study of spaces of non-positive curvature, Euclidean and hyperbolic space represent the two classically well understood extreme ends of the spectrum. More generally, in the literature a robust approach for studying particular spaces of interest is to identify particular directions, geodesics, or subspaces along which the space in question shares features in common with one of these two prototypes. In particular, with regard to identifying *hyperbolic type geodesics* in spaces of interest, or geodesics which share features in common with geodesics in hyperbolic space, there are various well studied precise notions including being Morse, being contracting, and being slim. Specifically, such studies have proven fruitful in analyzing right angled Artin groups [BC], Teichmüller space [B, BrF, BrM, BMM, Mos], the mapping class group [B], CAT(0) spaces [Sul, BD, BeF, Cha], and \( \text{Out}(F_n) \) [A] amongst others (See for instance [DMS, DS, KL, Osi, MM]).

A Morse geodesic \( \gamma \) is defined by the property that all quasi-geodesics \( \sigma \) with endpoints on \( \gamma \) remain within a bounded distance from \( \gamma \). A strongly contracting geodesic has the property that metric balls disjoint from the geodesic have nearest point projections onto the geodesic with uniformly bounded diameter. A geodesic is called slim if geodesic triangles with one edge along the geodesic are \( \delta \)-thin. It is an elementary fact that in hyperbolic space, or more generally \( \delta \)-hyperbolic spaces, all quasi-geodesics are Morse, strongly contracting, and slim. On the other hand, in product spaces such as Euclidean spaces of dimension two and above, there are no Morse, strongly contracting, or slim quasi-geodesics.

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Building on results in [Sul], in this paper we prove that the various aforementioned hyperbolic type properties are equivalent and moreover the quantifiers in the equivalences are explicit.

**Theorem 3.4 (Main Theorem).** Let $X$ be a CAT(0) space and $\gamma \subset X$ a quasi-geodesic. Then the following are equivalent:

1. $\gamma$ is $(b,c)$–contracting,
2. $\gamma$ is $C'$–strongly contracting,
3. $\gamma$ is $M$–Morse, and
4. $\gamma$ is $S$–slim

Moreover, any one of the four sets of constants $\{(b,c), C', M, S\}$ can be written in terms of any of the others.

Theorem 3.4 should be considered in the context of related theorems in [BeF, B, Cha, DMS, KL, Sul] among others. In particular, in [BeF] geodesics with property (2) are studied and in fact among other things it is shown that for the special case of $\gamma$ a geodesic $(2) \implies (4)$. In [Cha] geodesics with property (2) are studied and it is shown that $(2) \implies (3)$, an explicit proof of which also appears in [A]. In [DMS] geodesics with property (3) are studied. In [Sul] building on work of the previous authors it is shown that properties (1), (2), and (3) are equivalent, although the proof relies on limiting arguments and hence the constants of the equivalence could not be recovered.

As a corollary of Theorem 3.4 we highlight the following consequence, which in fact served as motivation for the results in this paper.

**Corollary 3.5** Let $X$ be a CAT(0) space, $\gamma \subset X$ a $C$–strongly contracting $(K', L')$–quasi-geodesic, and $f : X \to X$ a $(K, L)$ quasi-isometry. Then $f(\gamma)$ is $C'(C, K, L, K', L')$–strongly contracting quasi-geodesic.

In particular, Corollary 3.5 is very useful in [Cha] where it is used to show that self quasi-isometries of CAT(0) spaces give rise to continuous maps on Charney’s contracting boundary for CAT(0) spaces.

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2. **Background**

2.1. **Quasi-geodesics and CAT(0) spaces.**

**Definition 2.1** (quasi-geodesic). A $(K, L)$ quasi-geodesic $\gamma \subset X$ is the image of a map $\gamma : I \to X$ where $I$ is a connected interval in $\mathbb{R}$ (possibly all of $\mathbb{R}$) such that $\forall s, t \in I$ we have the following quasi-isometric inequality:

\[
\frac{|s - t|}{K} - L \leq d_X(\gamma(s), \gamma(t)) \leq K|s - t| + L
\]

We refer to the quasi-geodesic $\gamma(I)$ by $\gamma$, and when the constants $(K, L)$ are not relevant omit them.

CAT(0) spaces are geodesic metric spaces defined by the property that triangles are no “fatter” than the corresponding comparison triangles in Euclidean space. In particular, using this property one can prove the following lemma, see [BH, Section II.2] for details.

**Lemma 2.2.** Let $X$ be a CAT(0) space.

1. **(Projections onto convex subsets).** Let $C$ be a convex subset, complete in the induced metric, then there is a well-defined distance non-increasing nearest point projection map $\pi_C : X \to C$. In particular, $\pi_C$ is continuous. We will often consider the case where $C$ is a geodesic.
C2: (Convexity). Let $c_1 : [0, 1] \to X$ and $c_2 : [0, 1] \to X$ be any pair of geodesics parameterized proportional to arc length. Then the following inequality holds for all $t \in [0, 1]$:

$$d(c_1(t), c_2(t)) \leq (1 - t)d(c_1(0), c_2(0)) + td(c_1(1), c_2(1))$$

2.2. Hyperbolic type quasi-geodesics. In this section we define the hyperbolic types of quasi-geodesics we will consider in this paper. The following definition of Morse (quasi-)geodesics has roots in the classical paper [Mor]:

**Definition 2.3** (Morse quasi-geodesics). A (quasi-)geodesic $\gamma$ is called an $M$–Morse (quasi-)geodesic if for every $(K, L)$-quasi-geodesic $\sigma$ with endpoints on $\gamma$, we have $\sigma \subset N_M(K, L)(\gamma)$. That is, $\sigma$ is within a bounded distance, $M = M(K, L)$, from $\gamma$, with the bound depending only on the constants $K, L$. In the literature, Morse (quasi-)geodesics are sometimes referred to as stable quasi-geodesics.

The following generalized notion of contracting quasi-geodesics can be found for example in [B, BrM], and is based on a slightly more general notion of $(a,b,c)$–contraction found in [MM] where it serves as a key ingredient in the proof of the hyperbolicity of the curve complex.

**Definition 2.4** (contracting quasi-geodesics). A (quasi-)geodesic $\gamma$ is said to be $(b, c)$–contracting if $\exists$ constants $0 < b \leq 1$ and $0 < c$ such that $\forall x, y \in X$,

$$d_X(x, y) < bd_X(x, \pi_\gamma(x)) \implies d_X(\pi_\gamma(x), \pi_\gamma(y)) < c.$$ 

For the special case of a $(b, c)$–contracting quasi-geodesic where $b$ can be chosen to be 1, the quasi-geodesic $\gamma$ is called $c$–strongly contracting.

The following elementary lemma shows that given a $(b, c)$–contracting quasi-geodesic one can increase $b$ to be arbitrarily close to 1 at the expense of increasing $c$.

**Lemma 2.5.** If $\gamma$ is $(b, c)$–contracting, then for any arbitrarily small $\epsilon > 0$ $\gamma$ is $(1 - \epsilon, c'(\epsilon, b, c))$–contracting.

**Proof.** Notice that if $\gamma$ is $(b, c)$–contracting, then it is also $(b + b(1 - b), 2c)$–contracting. Similarly, it is also $(b + b(1 - b) + b(1 - b)^2, 3c)$–contracting. Iterating this process, the statement of the lemma follows, as for $0 < b < 1$ the sum of the geometric series $\sum_{i=0}^{\infty} b(1 - b)^i$ converges to 1. \(\square\)

Finally, the following definition of a slim quasi-geodesic is introduced in [BeF].

**Definition 2.6** (slim quasi-geodesics). A (quasi-)geodesic $\gamma$ is said to be $S$–slim if $\exists$ constant $S$ such that for all $x \in X$ and $y \in \gamma$, we have:

$$d(\pi_\gamma(x), [x, y]) \leq S.$$ 

Note that if $\gamma$ is a $S$–slim quasi-geodesic, then

$$|[x, \pi_\gamma(x)]| + |[\pi_\gamma(x), y]| - 2S \leq |[x, y]| \leq |[x, \pi_\gamma(x)]| + |[\pi_\gamma(x), y]|.$$ 

Moreover, if $z \in [x, y]$ is a point such that $d(y, z) = d(y, \pi_\gamma(x))$ (or similarly such that $d(x, z) = d(x, \pi_\gamma(x))$), then $d(z, \pi_\gamma(x)) \leq 2S$.

We conclude this section by citing a lemma relating contracting and slim geodesics.

**Lemma 2.7** ([BeF] Lemma 3.5). Let $\gamma$ be a $C$–strongly contracting geodesic in a CAT(0) space. Then $\gamma$ is $(3C + 1)$–slim.
3. Main Theorem and Proof

The following elementary lemma regarding the concatenation of geodesic segments will be useful in the proof of Theorem 3.4.

**Lemma 3.1.** For any triple of points $a, b, c$ in a CAT(0) space $X$, the concatenated path
\[ \phi = [a, \pi_{[b,c]}(a)] \cup [\pi_{[b,c]}(a), c], \]
is a $(3,0)$ quasi-geodesic.

**Proof.** We must show that $\forall x, y \in \phi$, the $(3,0)$–quasi-isometric inequality of Equation 2.4 is satisfied. Since $\phi$ is a concatenation of two geodesic segments, without loss of generality we can assume $x \in [a, \pi_{[b,c]}(a)], y \in [\pi_{[b,c]}(a), c]$. Since $x \in [a, \pi_{[b,c]}(a)]$ it follows that $\pi_{[b,c]}(x) = \pi_{[b,c]}(a)$, and hence $d(x, \pi_{[b,c]}(a)) \leq d(x, y)$. Let $d_\phi(x, y)$ denote the distance along $\phi$ between $x$ and $y$.

Then, the following inequality completes the proof:
\[
d(x, y) \leq d_\phi(x, y) = d(x, \pi_{[b,c]}(a)) + d(\pi_{[b,c]}(a), y) \\
\leq d(x, \pi_{[b,c]}(a)) + (d(\pi_{[b,c]}(a), x) + d(x, y)) \\
\leq 2d(x, \pi_{[b,c]}(a)) + d(x, y) \leq 3d(x, y)
\]
\[ \square \]

The following lemma and ensuing corollary will be useful in the proof of Theorem 3.4. Specifically, these results will be used to reduce arguments regarding quasi-geodesics to case of geodesics. The lemma is closely related to and should be compared with Lemma 3.8 of [BeF].

**Lemma 3.2.** Let $\gamma$ be an $M$–Morse, $C$–strongly contracting geodesic, and let $\gamma'$ be a $(K, L)$–quasi-geodesic with endpoints on $\gamma$. Then $\gamma'$ is $C'(C, M)$–strongly contracting.

Since nearest point projections onto quasi-geodesics is not uniquely determined, in the proof of the lemma we will use the convention that $\pi_{\gamma'}(x)$ represents an arbitrary element in the nearest point projection set of $x$ onto $\gamma'$. Additionally, when measuring distances between elements in nearest point projection sets, such as $d(\pi_{\gamma'}(x), \pi_{\gamma'}(y))$, we will use the convention that the distance is the supremum over all possible choices of elements in the nearest point projection sets. That is,
\[
d(\pi_{\gamma'}(x), \pi_{\gamma'}(y)) =: \sup\{d(x', y') \mid x' \in \pi_{\gamma'}(x), y' \in \pi_{\gamma'}(y)\}.
\]

**Proof.** First we will prove that $\forall z \in X, d(\pi_{\gamma'}(z), \pi_{\gamma'}(\pi_{\gamma}(z)))$ is bounded above in terms of the constants $C, M$. By Lemma 2.7 the geodesic $\gamma$ is $(3C+1)$-slim. Set the Morse constant $M(K, L) = M$. Consider the triangle $\triangle(z, \pi_{\gamma}(z), \pi_{\gamma}(\pi_{\gamma'}(z)))$. Since
\[
d(z, \pi_{\gamma}(\pi_{\gamma'}(z))) \leq d(z, \pi_{\gamma'}(z)) + M \leq d(z, \pi_{\gamma}(z)) + 2M,
\]
$(3C+1)$–slimness implies that
\[
||[\pi_{\gamma}(z), \pi_{\gamma}(\pi_{\gamma'}(z))]|| \leq 2M + 2(3C + 1).
\]
Since $\gamma' \subset N_M(\gamma)$, by the triangle inequality
\[
d(\pi_{\gamma}(z), \pi_{\gamma}(\pi_{\gamma'}(\pi_{\gamma}(z)))) \leq 2M.
\]
Combining Equations 3.1 and 3.2 by the triangle inequality we have
\[
d(\pi_{\gamma'}(\pi_{\gamma}(z)), \pi_{\gamma'}(\pi_{\gamma}(z))) \leq 4M + 2(3C + 1).
\]
Finally, using the Equation 3.3 in conjunction with the fact that $\gamma' \subset N_M(\gamma)$ and the triangle inequality, it follows that $\forall z \in X$,
\[
d(\pi_{\gamma'}(\pi_{\gamma}(z)), \pi_{\gamma'}(z)) \leq 6M + 2(3C + 1).
\]
Now assume we have \( x, y \in X \) such that \( d(x, y) < d(x, \pi_{\gamma'}(x)) \). We must show that we can bound \( d(\pi_{\gamma'}(x), \pi_{\gamma'}(y)) \) from above in terms of the constants \( C, M \). Since \( d(x, y) < d(x, \pi_{\gamma'}(x)) \leq d(x, \pi_{\gamma}(x)) + M \), using the facts that \( \gamma \) is \( C \)-strongly contracting and nearest point projections onto geodesics are distance non-increasing, we have that
\[
(3.5) \quad d(\pi_{\gamma}(x), \pi_{\gamma}(y)) \leq C + M.
\]
As above, using the fact that \( \gamma' \subset N_M(\gamma) \), in conjunction with Equation 3.5 and the triangle inequality, it follows that
\[
(3.6) \quad d(\pi_{\gamma'}(\pi_{\gamma}(x)), \pi_{\gamma'}(\pi_{\gamma}(y))) \leq C + 3M.
\]
Putting together Equations 3.4 and 3.6, the following completes the proof:
\[
d(\pi_{\gamma'}(x), \pi_{\gamma'}(y)) \leq d(\pi_{\gamma'}(x), \pi_{\gamma'}(\pi_{\gamma}(x))) + d(\pi_{\gamma'}(\pi_{\gamma}(x)), \pi_{\gamma'}(\pi_{\gamma}(y))) + d(\pi_{\gamma'}(\pi_{\gamma}(y)), \pi_{\gamma'}(y)) \leq (6M + 2(3C + 1)) + (C + 3M) + (6M + 2(3C + 1)) = 15M + 7C + 4
\]
\[\Box\]

As a corollary of Lemma 3.2, we have the following:

**Corollary 3.3.** If it’s true that a geodesic being \( M \)-Morse implies that the geodesic is \( C(M) \)-strongly contracting, then it’s also true that a quasi-geodesic being \( M' \)-Morse implies that the quasi-geodesic is \( C'(M') \)-strongly contracting.

**Proof.** Assume that if a geodesic is Morse then it is also \( C(M) \)-strongly contracting. Let \( \gamma' \) be an \( M' \)-Morse quasi-geodesic. Fix \( x \in X \) and \( x' \in \pi_{\gamma'}(x) \). Let \( y \in X \) be such that \( d(x, y) \leq d(x, x') \), and fix \( y' \in \pi_{\gamma'}(y) \). Notice that
\[
d(x', y') \leq d(x', x) + d(x, y) + d(y, y') \leq d(x', x) + d(x', x) + 2d(x', x) = 4d(x', x).
\]

Let \( \alpha_x \in \gamma' \) be any point preceding \( x' \) such that \( d(\alpha_x, x') \geq 4d(x', x) \). Similarly, let \( \beta_x \in \gamma' \) be any point following \( x' \) such that \( d(x', \beta_x) \geq 4d(x', x) \). Since \( \gamma' \) is an \( M' \)-Morse quasi-geodesic and because \([\alpha_x, \beta_x] \) is a geodesic with endpoints on \( \gamma' \), it follows that \([\alpha_x, \beta_x] \) is similarly \( 2M' \)-Morse. Then, by assumption \([\alpha_x, \beta_x] \) is \( C(M') \)-strongly contracting. By Lemma 3.2 it follows that \( \gamma'_{|[\alpha_x, \beta_x]} \) is \( C'(C'(M'), M') = C'(M') \)-strongly contracting. In particular, since for all \( y \in X \) such that \( d(x, y) \leq d(x, x') \), we know that \( \pi_{\gamma'}(y) \subset \gamma'_{|[\alpha_x, \beta_x]} \), it follows that \( d(\pi_{\gamma'}(x), \pi_{\gamma'}(y)) \leq C'(M') \).

Since for any starting \( x \in X \) we can preform this process of creating such an interval \([\alpha_x, \beta_x] \) and proceeding as above, it follows that the quasi-geodesic \( \gamma' \) is \( C'(M') \)-strongly contracting. \[\Box\]

We are now prepared to prove the main theorem.

**Theorem 3.4.** Let \( X \) be a \( \text{CAT}(0) \) space and \( \gamma \subset X \) a \((K,L)\)-quasi-geodesic. Then the following are equivalent:

1. \( \gamma \) is \( C' \)-strongly contracting,
2. \( \gamma \) is \((b,c)\)-contracting,
3. \( \gamma \) is \( M \)-Morse, and
4. \( \gamma \) is \( S \)-slim.

Moreover, any one of the four sets of constants \{\(b,c\), \(C', M\), \(S\)\} can be written purely in terms of the any of the others in conjunction with the quasi-isometry constant \((K,L)\).

**Proof.** By definition (1) \(\implies\) (2). The fact that (2) \(\implies\) (3) is a slight generalization of the well known “Morse stability lemma.” For an explicit proof see Lemma 3.3 in [Sul] (or similarly Lemma 5.13 in [A]). In order to complete the proof of the theorem we will provide an explicit proof that: (3) \(\implies\) (1), (1)+(3) \(\implies\) (4), and (4) \(\implies\) (2).
(3) $\implies$ (1): By Corollary 3.3 it suffice to prove (3) $\implies$ (1) in the special case of $\gamma$ a geodesic. Fix $x, y \in X, x' = \pi_\gamma(x), y' = \pi_\gamma(y)$ such that $d(x, x') < d(x, y)$. Set $A = d(x, x'), D = d(x', y')$.

In order to complete the proof of this step we will bound $D$. Let $z \in [x, y]$ be such that $d(x, z) = \frac{A}{4}$ (if $|[x, y]| \leq \frac{A}{4}$, then set $z = y$ and the rest of the proof follows without modification), and $z' \in [x', y'] \subset \gamma$ such that $d(x', z') = \frac{D}{4}$. Hence, $\frac{3A}{4} \leq d(z, z') \leq \frac{5A}{4}$.

Let $\rho_1 : [0, 1] \to X$ be the geodesic parameterized proportional to arc length joining $x' = \rho_1(0)$ and $x = \rho_1(1)$. Similarly, let $\rho_2 : [0, 1] \to X$ be the geodesic parameterized proportional to arc length joining $z' = \rho_2(0)$ and $z = \rho_2(1)$. Note that by property [C1] of Lemma 2.2 $A \geq D$. Set $s = \frac{D}{4A}$, so $s \in [0, 1]$. Applying property [C2] of Lemma 2.2 to the geodesics $\rho_1, \rho_2$, we have that

$$d(\rho_1(s), \rho_2(s)) \leq (1 - s)d(x', z') + sd(x, z) \leq (1 - s)\frac{D}{4} + s\frac{A}{4} \leq \frac{D}{4} + \frac{D}{4A} \frac{A}{4} = \frac{5D}{16}.$$

Since $\rho_1, \rho_2$ are geodesics realizing nearest point projection onto $\gamma$ putting together the facts that $d(\rho_1(s), \rho_2(s)) \leq \frac{5D}{16}$, $d(\rho_1(s), \gamma) = \frac{D}{4}$, and $d(\rho_2(s), \gamma) \geq \frac{3D}{4}$, it follows that

$$d([\rho_1(s), \rho_2(s)], \gamma) > \frac{D}{16}.$$  

Let $x'' \in [x', z']$ be such that $d(x', x'') = \frac{D}{32}$, and similarly let $z'' \in [x', z']$ be such that $d(z', z'') = \frac{D}{32}$. Let $u \in [\rho_1(s), \rho_2(s)]$ be the point closest to $\rho_2(s)$ such that $\pi_{[x', z']}(u) = x''$, and similarly, let $v \in [\rho_1(s), \rho_2(s)]$ be the point closest to $\rho_1(s)$ such that $\pi_{[x', z']}(v) = z''$. Let $\sigma$ be the shortest geodesic connecting $[x', \rho_1(s)]$ and $[u, v]$, and similarly let $\tau$ be the shortest geodesic connecting $[u, v]$ and $[\rho_2(s)]$. Notice that by property [C1] of Lemma 2.2 $\frac{D}{32} \leq \frac{D}{16} = |\tau|, |\sigma| \leq \frac{3D}{32}$. Let $u'$ denote the endpoint of $\sigma$ on $[u, v]$, and let $v'$ denote the endpoint of $\tau$ on $[u, v]$. Again using property [C1] of Lemma 2.2 it follows that $u'$ must precede $v'$ along $[u, v]$. Finally, let $t$ denote the endpoint of $\sigma$ on $\rho_1$ and let $w$ denote the endpoint of $\tau$ on $\rho_2$. See Figure 1 for an illustration of the situation.

In order to complete this step of the proof we will show that the concatenated path

$$\phi = \rho_1([x, t]) \cup \sigma \cup [u', v'] \cup \tau \cup \rho_2([w, z])$$

is a $(19,0)$–quasi-geodesic. Accordingly since $\gamma$ is M(K,L)–Morse it follows that $\phi \subset N_{M(19,0)}(\gamma)$.

However, since the segment $[u', v'] \subset [\rho_1(s), \rho_2(s)]$ is a nontrivial component of the concatenated path.
path φ, in light of Equation [3.7] it follows that $\frac{D}{16} \leq M(19,0)$. Hence it follows that $\gamma$ is $16M(19,0)$—strongly contracting, thus completing the proof of (3) $\implies$ (1).

Hence all that remains is to show that φ is a (19,0)–quasi-geodesic. We must show that $\forall a, b \in \phi$, that the (19,0)–quasi-isometric inequality of Equation [2.1] is satisfied. Since $\phi$ is a concatenation of geodesics, without loss of generality we can assume $a, b$ belong to different geodesic segments within $\phi$. Since there are 5 different geodesic segments in $\phi$, there are $\left(\frac{5}{3}\right) = 10$, cases to consider. By Lemma [3.1] we know that the (3,0)–quasi-isometric inequality is satisfied in the case where $a$ and $b$ belong to adjacent geodesic segments in the concatenation. There are six remaining cases to consider. We will consider each in turn after first recording some immediate inequalities regarding the lengths of the geodesic segments being concatenated in $\phi$. By definition $|[x', t]| = \frac{D}{4}$. Comparing the total length of $\rho_1$ and $\rho_2$ we know that $|[w, z']| \leq \frac{5D}{16}$. By property [C1] of Lemma 2.2 $\frac{3D}{16} \leq |[u', v']| \leq \frac{D}{4}$. Similarly as noted above, $\frac{D}{32} \leq |[t, u']|, |[v', w]| \leq \frac{3D}{32}$. Putting things together, we have the following three inequalities:

1. $a \in [x', t], b \in [u', v']$: By definition, in this case $|a, b| \leq d(a, b)$. Hence,

$$d(a, b) \leq d_\phi(a, b) = d(a, t) + |[t, u']| + d(u', b) \leq |[x', t]| + |[t, u']| + |[u', v']| \leq 8|[t, u']| + |[t, u']| + 8|[t, u']| = 17|[t, u']| \leq 17d(a, b)$$

2. $a \in [u', v'], b \in [w, z']$: By definition, in this case $|a, b| \leq d(a, b)$. Hence,

$$d(a, b) \leq d_\phi(a, b) = d(a, w) + |[w, v']| + d(v', b) \leq |[z', w]| + |[v', w]| + |[v', u']| \leq 8|[v', w]| + |[v', w]| + 10|[v', w]| = 19|[v', w]| \leq 19d(a, b)$$

3. $a \in [t, u'], b \in [v', w]$: By definition, in this case $|a, b| \leq d(a, b)$. Hence,

$$d(a, b) \leq d_\phi(a, b) = d(a, u') + |[t, u']| + d(u', b) \leq |[t, u']| + |[t, u']| + |[u', v']| \leq \frac{|[u', v']|}{2} + |[u', v']| + \frac{|[u', v']|}{2} = 2|[u', v']| \leq 2d(a, b)$$

4. $a \in [x', t], b \in [v', w]$: By property [C1] of Lemma 2.2 in this case $d(a, b) \geq \frac{7D}{32} - |[v', w]| \geq \frac{D}{8}$. Hence,

$$d(a, b) \leq d_\phi(a, b) = d(a, t) + |[t, u']| + |[u', v']| + d(v', b) \leq |[x', t]| + |[t, u']| + |[u', v']| + |[v', w]| \leq \frac{D}{4} + \frac{3D}{32} + \frac{D}{4} + \frac{3D}{32} = \frac{22D}{32} \leq \frac{11}{2}d(a, b)$$
(5) $a \in [t, u'], b \in [w, z']$ : As in the previous case, by property [C1] of Lemma 2.2 in this case $d(a, b) \geq \frac{D}{8}$. Hence,
\[
d(a, b) \leq d_\delta(a, b) = d(a, u') + |[u', v'|] + |[v', w]| + d(w, b) \leq |[t, u']| + |[u', v'|] + |[v', w]| + |[w, z']| \\
\leq \frac{3D}{32} + \frac{D}{4} + \frac{3D}{32} + \frac{10D}{32} = \frac{3D}{4} \\
\leq 6d(a, b)
\]

(6) $a \in [x', t], b \in [w, z']$ : By property [C1] of Lemma 2.2 in this case $d(a, b) \geq \frac{D}{4}$. Hence,
\[
d(a, b) \leq d_\delta(a, b) = d(a, t) + |[t, u']| + |[u', v']| + |[v', w]| + d(w, b) \\
\leq |[x', t]| + |[t, u']| + |[u', v']| + |[v', w]| + |[w, z']| \\
\leq \frac{D}{4} + \frac{3D}{32} + \frac{D}{4} + \frac{3D}{32} + \frac{10D}{32} = D \\
\leq 4d(a, b)
\]

(1)+(3) $\implies$ (4): Fix $x \in X$, $x' \in \pi_\gamma (x)$, and $y \in \gamma$. Let $z = \pi_{[x, y]} (x')$, and let $z' \in \pi_\gamma (z)$. Consider the nearest point projection of $[x, x']$ onto $[x, y]$. Since $\pi_{[x, y]} (x) = x$, and $\pi_{[x, y]} (x') = z$, by property [C1] of Lemma 2.2 it follows that $d(x, z) \leq d(x', x)$. By $C'$-contraction of $\gamma$, it follows that $d(x', z') < C'$. Furthermore, by Lemma 3.1, the concatenated path $[x', z] \cup [z, y]$ is a $(3,0)$-quasi-geodesic. In particular, it follows that $d(x', z)$ is bounded above by the Morse constant $M(3,0)$. Hence, $d(x', [x, y]) = d(x', z) \leq d(x', z') + d(z', y) \leq C' + M(3,0)$. Thus, $\gamma$ is $(C' + M(3,0))$-slimming, thus completing this step of the proof. See Figure 2.

\[
\text{Figure 2. (1)+(3) } \implies \text{ (4).}
\]

(4) $\iff$ (2): Assume $\gamma$ is an S-slim, $(K,L)$–quasi-geodesic. Fix $x \in X$, $x' \in \pi_\gamma (x)$. Let $y \in X$ be any point such that $d(x, y) \leq \frac{d(x,x')}{2K}$, and fix any $y' \in \pi_\gamma (y)$. We will show that $d(x', y')$ is bounded above by a constant $B(S,K,L)$, thus showing that $\gamma$ is $(\frac{1}{2K} B(S,K,L))$-contracting.

Consider the function $f : X \rightarrow \mathbb{R}$ defined by $f(a) = d(x', a) - d(y', a)$.Restricting the function $f$ to $\gamma_{[x',y']}$ the function can have jump discontinuities of at most $2L$, and hence by the intermediate value theorem, $\exists \zeta \in \gamma$ such that $d(y', \zeta) - L \leq d(x', \zeta) \leq d(y', \zeta) + L$. See Figure 3. Without loss of generality we can assume $d(x', \zeta) \geq 2(S + L)$, for if not, then $d(x', y') \leq 4S + 6L$, in which case we are done. Furthermore since $\gamma$ is a $(K,L)$–quasi-geodesic, it follows that
\[
2d(\zeta, x') + 2L \leq Kd(x', y') + L \implies d(\zeta, x') \leq \frac{Kd(x', y') - L}{2} \leq \frac{Kd(x', y')}{2}
\]

(3.8)
Let $x'' \in [x, \zeta]$ such that $d(\zeta, x'') = d(\zeta, x')$. Similarly, let $y'' \in [y, \zeta]$ such that $d(\zeta, y'') = d(\zeta, y')$. By the remarks following the definition of $S$–slim, $d(x', x'') + d(y', y'') \leq 2S$. Let $y'' \in [\zeta, y]$ be such that $d(y'', \zeta) = \frac{d(\zeta, x'') d(y, \zeta)}{d(x, \zeta)}$. Comparing the lengths of $[\zeta, x]$ and $[\zeta, y]$, it follows that $d(y'', y''') \leq \frac{d(\zeta, x'')}{2K} + 2L$.

Applying CAT(0) thinness of triangles to the triangle $\triangle (x, y, \zeta)$, it follows that $|d(x', x'')| \leq \frac{d(\zeta, x'') d(x, x')}{d(x, \zeta)} \leq \frac{d(\zeta, x'')}{2K}$.

Putting things together,

$$d(x', y') \leq d(x', x'') + d(x'', y''') + d(y'', y')$$

$$\leq 2S + \frac{d(\zeta, x'')}{2K} + \left( \frac{d(\zeta, x'')}{2K} + 2L \right) + 2S = 4S + 2L + \frac{d(\zeta, x'')}{K}$$

$$\leq 4S + 2L + \left( \frac{d(x', y')}{2} \right) \quad \text{[by Equation 3.8]}

\Rightarrow$$

$$d(x', y') \leq 8S + 4L$$

Setting $B(S, L) = \max\{4S + 6L, 8S + 4L\}$ it follows that $\gamma$ is $(\frac{1}{2K}, B(S, L))$-contracting, thus completing the proof.

Notice that of the four equivalent definitions of hyperbolic type quasi-geodesics considered in Theorem 3.4 the Morse version is particular well suited with regard to quasi-isometries. In particular, let $\gamma$ be a $M$–Morse quasi-geodesic. Then for $f : X \to Y$ a $(K, L)$–quasi-isometry between CAT(0) spaces $X$ and $Y$, by definition $f(\gamma)$ is an $M'(K, L)$–Morse quasi-geodesic. In light of Theorem 3.4 we immediately obtain the following corollary, which as noted in the introduction has application in [Cha]:

**Corollary 3.5.** Let $X$ be a CAT(0) space, $\gamma \subset X$ a $C$–strongly contracting $(K', L')$–quasi-geodesic, and $f : X \to X$ a $(K, L)$ quasi-isometry. Then $f(\gamma)$ is $C'(C, K, L, K', L')$–strongly contracting quasi-geodesic.

**References**


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