SEPARATING CURVE COMPLEX OF THE GENUS TWO SURFACE: QUASI-DISTANCE FORMULA AND HYPERBOLICITY

HAROLD SULTAN

Abstract. We prove that the separating curve complex of $S_{2,0}$ satisfies a quasi-distance formula akin to the quasi-distance formulas for the marking and pants complexes in [6], and the disk and arc complexes in [7]. The proof uses basic properties of Farey graphs in conjunction with tools of Masur–Minsky [5, 6] and Masur–Schleimer [7]. As a corollary, we provide an alternative proof that $C_{sep}(S_{2,0})$ is $\delta$-hyperbolic, a fact implicit in the work of Brock–Masur [2], as well as explicit in recent work of Ma [4].

1. Introduction

In recent years, the curve complex and natural relatives thereof have been extensively featured in the geometric group theory literature. These natural combinatorial complexes have proven to be extremely useful tools with applications to a variety of settings including notably the study of mapping class groups and Teichmüller space. See for instance [3] for a broad overview of the topics and close relations to various natural combinatorial complexes. In this context, we study the coarse geometry of the separating curve complex of the genus two surface.

In studying the coarse geometry of natural combinatorial complexes, being able to approximate distance is of fundamental interest. A prominent idea in [7] is that distance in a combinatorial complex can sometimes be approximated by summing over the distances in the subsurface projections to the curve complexes of certain subsurfaces. Specifically, the set of subsurfaces summed over, 2010 Mathematics Subject Classification. Primary: 20F65, Secondary: 57M50.
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or holes, are defined by the property that every vertex of the combinatorial complex has nontrivial subsurface projections into their curve complexes. Such an approximation, when it exists, is called a quasi-distance formula. In fact, in their groundbreaking paper [6], Masur–Minsky develop a notion of hierarchies which in particular provides examples of quasi-distance formulas in the marking and pants complexes.

The ideas in this paper are similar to, as well as motivated by, work of Masur–Schleimer in [7]. Using ideas implicit in [1], in [7] Masur–Schleimer axioms are established which are proven to be sufficient for ensuring that a combinatorial complex satisfies a quasi-distance formula and is $\delta$-hyperbolic. In particular, in [7] verification of the Masur–Schleimer axioms are used prove that the disk complex and the arc complex satisfy quasi-distance formulas and are $\delta$-hyperbolic. Unfortunately, as we will see, one of the Masur–Schleimer axioms fails in the case of $C_{sep}(S^2, 0)$. Nonetheless, in this paper we are able to show by a direct argument that $C_{sep}(S^2, 0)$ does in fact satisfy a quasi-distance formula. Furthermore, carefully considering the Masur–Schleimer argument, as a corollary we also obtain that $C_{sep}(S^2, 0)$ is $\delta$-hyperbolic. Specifically, we prove the following main theorem:

**Theorem 1.1.** The combinatorial complex $C_{sep}(S^2, 0)$ satisfies a quasi-distance formula. That is, if we let $NS$ denote the set of all nonseparating essential subsurfaces of $S^2$. Then there is a constant $K_0$ such that for all $k \geq K_0$ there exists constants $K(k), L(k)$ such that $\forall \alpha, \beta \in C_{sep}(S^2, 0)$ we have the following quasi-isometric relation:

$$\frac{1}{K} \sum_{Y \in NS} \{d_{C(Y)}(\alpha, \beta)\} \leq d_{C_{sep}(S^2, 0)}(\alpha, \beta) \leq \frac{1}{K} \sum_{Y \in NS} \{d_{C(Y)}(\alpha, \beta)\} + L,$$

where the threshold function $\{f(x)\}_k$ is defined to be $f(x)$ if $f(x) \geq k$, and 0 otherwise. Moreover, the combinatorial complex $C_{sep}(S^2, 0)$ is $\delta$-hyperbolic.

It is well known that the curve complex $C(S)$ is $\delta$-hyperbolic, see [5]. On the other hand, the separating curve complex $C_{sep}(S)$ in general is not $\delta$-hyperbolic. In fact, for all closed surfaces $S = S_{g, 0}$ with genus $g \geq 3$, $C_{sep}(S)$ contains natural quasi-isometric embeddings of $\mathbb{Z}^2$; an obstruction to hyperbolicity, [8]. For $S_{2, 0}$ however, there are no such natural non-trivial quasi-flats. Given this context, Schleimer conjectures that $C_{sep}(S_{2, 0})$ is $\delta$-hyperbolic; see [8] Conjecture 2.48. In this paper we provide a proof in the affirmative of this conjecture. To be sure, the fact that $C_{sep}(S_{2, 0})$ is $\delta$-hyperbolic is implicit in [2] as well as explicit in [4].
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2. Preliminaries

2.1. Coarse Geometry. When studying large scale geometry, in place of the usual notions of functions and isometries it is often useful to consider the notions of coarsely well-defined maps and quasi-isometries. The latter are natural large scale generalizations of the former.

Definition 2.1 (coarsely well defined). Given metric spaces \( X, Y \), a map \( f : X \to 2^Y \setminus \emptyset \) is coarsely well-defined if \( \exists \) a constant \( C \) such that \( \forall x \in X, \text{diam}_Y(f(x)) < C \).

Given a coarsely well defined map \( f : X \to 2^Y \), by abuse of notation we will sometimes consider it as a map \( f : X \to Y \) obtained by assigning to each \( x \in X \) an arbitrarily selected element \( y \in f(x) \).

Definition 2.2 (quasi-isometry). Given metric spaces \( (X, d_X), (Y, d_Y) \), a map \( f : (X, d_X) \to (Y, d_Y) \) is called a \((K, L)\) quasi-isometric embedding of \( X \) into \( Y \) if there exist constants \( K \geq 1, L \geq 0 \) such that for all \( x, x' \in X \) the following inequality holds:

\[
K^{-1} d_X(x, x') - L \leq d_Y(f(x), f(x')) \leq K d_X(x, x') + L.
\]

If in addition, the map \( f \) is roughly onto, i.e. a fixed neighborhood of the image is the entire codomain, \( f \) is called a quasi-isometry. Two metric spaces are called quasi-isometric if there exists a quasi-isometry between them. The special case of a quasi-isometric embedding with domain a line (segment, ray, or bi-infinite) is a quasi-geodesic.

Remark 2.3. To simplify notation, in place of Equation 2.1 we sometimes write

\[
d_X(x, x') \approx_{K, L} d_Y(y, y').
\]

Similarly, we write \( d_X(x, x') \preceq_{K, L} d_Y(y, y') \) to imply \( d_X(x, x') \leq K d_Y(y, y') + L \). When the constants \( K, L \) are not important, they may be omitted.
2.2. Curve Complex and Separating Curve Complex. Let $S = S_{g,n}$ be a compact surface of genus $g$ with $n$ boundary components. The complexity of $S$, denoted $\xi(S)$, is a topological invariant defined to be $3g - 3 + n$. An isotopy class of a simple closed curve $\gamma$ on $S$ is essential if it does not have a representative bounding a disk or an annulus. We will only consider hyperbolic compact surfaces, and to streamline the exposition we will use the term curve to refer to a geodesic representative of an isotopy class of an essential simple closed curve.

A multicurve is a (possibly empty) set of curves which are pairwise disjoint. For $S$ any surface with positive complexity, the curve complex of $S$, denoted $C(S)$, is the simplicial complex obtained by associating to each curve a 0-cell, and more generally a $k$–cell to each multicurve consisting of $k+1$ curves. In the special case of low complexity surfaces which do not admit disjoint curves, we relax the notion of adjacency to allow edges between vertices corresponding to curves that intersect minimally on the given surface. Along similar lines, given a curve $\gamma \subset S$, let $N(\gamma)$ denote a regular annular neighborhood of $\gamma$. Then, the annular complex $C(\gamma)$ is defined as follows: vertices correspond to isotopy classes, relative to the boundary, of arcs connecting the two boundary components of $N(\gamma)$, and edges connect isotopy classes (rel. boundary) of arcs which have representatives with disjoint interiors. The annular complex measures the twisting of arcs around the core curve $\gamma$, and is quasi-isometric to $\mathbb{Z}$. In fact, for any fixed points $x$ and $y$ on different boundary components of $N(\gamma)$, and any arc $\alpha \in C(\gamma)$ there exists an adjacent arc $\beta \in C(\gamma)$ with endpoints $x, y$.

A (multi)curve $\gamma \subset S$ is said to be separating if $S \setminus \gamma$ consists of a disjoint union of at least two connected essential subsurfaces, and nonseparating otherwise. Given this distinction, we define the separating curve complex, denoted $C_{\text{sep}}(S)$, to be the restriction of the curve complex to the subset of separating curves. For certain low complexity surfaces such as $S_{2,0}$, as defined $C_{\text{sep}}(S)$ is a totally disconnected space. Accordingly, in such circumstances we relax the definition of connectivity and define two separating curves to be connected by an edge if the curves intersect minimally on the given surface. More generally, in such low complexity situations, a set of $k + 1$ separating curves form a $k$–simplex if the separating curves pairwise intersect minimally. For our purposes we will only be interested in the 1–skeleton of $C_{\text{sep}}(S)$. In particular, for the case of $S_{2,0}$, two separating curves in $C_{\text{sep}}(S_{2,0})$ are connected if and only if they intersect four times.

2.3. Essential Subsurfaces, Projections. An essential subsurface $Y$ of a surface $S$ is a disjoint union of complexity at least one subsurface which is a union of (not necessarily all) complementary components of a multicurve. Note that due to the requirement that connected components of essential subsurface have complexity at least one, annuli are not essential subsurfaces nor are they connected components of essential subsurfaces.
As multicurves are in fact sets of isotopy classes of essential simple closed curves, essential subsurfaces are only defined up to isotopy. However, in light of our assumption that curves always refer to geodesic representatives of their isotopy classes, we will similarly assume fixed representatives of all essential subsurfaces by assuming that all boundary components of essential subsurfaces are geodesics. An essential subsurface $Y \subseteq S$ is proper if $Y \subsetneq S$. Two essential surfaces $W, V \subset S$ are disjoint if they have empty intersection, and intersecting otherwise. We are implicitly using the fact that two geodesics on hyperbolic surfaces never form a bigon, $[3]$.

An essential subsurface $Y \subset S$ is called separating if the multicurve $\partial Y$ contains a separating multicurve, and nonseparating otherwise. In particular, note that the entire surface is always a nonseparating essential subsurface.

Figure 1. Examples of subsurface projections.

Given a multicurve $\alpha \subset C(S)$ and a connected essential subsurface $Y \subset S$ such that $\alpha$ intersects $Y$, we can define the projection of $\alpha$ to $2C(Y)$, denoted $\pi_{C(Y)}(\alpha)$, to be the collection of vertices in $C(Y)$ obtained by surgering the arcs of $\alpha \cap Y$ along $\partial Y$ to obtain simple closed curves in $Y$. Specifically, the intersection $\alpha \cap Y$ consists of a (possibly empty) submulticurve $\beta \subset \alpha$ contained in $Y$ as well as a disjoint union of arc subsegments of $\alpha$ with the endpoints of the arcs on boundary components of $Y$. We define the projection $\pi_{C(Y)}(\alpha) \subset C(Y)$ to be the union of the submulticurve $\beta$ in conjunction with all curves obtained by the following process. For an arc $\alpha \cap Y$ consider the union of the arc and the components of $\partial Y$ incident to the endpoints of the arc. Then take a regular neighborhood of this union and define the subsurface projection $\pi_{C(Y)}(\alpha)$ to include all essential curves in the boundary of this regular neighborhood. See Figure 1
for an example. In [6] is is shown that subsurface projections are coarsely well-defined. Note that the projection $\pi_C(Y)$ is only defined on curves intersecting $Y$. To simplify notation, when measuring distance in the image subsurface complex, we write $d_C(Y)(\alpha_1,\alpha_2)$ as shorthand for $d_C(Y)(\pi_C(Y)(\alpha_1),\pi_C(Y)(\alpha_2))$. In particular, for this distance to be well-defined, $\alpha_1$ and $\alpha_2$ must both intersect $Y$.

2.4. Combinatorial Complexes. A combinatorial complex for $\text{MCG}(S)$, or for short a combinatorial complex, $G(S)$, is any graph with vertices defined in terms of multicurves on a surface and edge relations defined in terms of upper bounds on intersection numbers between the corresponding multicurves. In addition, we will assume that combinatorial complexes admit an isometric action of the mapping class group, $\text{MCG}(S)$. Examples of combinatorial complexes of $\text{MCG}(S)$ include the separating curve complex, the arc complex, the pants complex, the marking complex, as well as many others in the literature.

A hole for $G(S)$ is defined to be any connected essential subsurface or annulus such that every vertex of the combinatorial complex has non-trivial subsurface projection into it. For example, for the arc complex holes are precisely all connected subsurfaces such that $\partial S \subset \partial Y$. On the other hand, for the complex $C_{\text{sep}}(S_{2,0})$ holes are precisely the set of nonseparating essential subsurfaces.

The central idea in [7], which is also implicit in [1], is that distance in a combinatorial complex is approximated by summing over the distances in the subsurface projections to the curve complexes of holes. In particular, if a complex has disjoint holes then the complex admits non-trivial quasi-flats, and hence cannot be $\delta$-hyperbolic. Conversely, if a combinatorial complex has the property that no two holes are disjoint, then assuming a couple of additional Masur–Schleimer axioms the complex is $\delta$-hyperbolic.

2.5. Marking Complex and Hierarchy paths. A complete marking $\mu$ on $S$ is a collection of base curves and transverse curves subject to the following conditions:

1. The set of base curves $\{\gamma_1, ..., \gamma_n\}$ forms a top-dimensional simplex in $C(S)$. Equivalently $n = \xi(S)$.

2. For each base curve $\gamma_i$, let $S_{\gamma_i}$ denote the unique connected complexity one essential subsurface in the complement $S \setminus \gamma_i$. Then, each base curve $\gamma_i$ has a corresponding transversal curve $t_i$, transversely intersecting $\gamma_i$, such that $t_i$ intersects $\gamma_i$ once if $S_{\gamma_i}$ is topologically $S_{1,1}$ and twice if $S_{\gamma_i}$ is topologically $S_{0,4}$.

A complete marking $\mu$ is said to be clean if in addition each transverse curve $t_i$ is disjoint from all other base curves $\gamma_j$. Two complete markings $\mu, \mu'$ are compatible if they have the same base curves and moreover $\forall i$, the distance in the annular complex $d_C(\gamma_i)(t_i, t'_i)$ is minimal over all choices of $t'_i$. See [6] for technical details regarding the distance in annular complex. For our purposes
it suffices to use the fact that traveling in the annular complex is accomplished by taking an arc in a regular neighborhood of the annulus and Dehn twisting it about the core curve of the annulus. In [6] it is shown that there is a bound, depending only on the topological type of $S$, on the number of clean complete markings which are compatible with any given complete marking.

Let $\mu$ denote a clean complete marking with curve pair data $(\gamma_i, t_i)$, then we define an elementary move to be one of the following two operations applied to the marking $\mu$:

(1) Twist: For some $i$, we replace $(\gamma_i, t_i)$ with $(\gamma_i, t'_i)$ where $t'_i$ is the result of one full or half twist (when possible) of $t_i$ around $\gamma_i$.

(2) Flip: For some $i$ we interchange the base and transversal curves. After a flip move, the resulting complete marking may no longer be clean, in which case as part of the flip move we then replace the non-clean complete marking with a compatible clean complete marking. Since there is a uniform bound on the number of clean complete markings which are compatible with any given complete marking, a flip move is coarsely well-defined.

The Marking Complex, $\mathcal{M}(S)$, is defined to be the graph formed by taking clean complete markings of $S$ to be vertices and connecting two vertices by an edge if they differ by an elementary move.

In [6] a 2-transitive family of quasi-geodesics in $\mathcal{M}(S)$ called resolutions of hierarchies are developed. In broad strokes, hierarchies are defined inductively as a union of geodesics of multicurves in the curve complexes of essential subsurfaces or annuli, while resolutions of hierarchies are quasi-geodesics in the marking complex associated to a hierarchy. By abuse of notation, throughout this paper we will refer to resolutions of hierarchies as hierarchies. The construction of hierarchies is technical, although for our purposes the following theorem recording some of their properties suffices.

**Theorem 2.4** ([6] Section 4; Hierarchies). For $S = S_{g,n}$, and for all $\mu, \nu \in \mathcal{M}(S)$, there exists a hierarchy path $\rho = \rho(\mu, \nu) : [0, n] \to \mathcal{M}(S)$ with $\rho(0) = \mu$, $\rho(n) = \nu$. Moreover, $\rho$ is a quasi-geodesic with constants depending only on the topological type of $S$, with the following properties:

H1: The hierarchy $\rho$ shadows a $C(S)$ geodesic of multicurves, $g_S$, from a multicurve $a \subset \text{base}(\mu)$ to a multicurve $b \subset \text{base}(\nu)$, called the main geodesic of the hierarchy. That is, there is a monotonic map $\phi : \rho \to g_S$ such that $\forall i$, $\phi(\rho(i)) \subset \text{base}(\rho(i))$.

H2: There is a constant $M_1$ such that if an essential subsurface or an annulus $Y \subset S$ satisfies $d_C(Y)(\mu, \nu) > M_1$, then there is a maximal connected interval $I_Y = [t_{Y,1}, t_{Y,2}]$ and a geodesic of multicurves $g_Y$ in $C(Y)$ from a submulticurve in $\text{base}(\rho(t_{Y,1}))$ to a submulticurve in $\text{base}(\rho(t_{Y,2}))$ such that for all $t_{Y,1} \leq t \leq t_{Y,2}$, $\partial Y$ is a submulticurve
in base($\rho(t)$) and $\rho|_{I_Y}$ shadows the geodesic $g_Y$. Such a subsurface $Y$ is called a component domain of $\rho$. By convention the entire surface $S$ is always considered a component domain.

The next theorem contains a quasi-distance formula for $M(S)$, which serves as both the motivation for as well an important ingredient in proving Theorem 1.1.

**Theorem 2.5** ([6] Theorem 6.12; Quasi-Distance Formula). For $S = S_{g,n}$, there is constant $C_0$ such that for all $c \geq C_0$ there exists constants $K(c), L(c)$ such that $\forall \alpha, \beta \in M(S)$ we have the following quasi-isometry:

$$\sum_{Y \subseteq S} \{d_C(Y)(\alpha, \beta)\}_c \approx_{K, L} d_{M(S)}(\alpha, \beta),$$

where the sums are over all connected essential subsurfaces $Y$ or annuli.

**Remark 2.6.** Note that holes for the marking complex are precisely all essential subsurfaces $Y$ or annuli. Hence the sums in Theorem 2.5 are sums over all holes.

![Figure 2. Farey Graph with some vertices labeled.](image)

2.6. **Farey Graph.** The Farey graph is a classical graph which has vertices corresponding to elements of $\mathbb{Q} \cup \{\infty = \frac{1}{0}\}$, and edges between two rational numbers in lowest terms $\frac{p}{q}$ and $\frac{r}{s}$ if $|qs - qr| = 1$. The Farey graph can be drawn as...
an ideal triangulation of the unit disk as in Figure 2. A nice feature of the Farey graph is the so called Farey addition property which ensures that if rational number \( \frac{p}{q} \) and \( \frac{r}{s} \) are connected in the Farey graph, then there is an ideal triangle in the Farey graph with vertices \( \frac{p}{q}, \frac{r}{s}, \) and \( \frac{p+r}{q+s} \). The curve complexes \( C(S_{0,4}) \) and \( C(S_{1,1}) \) are isomorphic to the Farey graph. The isomorphism is given by sending the meridional curve of the surfaces to \( \frac{1}{0} \), the longitudinal curve of the surfaces to \( \frac{0}{1} \), and more generally sending the \((p, q)\) curve to \( \frac{p}{q} \).

3. Proof of Theorem 1.1

In Subsection 3.1 we will show that \( C_{sep}(S_{2,0}) \) has a quasi-distance formula as in Theorem 1.1. Then, in Subsection 3.2, using the quasi-distance formula for \( C_{sep}(S_{2,0}) \) we show that the Masur–Schleimer proof for \( \delta \)-hyperbolicity of a combinatorial complex found in [7] applies to \( C_{sep}(S_{2,0}) \), thus proving that \( C_{sep}(S_{2,0}) \) is \( \delta \)-hyperbolic.

3.1. \( C_{sep}(S_{2,0}) \) has a quasi-distance formula. We begin by recalling a lemma of [7] which in particular ensures a quasi-lower bound for a quasi-distance formula for \( C_{sep}(S_{2,0}) \).

Lemma 3.1 (Theorem 5.10). For \( S = S_{g,n} \), and \( G(S) \) any combinatorial complex, there is constant \( C_0 \) such that for all \( c \geq C_0 \) there exists constants \( K(c), L(c) \) such that for all \( \alpha, \beta \in G(S) \) we have the following relationship:

\[
\sum_{Y \text{ a hole for } G(S)} \{d_{C(Y)}(\alpha, \beta)\} c \lesssim d_{C(S)}(\alpha, \beta).
\]

In light of Lemma 3.1 to prove a quasi-distance formula for \( C_{sep}(S_{2,0}) \) it suffices to obtain a quasi-upper bound on \( C_{sep}(S_{2,0}) \) distance in terms of the sum of subsurface projections to holes. As motivated by [7], our approach for doing so will be by relating markings to separating curves and more generally marking paths to paths in the separating curve complex. In the rest of this subsection let \( S = S_{2,0} \).

Let \( \mu \in \mathcal{M}(S) \). Presently we describe a coarsely well-defined mapping

\[
\phi: \mathcal{M}(S) \rightarrow 2^C_{sep}(S)
\]

If base(\( \mu \)) contains a separating curve \( \gamma_i \), then we define \( \phi(\mu) \) to contain \( \gamma_i \). On the other hand, if all three base curves of \( \mu, \gamma_1, \gamma_2, \gamma_3 \), are nonseparating curves, then for any \( i, j, k \in \{1, 2, 3\} \), \( i \neq j \neq k \neq i \), denote the essential subsurface \( S_{i,j} := S \setminus \gamma_i, \gamma_j \simeq S_{0,4} \). Note that \( C(S_{i,j}) \) is a Farey graph containing the adjacent curves \( \gamma_k \) and \( t_k \). Let \( o_k \) be a curve in \( S_{i,j} \) such that \( \gamma_k, t_k, o_k \) form a triangle in \( C(S_{i,j}) \). Note that \( o_k \) is not uniquely determined by this condition; in fact, there are exactly two possibilities for \( o_k \), which we denote \( o_k^\pm \). Note that \( d_{C(S_{i,j})}(o_k^+, o_k^-) = 2 \). Then, in this case (assuming that none of the base
curves of \( \mu \) are separating curves), define \( \phi(\mu) \) to contain all the curves in the set \( \{ t_i o_k^\pm, t_j, o_j, t_k, o_k^\pm \} \) which are separating curves of \( S \). The following lemma ensures that the mapping \( \phi \) always has non-trivial image.

**Lemma 3.2.** With the notation from above, assume \( \text{base}(\mu) \cap \mathcal{C}_{\text{sep}}(S) = \emptyset \), and let \( \gamma_k, t_k, o_k \) form a triangle in the Farey graph \( \mathcal{C}(S_{i,j}) \). Then one and only one of the curves \( \gamma_k, t_k \) and \( o_k \) is separating curves of \( S \).

**Proof.** The subsurface \( S_{i,j} \) has four boundary components which glue up in pairs inside the ambient surface \( S \). Any curve \( \alpha \in \mathcal{C}(S_{i,j}) \) gives rise to a partition of the four boundary components of \( S_{i,j} \) into pairs given by pairing boundary components in the same connected component of \( S_{i,j} \setminus \alpha \). In total there are three different ways to partition the four boundary components of \( S_{i,j} \) into pairs. Precisely one of the three partitions has the property that any curve that gives rise to the given partition is a separating curve. The partition corresponding to a \( (p,q) \)-curve on \( S_{i,j} \) is entirely determined by \( \frac{p \mod 2}{q \mod 2} \), with \( \frac{0}{0} = \frac{1}{1} \). Using the Farey addition property it follows that each triangle in the Farey graph has exactly one representative from each equivalence class. The lemma follows. \( \square \)

![Figure 3](image)

**Figure 3.** A marking \( \mu \) on \( S = S_{2,0} \) with no separating curves.

Notice that \( d_{\mathcal{C}_{\text{sep}}(S)}(t_1, t_2) = 1 \).

The following theorem ensures that the mapping \( \phi: \mathcal{M}(S) \to \mathcal{C}_{\text{sep}}(S) \) is coarsely well-defined.

**Theorem 3.3.** Using the notation from Claim 3.2 above, let \( t_i, t_j \) be transversals which are separating curves. Then \( t_i \) and \( t_j \) are connected in the separating curve complex \( \mathcal{C}_{\text{sep}}(S) \). Similarly, if \( t_i \) and \( o_j \), or \( o_i \) and \( o_j \) are separating curves the same result holds.
Proof. We will prove the first case; the similar statement follows from the same proof. Specifically, we will show that the separating curves \( t_i, t_j \) intersect four times. Up to action of \( \text{MCG}(S) \), there is only one picture for a marking \( \mu \) which does not contain a separating base curve, as presented in Figure 3. Without loss of generality we can assume \( t_i = t_1 \) and \( t_j = t_2 \). Notice that in the subsurface \( S_{2,3} \), as in Figure 3, the base curve \( \gamma_1 \) corresponds to the meridional curve \( \frac{1}{l} \), and similarly in the subsurface \( S_{1,3} \) the base curve \( \gamma_2 \) also corresponds to the meridional curve \( \frac{1}{l} \). Since \( t_1 \) is connected to \( \gamma_1 \) in the Farey graph \( \mathcal{C}(S_{2,3}) \) it follows that \( t_1 \in \mathcal{C}(S_{2,3}) \) is a curve of the form \( \frac{1}{n} \) for some integer \( n \). Similarly, \( t_2 \in \mathcal{C}(S_{1,3}) \) is a curve of the form \( \frac{m}{n} \) for some integer \( m \). As in the examples in Figure 3 it is easy to draw representatives of the two curves \( t_1 \) and \( t_2 \) which intersect four times. □

The following lemma says that our coarsely well-defined mapping \( \phi \) which associates a separating curve to a clean complete marking is natural with respect to elementary moves in the marking complex.

**Lemma 3.4.** If \( d_{\mathcal{M}(S)}(\mu, \nu) \leq 1 \) then \( \phi(\mu) \cap \phi(\nu) \neq \emptyset \).

**Proof.** First assume \( \mu \) and \( \nu \) differ by a twist move applied to the pair \((\gamma_i, t_i)\). If \( \mu \) has a separating base curve, then so does \( \nu \) as twists do not affect base curves. Then, by definition this separating base curve is in the intersection \( \phi(\mu) \cap \phi(\nu) \).

On the other hand, if \( \mu \) has no separating base curves, then by Claim 3.2 either \( t_j \) or \( o_j \), for \( i \neq j \), is a separating curve. In either case this separating curve is in the intersection \( \phi(\mu) \cap \phi(\nu) \), and we are also done.

Next assume \( \mu \) and \( \nu \) differ by a flip move applied to the pair \((\gamma_i, t_i)\). Recall that after the flip move is performed one must pass to a compatible clean marking. Specifically, if \( \mu = \{ (\gamma_i, t_i), (\gamma_j, t_j), (\gamma_k, t_k) \} \). Then \( \nu = \{ (t_i, \gamma_i), (\gamma_j, t'_j), (\gamma_k, t'_k) \} \), where the transversals \( t'_j, t'_k \) are obtained by passing to a compatible clean marking if necessary. If \( \gamma_i, \gamma_j \) or \( \gamma_k \) is a separating base curve, we are done. Finally, if none of the base curves of \( \mu \) are separating curves, then we are also done as again by Claim 3.2 either \( t_j \) or \( o_j \), is a separating curve. In either case this separating curve is in the intersection \( \phi(\mu) \cap \phi(\nu) \). □

Considering our coarsely well-defined mapping \( \phi: \mathcal{M}(S) \rightarrow C_{\text{sep}}(S) \), in conjunction with Lemma 3.4 we have the following procedure for finding a path between any two separating curves. Given \( \alpha, \beta \in C_{\text{sep}}(S) \), complete the separating curves into clean complete markings \( \mu \) and \( \nu \) such that \( \alpha \in \text{base}(\mu) \) and \( \beta \in \text{base}(\nu) \). Then construct a hierarchy path \( \rho \) in \( \mathcal{M}(S) \) between \( \mu \) and \( \nu \). Applying the mapping \( \phi \) to our hierarchy path \( \rho \), yields a path of separating curves in \( C_{\text{sep}}(S) \) between the separating curves \( \alpha \) and \( \beta \). By construction, the length of the obtained path in \( C_{\text{sep}}(S) \) between the separating curves \( \alpha \) and \( \beta \) has length quasi-bounded above by the length of the marking path \( \rho \). In fact, in the following corollary we will use this procedure to obtain a quasi-upper
bound on $C_{\text{sep}}(S_{2,0})$ distance in terms of the sum of subsurface projection to holes. Note that together with Lemma 3.1 Corollary 3.5 gives a quasi-distance formula for $C_{\text{sep}}(S)$, thus completing the proof of theorem 1.1. Recall that the set of set of holes for $C_{\text{sep}}(S)$ is precisely the set of all nonseparating essential subsurfaces, or equivalently all essential subsurfaces whose boundary does not contain a separating curve.

**Corollary 3.5.** For $S = S^2_0$, there is a constant $K_0$ such that $\forall k \geq K_0$ there exists quasi-isometry constants such that $\forall \alpha, \beta \in C_{\text{sep}}(S)$:

$$d_{C_{\text{sep}}(S)}(\alpha, \beta) \lesssim \sum_{Y \text{ a hole for } C_{\text{sep}}(S)} \{d_{C}(Y)(\alpha, \beta)\}_k.$$  

**Proof.** As noted, we have a quasi-upper bound on $d_{C_{\text{sep}}(S)}(\alpha, \beta)$ given by the length of a hierarchy path $\rho: [0, n] \rightarrow M(S)$ such that $\alpha \in \text{base}(\rho(0))$ and $\beta \in \text{base}(\rho(n))$. In other words, by Theorems 2.4 and 2.5 we have already have a quasi-upper bound of the form:

$$d_{C_{\text{sep}}(S)}(\alpha, \beta) \lesssim \sum_{\xi(\gamma) \geq 1, \text{ or } Y \text{ an annulus}} \{d_{C}(Y)(\alpha, \beta)\}_k.$$  

Hence, it suffices to show that for all components domains $Y$ in the above sum which are not holes of $C_{\text{sep}}(S)$, there is a uniform bound on $\text{diam}_{C_{\text{sep}}(S)}(\phi(I_Y))$, where $I_Y = [t_{Y,1}, ..., t_{Y,m}]$ is as in property [H2] of Theorem 2.4. Since holes for $C_{\text{sep}}(S)$ consist of all nonseparating essential subsurfaces, we can assume $Y$ is either an annulus or a separating essential subsurface. Furthermore, since $\phi$ is coarsely well-defined it suffices to show that for any such $Y$ that the intersection $\bigcap_{j=1}^{m} \phi(\rho(t_{Y,j}))$ is nonempty.

First consider the case of $Y$ an annulus. In this case, the subpath of $\rho$ in the marking complex corresponding to $I_Y$ consists of a sequence of clean complete markings $\rho(t_{Y,1}), ..., \rho(t_{Y,m})$, such that any adjacent markings $\rho(t_{Y,j}), \rho(t_{Y,j+1})$ differ by a twist move along a fixed base curve $\gamma_i$ (which is the the core of the annulus $Y$). Then, exactly as in the first paragraph of the proof of Lemma 3.4 it follows that $\bigcap_{j=1}^{m} \phi(\rho(t_{Y,j}))$ is nonempty, thus completing the proof for $Y$ an annulus.

On the other hand, if $Y$ is a separating essential subsurface, or equivalently $\partial Y$ contains a separating curve $\alpha \in C_{\text{sep}}(S)$, then by Theorem 2.4 property [H2], the separating curve $\alpha$ is contained as a base curve in all markings $\rho(t_{Y,j})$ for all $j \in \{1, ..., m\}$. In particular, $\alpha \in \bigcap_{j=1}^{m} \phi(\rho(t_{Y,j}))$. This completes the proof. \qed

3.2. $C_{\text{sep}}(S_{2,0})$ is $\delta$-hyperbolic. In Section 13 of [7], sufficient axioms are established for implying a combinatorial complex admits a quasi-distance formula and furthermore is $\delta$-hyperbolic. The first axiom is that no two holes for the combinatorial complex are disjoint. This is easily verified for $C_{\text{sep}}(S_{2,0})$. 
The rest of the axioms are related to the existence of an appropriate marking path \( \{ \mu_i \}_{i=0}^{\infty} \subset \mathcal{M}(S) \) and a corresponding well suited combinatorial path \( \{ \gamma_i \}_{i=0}^{\infty} \subset C(S) \). In particular, there is a strictly increasing reindexing function \( r : [J,K] \to [0, N] \) with \( r(J) = 0 \) and \( r(K) = N \). In the event that one uses a hierarchy as a marking path, the rest of the axioms can be simplified as follows:

1. (Combinatorial) There is a constant \( C_1 \) such that for all \( i \),
   \[
   d_{C(Y)}(\gamma_i, \mu_{r(i)}) < C_1
   \]
   for every hole \( Y \), and moreover \( d_{C(S)}(\gamma_i, \gamma_{i+1}) < C_1 \).
2. (Replacement) There is a constant \( C_2 \) such that:
   - [R1] If \( Y \) is a hole and \( r(i) \in I_Y \), then there is a vertex \( \gamma' \in G(S) \)
     with \( \gamma' \subset Y \) and \( d_{G(S)}(\gamma, \gamma') < C_2 \).
   - [R2] If \( Y \) is a non-hole and \( r(i) \in I_Y \), then there is a vertex
     \( \gamma' \in G(S) \) with \( \gamma' \subset Y \) or \( \gamma' \subset Y \) \( \gamma \) and \( d_{G(S)}(\gamma, \gamma') < C_2 \).
3. (Straight) There exist constants such that for any subinterval \( [p, q] \subset [0, K] \) with the property that \( d_{C(Y)}(\mu_{r(p)}, \mu_{r(q)}) \) is uniformly bounded for all non-holes, then \( d_{C(S)}(\gamma_p, \gamma_q) < d_{C(S)}(\gamma_p, \gamma_q) \).

Presently we will show that in the case of the separating curve complex \( C_{sep}(S_{2,0}) \) all of above axioms with the exception of axiom [R2] hold. Let \( \rho = \{ \mu_i \}_{i=0}^{\infty} \) be a hierarchy path between two clean complete markings each containing a separating base curve. Then, as in the proof of Corollary 3.5, define the combinatorial path \( \{ \gamma_i \}_{i=0}^{\infty} \subset C_{sep}(S_{2,0}) \) using the coarsely well-defined map \( \phi : \mathcal{M}(S_{2,0}) \to C_{sep}(S_{2,0}) \) applied to the hierarchy \( \rho \). Let the reindexing function \( r \) be defined by:

\[
r(i) = \max_{[0,N]} \{ j | \gamma_i \in \phi(\mu_j) \}.
\]

Given this setting, the first clause of the combinatorial axiom is immediate from the definition of \( \phi \), while the moreover clause follows from Lemma 3.4 and the fact that \( \phi \) is coarsely well-defined. Similarly, the straight axiom follows from the properties of hierarchy paths of Theorem 2.4. Replacement axiom [R1] also holds for if \( Y \) is a hole, then \( \partial Y \) contains at most two nonseparating curves. Then for all markings \( \mu \in I_Y \), \( base(\mu) \) contains the at most two nonseparating curves \( \partial Y \). Let \( \gamma_i \) be a base curve of \( \mu \) not in \( \partial Y \). Then, by Claim 3.2, one of \( \gamma_i, t_i, \) or \( o_i \) is a separating curve properly contained in \( Y \). Claim 3.2 ensures that exactly one of the three curves \( \gamma_i, t_i, \) or \( o_i \) is a separating curve. On the other hand, axiom [R2] fails as if \( Y \) is an essential subsurface which is a nonhole then perforce by topological considerations \( \partial Y \in C_{sep}(S_{2,0}) \). In this case, there cannot exist any separating curve properly contained in either \( Y \) or \( S_{2,0} \setminus Y \).

Nonetheless, while the Masur–Schleimer axioms fail due to the failure of axiom [R2], Masur and Schleimer’s proof that a combinatorial complex satisfying the Masur-Schleimer axioms is \( \delta \)-hyperbolic carries through in the case
of \( C_{\text{sep}}(S_2, 0) \). In Section 14 of \[7\], Masur–Schleimer prove that a combinatorial complex satisfying the Masur–Schleimer axioms satisfies a quasi-distance formula. Then, in Section 20 of \[7\], they go on to prove that in addition a combinatorial complex satisfying the Masur–Schleimer axioms is also \( \delta \)-hyperbolic. To be sure, in Section 20 in the course of proving \( \delta \)-hyperbolicity, the replacement axiom is not in fact ever directly needed. Instead, the replacement axiom is only used in Section 14 to prove the existence of a quasi-distance formula, and then this formula in turn is used in Section 20 along with other axioms to prove \( \delta \)-hyperbolicity. Consequently, since in this paper we have obtained a direct proof of the existence of a quasi-distance formula for \( C_{\text{sep}}(S_2, 0) \), it follows from the argument in Section 20 of \[7\] that \( C_{\text{sep}}(S_2, 0) \) is \( \delta \)-hyperbolic.

3.3. A quasi-distance formula for \( C_{\text{sep}}(S) \) in general? Considering the arguments in Section 3.1 naive consideration suggests appropriate modified may provide a proof of a quasi-distance formula for \( C_{\text{sep}}(S) \) in general. However, this is certainly not immediate. Specifically, an explicit construction in \[9\] implies that, for high enough genus, there exist clean complete markings of closed surfaces which are arbitrarily far (with respect to elementary moves) from any clean complete marking containing a separating base or transversal curve. This is in stark contrast with the situation in \( C_{\text{sep}}(S_2, 0) \), for which we make strong use of the fact that any clean complete marking is distance at most one from a clean complete marking containing a separating base or transversal curve.

REFERENCES