MATH 101A: HOMEWORK

1. Homework 1. Answers

1.1. Prove that a finite monoid with the left cancellation property is a group. (A monoid M has the *left cancellation property* if for all $x, y, z \in M$, $xy = xz$ implies $y = z$. Right cancellation is analogous.)

There are basically two proofs that students gave. I had in mind the second proof.

Proof 1. Let $x \in M$. Then left multiplication by x gives a mapping

$$\lambda_x : M \to M$$

which is $1 - 1$ by left cancellation. Since M is finite, λ_x must also be onto. Therefore, there exists $y \in M$ so that $\lambda_x(y) = xy = e$. So, every element has a right inverse. This implies that M is a group. □

Proof 2. Let $x \in M$. Then the powers $x^n \in M$ for positive integers n cannot all be different since M is finite. Therefore, $x^n = x^m$ for some $m > n > 0$. Writing this as

$$x^n e = x^n = x^m = x^n x^{m-n}$$

we see by left cancellation that $x^{m-n} = e$. Since $m - n > 0$ we can write this as

$$x(x^{m-n-1}) = e = (x^{m-n-1})x.$$

This shows that x^{m-n-1} is a two-sided inverse for x. Therefore, every element of M has a two-sided inverse. So, M is a group. □

1.2. Find an example of a solvable group which is not metabelian. (A group G is *metabelian* if it has an abelian normal subgroup A so that G/A is also abelian.)

Almost everybody had the same example $G = S_4$. I had in mind the group $T(5, \mathbb{Z})$.

Proof. The easy part was to show that S_4 is solvable. The hard part was to show that it is not metabelian. The key point that several of you got is that, in order to show that a group is not metabelian, you need

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to find four elements of the group: a, b, c, d so that the commutator of commutators is nontrivial:

$$[[a, b], [c, d]] \neq e$$

In a metabelian group, all commutators lie in G' which is abelian. Therefore all commutators commute with each other. The calculation

$$[[(12), (13)], [(12), (14)]] = [(123), (124)] = (12)(34) \neq e$$

therefore shows that S_4 is not metabelian. $\square$