MATH 101A: HOMEWORK

2. Homework 2

2.1. If G acts on a set X show that the intersection of all stabilizer subgroups is a normal subgroup $N \trianglelefteq G$. Show that the quotient group G/N acts on X in an *effective* way, i.e., every element of G/N moves at least one element of X. Correction: Every nontrivial element of G/N moves at least one element of X. Give an example where N and G/N are both nontrivial.

2.1.1. *N is normal.* To show that N is normal it suffices to show that $gNg^{-1} \leq N$. (As several students pointed, this implies that $gNg^{-1} = N$.) To show this, take any $n \in N$. Then n lies in every stabilizer. In particular $n \in G_{g^{-1}x}$. So, $n(g^{-1}x) = g^{-1}x$, which implies that

$$\left(gng^{-1}\right)x = gn(g^{-1}x) = g(g^{-1}x) = ex = x$$

Therefore $gng^{-1} \in G_x$ for every $x \in X$ which implies that $gng^{-1} \in N$.

2.1.2. *G/N acts effectively on X.* This is actually two statements. It says that G/N acts on X and the action is effective. Students gave two different proofs by computation or by the universal property of the quotient.

Proof 1. First, you get an action of G/N on X by

$$gNx = gx$$

This is well defined since $gx = hx$ for any two elements g, h of the same coset. (If $gN = hN$ then $g^{-1}h \in N$. So $g^{-1}hx = x$ which implies that $gx = hx$ for all $x \in X$.) It is clear that this defines an action.

Let gN be a nontrivial element of G/N. Thus $g \notin N$. Then, by definition of N, this implies that $gx \neq x$ for some $x \in X$. Therefore, $gNx = gx \neq x$ and gN moves x. So the action of G/N on X is effective. □

Proof 2. The action of G on X is, by definition, a homomorphism

$$\pi : G \to Perm(X)$$

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N is the set of elements of G which fix every element of X. Therefore, N is the kernel of π. This implies that there is an induced homomorphism

$$\overline{\pi} : G/N \to Perm(X)$$

given by $\overline{\pi}(gN) = \pi(g)$. The kernel of the induced homomorphism is the identity eN of G/N. Therefore, the action of G/N on X is effective. □

2.1.3. *Nontrivial example.* Several students gave the general example of a nonabelian p-group P acting on itself by conjugation. The kernel of this action is the center of P which we know is nontrivial. The quotient $P/Z(P)$ is also nontrivial since P is nonabelian.

Most students gave the specific example of the dihedral group D_4 (group of symmetries of the square). Since this is nonabelian of order 8, it is a special case of this.

Some students took nonabelian groups acting on abelian but not central normal subgroups by conjugation. However, they did not state it in such generality.

2.2. Let G act on the set S of all subgroups of G by conjugation. Show that the functions $d : S \to S$ and $c : S \to S$ given by $d(H) = H'$ (the commutator subgroup) and $c(H) = C_G(H)$ (the centralizer of H in G) are morphisms of G-sets.

Everybody realized that the problem is to show that $gH'g^{-1} = (gHg^{-1})'$ and $C_G(gHg^{-1}) = gC_G(H)g^{-1}$ for all $g \in G, H \le G$.

2.2.1. $gH'g^{-1} = (gHg^{-1})'$. Many students did not realize that the elements of H' are *products* of commutators. Thus

$$H' = \langle [x, y] | x, y \in H \rangle$$

and

$$\begin{aligned} gH'g^{-1} &= g\langle [x, y] | x, y \in H \rangle g^{-1} = \left\langle g[x, y]g^{-1} | x, y \in H \right\rangle \\ &= \left\langle [gxg^{-1}, gyg^{-1}] | x, y \in H \right\rangle = \left(gHg^{-1}\right)'. \end{aligned}$$

2.2.2. $C_G(gHg^{-1}) = gC_G(H)g^{-1}$. Most student got this correct:

$$\begin{aligned} x \in C_G(gHg^{-1}) &\iff xghg^{-1} = ghg^{-1}x \ \forall h \in H \\ &\iff g^{-1}xgh = hg^{-1}xg \ \forall h \in H \\ &\iff g^{-1}xg \in C_G(H) \\ &\iff x \in gC_G(H)g^{-1}. \end{aligned}$$

Therefore, $C_G(gHg^{-1}) = gC_G(H)g^{-1}$.