2. Homework 2

2.1. If $G$ acts on a set $X$ show that the intersection of all stabilizer subgroups is a normal subgroup $N \trianglelefteq G$. Show that the quotient group $G/N$ acts on $X$ in an effective way, i.e., every element of $G/N$ moves at least one element of $X$. Correction: Every nontrivial element of $G/N$ moves at least one element of $X$. Give an example where $N$ and $G/N$ are both nontrivial.

2.1.1. $N$ is normal. To show that $N$ is normal it suffices to show that $gNg^{-1} \leq N$. (As several students pointed, this implies that $gNg^{-1} = N$.) To show this, take any $n \in N$. Then $n$ lies in every stabilizer. In particular $n \in G_{g^{-1}x}$. So, $n(g^{-1}x) = g^{-1}x$, which implies that

$$(gng^{-1})x = gn(g^{-1}x) = g(g^{-1}x) = ex = x$$

Therefore $gng^{-1} \in G_x$ for every $x \in X$ which implies that $gng^{-1} \in N$.

2.1.2. $G/N$ acts effectively on $X$. This is actually two statements. It says that $G/N$ acts on $X$ and the action is effective. Students gave two different proofs by computation or by the universal property of the quotient.

Proof 1. First, you get an action of $G/N$ on $X$ by

$$gNx = gx$$

This is well defined since $gx = hx$ for any two elements $g, h$ of the same coset. (If $gN = hN$ then $g^{-1}h \in N$. So $g^{-1}hx = x$ which implies that $gx = hx$ for all $x \in X$.) It is clear that this defines an action.

Let $gN$ be a nontrivial element of $G/N$. Thus $g \notin N$. Then, by definition of $N$, this implies that $gx \neq x$ for some $x \in X$. Therefore, $gNx = gx \neq x$ and $gN$ moves $x$. So the action of $G/N$ on $X$ is effective.

Proof 2. The action of $G$ on $X$ is, by definition, a homomorphism

$$\pi : G \rightarrow \text{Perm}(X)$$

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\[ N \text{ is the set of elements of } G \text{ which fix every element of } X. \] Therefore, \( N \) is the kernel of \( \pi \). This implies that there is an induced homomorphism
\[ \bar{\pi} : G/N \to \text{Perm}(X) \]
given by \( \bar{\pi}(gN) = \pi(g) \). The kernel of the induced homomorphism is the identity \( eN \) of \( G/N \). Therefore, the action of \( G/N \) on \( X \) is effective. \( \square \)

2.1.3. Nontrivial example. Several students gave the general example of a nonabelian \( p \)-group \( P \) acting on itself by conjugation. The kernel of this action is the center of \( P \) which we know is nontrivial. The quotient \( P/Z(P) \) is also nontrivial since \( P \) is nonabelian.

Most students gave the specific example of the dihedral group \( D_4 \) (group of symmetries of the square). Since this is nonabelian of order 8, it is a special case of this.

Some students took nonabelian groups acting on abelian but not central normal subgroups by conjugation. However, they did not state it in such generality.

2.2. Let \( G \) act on the set \( S \) of all subgroups of \( G \) by conjugation. Show that the functions \( d : S \to S \) and \( c : S \to S \) given by \( d(H) = H' \) (the commutator subgroup) and \( c(H) = C_G(H) \) (the centralizer of \( H \) in \( G \)) are morphisms of \( G \)-sets.

Everybody realized that the problem is to show that \( gH'g^{-1} = (gHg^{-1})' \) and \( C_G(gHg^{-1}) = gC_G(H)g^{-1} \) for all \( g \in G, H \leq G \).

2.2.1. \( gH'g^{-1} = (gHg^{-1})' \). Many students did not realize that the elements of \( H' \) are products of commutators. Thus
\[ H' = \langle [x, y] | x, y \in H \rangle \]
and
\[ gH'g^{-1} = g \langle [x, y] | x, y \in H \rangle g^{-1} = \langle g[x, y]g^{-1} | x, y \in H \rangle \]
\[ = \langle [gxg^{-1}, gyg^{-1}] | x, y \in H \rangle = (gHg^{-1})' . \]

2.2.2. \( C_G(gHg^{-1}) = gC_G(H)g^{-1} \). Most student got this correct:
\[ x \in C_G(gHg^{-1}) \iff xghg^{-1} = gh^{-1}x \forall h \in H \]
\[ \iff g^{-1}xg = hg^{-1}xg \forall h \in H \]
\[ \iff g^{-1}xg \in C_G(H) \]
\[ \iff x \in gC_G(H)g^{-1} . \]

Therefore, \( C_G(gHg^{-1}) = gC_G(H)g^{-1} \).