MATH 101A: HOMEWORK

3. Homework 3 answers

The following problems were due Thursday (9/20/7).

3.1. Show that every subgroup of G containing $N_G(P)$ is self-normalizing. [This means $H = N_G(H)$ for every H which contains $N_G(P)$.]

Students gave three different proof.

Proof. My favorite proof uses 3.2. Since $H \trianglelefteq N_G(H)$ and P is a Sylow p-subgroup of $N_G(H)$ it follows from the next problem that $N_G(H) = HN_G(P) = H$ where we use that fact that

$$N_{N_G(H)}(P) = N_G(P) \cap N_G(H) = N_G(P)$$

since $N_G(P) \leq H \leq N_G(H)$. □

Proof. Let $g \in N_G(H)$. Then $Q = gPg^{-1} \leq gHg^{-1} = H$. So, Q is a Sylow p-subgroup of H and, by Sylow, there is an element $h \in H$ so that $P = hQh^{-1} = hgPg^{-1}h^{-1}$. But this implies that $k = hg \in H_G(P) \leq H$. So $g = h^{-1}k \in H$. This shows that $N_G(H) \leq H$. Since $H \leq N_G(H)$, they must be equal. □

Finally, there is the counting argument which is the proof I had in mind. It is more complicated. But I like it.

Proof. G acts on the set of Sylow p-subgroups of G by conjugation. The stabilizer of P is $N_G(P)$. So, by the orbit-stabilizer formula, the number of Sylow p-subgroups of G is equal to $|G : N_G(P)|$. Applying this to the group H we see that the number of Sylow p-subgroups of H is equal to $|H : N_G(P)|$. For any Sylow p-subgroup Q of $N_G(H)$ there is an element $x \in N_G(H)$ so that $Q = xPx^{-1} \leq xHx^{-1} = H$. Therefore, $N_G(H)$ contains the same number of Sylow p-subgroups as H and this number is

$$|N_G(H) : N_G(P)| = |H : N_G(P)|.$$

The ratio is $|N_G(H) : H| = 1$. Therefore, $N_G(H) = H$. □

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3.2. If $K \trianglelefteq G$ and P is a Sylow subgroup of K then $KN_G(P) = G$. [This follows from the fact that all conjugates of P lie in K.]

Almost everyone gave the same proof.

Proof. Let $g \in G$. Then $gPg^{-1} \leq gKg^{-1} = K$. Therefore, gPg^{-1} is a Sylow p-subgroup of K which implies that $gPg^{-1} = kPk^{-1}$ for some $k \in K$. But then $g^{-1}kPk^{-1}g = P$ making $h = g^{-1}k$ an element of $N_G(P)$. So, $g = kh^{-1} \in KN_G(P)$. Since $g \in G$ was arbitrary, this shows that $G = KN_G(P)$. □

3.3. Skip this. If each Sylow subgroup of G is normal then G is the product of its Sylow subgroups.

3.4. Show that $T(n, \mathbb{Z})$ is nilpotent.

Let T_k be the group of all upper triangular matrices (x_{ij}) so that $x_{ii} = 1$ and $x_{ij} = 0$ for $0 < j - i < k$. For example, T_1 is the group of upper triangular matrices.

Lemma 3.1. $[T_i, T_j] \subseteq T_{i+j}$.

Given this lemma we see that T_k/T_{k+1} lies in the center of T/T_{k+1} since $[T_k, T] = [T_k, T_1] \subseteq T_{k+1}$. Therefore,

$$T = T_1 \trianglerighteq T_2 \trianglerighteq T_3 \trianglerighteq \cdots \trianglerighteq T_n = \{I_n\}$$

is a central series proving that the upper triangular matrix group is nilpotent. Therefore, it suffices to prove the lemma. (You don't need to prove separately that T_i is normal in T_1.)

Proof. The lemma can be rephrased as follows. Let $X \in T_i, Y \in T_j$ then

$$XY \equiv YX \mod T_{i+j}$$

But, equivalence module T_{i+j} is the same as saying that $XY - YX \in U_{i+j}$ where U_k is the set of all upper triangular matrices (x_{ij}) so that $x_{ij} = 0$ for all $j - i < k$ (including the case $i = j$).

But $X = I + A, Y = I + B$ where $A \in U_i, B \in U_j$. So,

$$XY - YX = (I + A + B + AB) - (I + B + A + BA) = AB - BA$$

This lies in U_{i+j} since both AB and BA lie in U_{i+j}. □