MATH 101A: HOMEWORK

4. Homework 4

4.1. Prove the properties of the pull-back, namely, if P is the pull-back of the diagram

$$G \to K \leftarrow H$$

then

(1) The kernel of $f : G \to K$ is isomorphic to the kernel of $p_2 : P \to H$.
(2) If $H \to K$ is onto, there is a bijection $H/p_2(P) \cong K/f(G)$.

Everyone got the first part correct. So, I won't comment on that. On the second part, everyone made one of the following two mistakes.

(1) Most students assumed that $f(G) \trianglelefteq K$. Then they proved that you get a homomorphism $H \to K/f(G)$ with kernel $p_2(P)$.
(2) Other students avoided this mistake and showed that there is a bijection between these two coset spaces:

$$\phi : H/p_2(P) \to K/f(G)$$

and defined the mapping by $\phi(hp_2(P)) = f'(h)f(G)$. The mistake is that you need to show that ϕ is *well-defined*. The reason this is necessary is that the letter h is not actually given. Cosets can be written in several ways and $h_1p_2(P) = h_2p_2(P)$ iff $h_2^{-1}h_1 \in p_2(P)$. By definition of P this means there is a $g \in G$ so that $(g, h_2^{-1}h_1) \in P$, i.e., $f(g) = f'(h_2^{-1}h_1) = f'(h_2)^{-1}f'(h_1)$. This implies that $f'(h_1)f(G) = f'(h_2)f(G)$. So, ϕ does not depend on the choice of h and is therefore well-defined.

4.2. Define the arrow category $\mathcal{A}r(\mathcal{G}ps)$ and show that the kernel

$$(f : G \to H) \mapsto \ker f$$

is a functor

$$\ker : \mathcal{A}(\mathcal{G}ps) \to \mathcal{G}ps.$$

Can you find the adjoint functor?

The morphisms in this category are commuting squares. I.e., a morphism $(\phi, \psi) : (f : G \to H) \to (f' : G' \to H')$ is a pair of morphisms

Date: October 15, 2007.

$\phi : G \to G', \psi : H \to H'$ so that $f' \circ \phi = \psi \circ f$. The functor $F(f) = \ker f$ takes this morphism to the induced homomorphism

$$F(\phi, \psi) : \ker f \to \ker f'$$

given by the restriction of $\phi : G \to G'$ to $\ker f \subseteq G$.

The adjoint functor is $A(G) = (G \to e)$. This has the property that

$$\mathrm{Hom}_{\mathcal{A}r}((G \to e), (f : X \to Y)) \cong \mathrm{Hom}_{\mathcal{G}ps}(G, \ker f)$$

since any morphism $(G \to e) \to (f : X \to Y)$ sends G into the kernel of f.

4.3. Suppose that p is prime and n is a positive integer relatively prime to p. Then show that $\mathbb{Z}_p$ is uniquely divisible by n in the sense that for any $x \in \mathbb{Z}_p$ there exists a unique $y \in \mathbb{Z}_p$ so that $ny = x$ where ny is defined to be $y + y + \cdots + y$ (n times). You may use the fact that n, p^i relatively prime implies there exist integers a_i, b_i so that

$$a_i p^i + b_i n = 1.$$

Don't use the ring structure of $\mathbb{Z}_p$. Use just the additive groups structure (as the inverse limit of the additive groups $\mathbb{Z}/p^i$).

If $x = (x_1, x_2, \cdots) \in \mathbb{Z}_p$ then $x_i \in \mathbb{Z}/p^i$ and $y_i = b_i x_i$ is the unique element of $\mathbb{Z}/p^i$ with the property that $ny_i = x_i$ in $\mathbb{Z}/p^i\mathbb{Z}$. So, the only thing to prove is that $y = (y_1, y_2, \cdots)$ is an element of $\mathbb{Z}_p$. In other words, $y_i \equiv y_{i-1}$ modulo p^{i-1}. But this follows from the uniqueness of y_{i-1}.