MATH 101A: HOMEWORK

5. Homework 5

The following problems are due next Thursday (Oct 18). I will post the answers after I grade it.

5.1. If R is any ring, show that there is another ring given by

$$S = R \oplus R$$

with addition given coordinatewise ($(a,x)+(b,y) = (a+b, x+y)$) and multiplication given by

$$(a,x)(b,y) = (ab, ay + xb).$$

5.2. Suppose that $\mathcal{C}$ is a preadditive category (i.e., Hom sets are additive groups and composition is biadditive). Let $E_1, \cdots, E_n$ be n objects of $\mathcal{C}$. Then show that

$$R = \oplus_{i,j} \mathrm{Hom}_{\mathcal{C}}(E_i, E_j)$$

is a ring. Show that there are orthogonal idempotents $e_i \in R$ so that

$$\mathrm{Hom}_{\mathcal{C}}(E_i, E_j) = e_j R e_i.$$

5.3. Suppose that K is a field and $R = \mathcal{M}_n(K)$ is the ring of $n \times n$ matrices with coefficients in K. Determine if the set of $n \times n$ upper triangular matrices (with zero on the diagonal) with coefficients in K is an ideal and prove it.

5.4. Show that the ring of p-adic integers $\mathbb{Z}_p$ has a nontrivial third root of unity if and only if p is congruent to 1 modulo 3. You can use the formula for the units in $\mathbb{Z}/n$ proved in the book.

Date: October 13, 2007.