MATH 101A: HOMEWORK

5. Answers to Homework 5

The following problems were due Thursday (Oct 18).

5.1. If R is any ring, show that there is another ring given by

$$S = R \oplus R$$

with addition given coordinatewise ($(a, x) + (b, y) = (a + b, x + y)$) and multiplication given by

$$(a, x)(b, y) = (ab, ay + xb).$$

This was a straightforward problem that almost everyone got. There is also a theoretical intepretation: This is the quotient ring

$$S = R[T]/(T^2).$$

5.2. Suppose that $\mathcal{C}$ is a preadditive category (i.e., Hom sets are additive groups and composition is biadditive). Let $E_1, \cdots, E_n$ be n objects of $\mathcal{C}$. Then show that

$$R = \oplus_{i,j} \operatorname{Hom}_{\mathcal{C}}(E_i, E_j)$$

is a ring. Show that there are orthogonal idempotents $e_i \in R$ so that

$$\operatorname{Hom}_{\mathcal{C}}(E_i, E_j) = e_j R e_i.$$

Since R is a direct sum of additive groups, it is an additive group. The standard correct answer for the multiplicative structure is to write the elements of R as matrices (f_{ij}) where

$$f_{ij} \in \operatorname{Hom}_{\mathcal{C}}(E_j, E_i)$$

and use matrix multiplication:

$$(fg)_{ij} = \sum_k f_{ik} \circ g_{kj}$$

However, two of you came up with a nonstandard but still correct way to multiply elements of R and this method also applies to matrix multiplication!!

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The nonstandard method is:

$$(fg)_{ij} = f_{ii} \circ g_{ij} + f_{ij} \circ g_{jj}$$

for $i \neq j$ and

$$(fg)_{ii} = f_{ii} \circ g_{ii}.$$

After asking everybody and thinking for several days, I finally found the theoretical interpretation: Matrices can be made into a graded ring and this nonstandard ring is the graded matrix ring truncated at degree 2.

A *graded ring* is ring R with a direct sum decomposition as an additive group:

$$R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$$

so that

$$R_i R_j \subseteq R_{i+j}.$$

Elements of R_n are called *graded homogeneous of degree* n. For example polynomial rings $R = F[T]$ are graded in an obvious way so that $R_n \cong F$ for each $n \geq 0$.

Graded rings have the property that, for each $n > 0$, we get an ideal:

$$R_(n) := R_n \oplus R_{n+1} \oplus R_{n+2} \oplus \cdots$$

with quotient ring

$$R/R_{(n)} = R_0 \oplus R_1 \oplus \cdots \oplus R_{n-1}$$

For the polynomial ring $F[T]_{(n)}$ is the ideal generated by T^n.

For our example $R = \oplus \operatorname{Hom}_{\mathcal{C}}(E_i, E_j)$, we can form another graded ring $S = \oplus S_n$ where

$$S_0 = \bigoplus \operatorname{Hom}_{\mathcal{C}}(E_i, E_i)$$

$$S_1 = \bigoplus_{i \neq j} \operatorname{Hom}_{\mathcal{C}}(E_i, E_j)$$

and $S_n = R$ for $n \geq 2$. Using the standard method we get:

$$S_n S_m \subseteq S_{n+m}$$

Therefore $S = \oplus S_n$ is a graded ring. The quotient ring

$$S/S_{(2)} = S_0 \oplus S_1$$

is additively the same as R with the same idempotents and the same action of the idempotents on S and the same $e_j S e_i = e_j R e_i$. In other words, everything works as required by the homework problem!

A very interesting answer, thank you Jennifer and Yurong!

5.3. Suppose that K is a field and $R = \mathcal{M}_n(K)$ is the ring of $n \times n$ matrices with coefficients in K. Determine if the set of $n \times n$ upper triangular matrices (with zero on the diagonal) with coefficients in K is an ideal and prove it.

This is not an ideal because it is not closed under multiplication by a matrix which has nonzero entries below the diagonal. However, if we change the problem as I asked you to do by email, the upper triangular matrices form a ring and the strictly upper triangular matrices form an ideal in that ring.

5.4. Show that the ring of p-adic integers $\mathbb{Z}_p$ has a nontrivial third root of unity if and only if p is congruent to 1 modulo 3. You can use the formula for the units in $\mathbb{Z}/n$ proved in the book.

If p is congruent to 1 mod 3 then $U(\mathbb{Z}/p)$ contains an element of order 3. This is represented by some integer a which is not congruent to 1 mod p. Then

$$(a, a^p, a^{p^2}, \cdots)$$

is a nontrivial 3rd root of unity in $\mathbb{Z}_p$. You need the following lemma:

Lemma 5.1. *If $x \equiv y$ modulo p^i then $x^p \equiv y^p$ modulo p^{i+1}.*

Conversely, you need to show that $\mathbb{Z}_3$ does not contain a nontrivial 3rd root of unity. (It is clear that, if $p \equiv 2$ modulo 3, then $\mathbb{Z}_p$ contains no nontrivial 3rd root of unity since

$$U(\mathbb{Z}_p) = \lim U(\mathbb{Z}/(p^k))$$

and $U(\mathbb{Z}/p^k)$ has order $\varphi(p^k) = p^{k-1}(p-1)$ which is not divisible by 3 unless either $p = 3$ or $p \equiv 1$ mod 3.)

To exclude the case $p = 3$ (which none of you did), you need to know that $U(\mathbb{Z}/p^k)$ is cyclic when $p \neq 2$. So, when $p = 3$, there are exactly two nontrivial 3rd roots of unity in $\mathbb{Z}/3^k$ and both of them go to 1 in $\mathbb{Z}/3^{k-1}$. So, if

$$x = (a_1, a_2, \cdots) \in \mathbb{Z}_3$$

is a 3rd root of 1 then $a_k^3 \equiv 1$ mod p^k implies that $a_{k-1} \equiv 1$ mod p^{k-1} for all k. So, $x = 1$ and there are no nontrivial 3rd roots of unity in $\mathbb{Z}_3$.