MATH 101A: HOMEWORK

6. HOMEWORK 6

6.1. (Ex. 3 on p. 115) Show that the localization of a ring at a prime ideal is a local ring.

Let S be the complement of a prime ideal $\mathfrak{p}$. Then

1. S is a multiplicative set containing no zero divisors. So, $R_\mathfrak{p} = S^{-1}R$ is a nonzero ring.
2. $S^{-1}\mathfrak{p} = \{x/s \,|\, x \in \mathfrak{p}, s \in S\}$ is an ideal in $S^{-1}R$.
3. Any element of $S^{-1}R$ which is not in $S^{-1}\mathfrak{p}$ has the form s/t where $s, t \in S$. These elements are invertible (with inverse t/s).
4. So, $S^{-1}\mathfrak{p}$ is the unique maximal ideal and $S^{-1}R$ is a local ring.

6.2. If R is a domain, show that the ideal in $R[X,Y]$ generated by $X^2 - Y^3$ is prime. [Hint: Consider the ring homomorphism $\phi : R[X,Y] \to R[T]$ given by $\phi(X) = T^3, \phi(Y) = T^2$.]

Since $R[T]$ is a domain, the kernel of ϕ is a prime ideal. Since $\phi(X^2) = \phi(Y^3)$, the ideal $I = (X^2 - Y^3)$ is contained in $\ker \phi$. We just need to show that they are equal.

Suppose that $f(X,Y) \in \ker \phi$. Then we have to show that $f \in I$. Working modulo I, any term with X^2 can be converted into Y^3. So,

$$f(X,Y) \equiv g(Y) + Xh(Y)$$

modulo I where $g(Y), h(Y) \in R[Y]$. However, $\phi(f) = 0$ implies that

$$g(T^2) + T^3 h(T^2) = 0.$$

But $g(T^2)$ has only even powers of T and $T^3 h(T^2)$ has only odd powers of T. So, they must both be zero. So $g = 0 = h$ which makes $f \equiv 0$ modulo I. I.e., $\ker \phi = I$.

Most students proved this directly (namely that fact that $\ker \phi \subseteq I$). The shortest method was to first note that

$$(X^2 - Y^3)(X^{j+2n}Y^k + X^{j+2n-2}Y^{k+3} + \cdots + X^j Y^{k+3n}) = X^{j+2n+2}Y^k - X^j Y^{k+3n+3}$$

This implies that any two monomials in X, Y which map to the same power of T are congruent module the ideal I.

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6.3. (Ex. 9 on p. 115) If $i = \sqrt{-1}$ show that $\mathbb{Z}[i]$ is a PID and hence a UFD. What are the units?

The key point is that any complex number $x+iy$ is within a distance of $\frac{\sqrt{2}}{2}$ of an element of $\mathbb{Z}[i]$, namely $a+ib$ where a is the closest integer to x and b is the closest integer to y. Algebraically, we take the *norm* $N : \mathbb{Q}(i) \to \mathbb{Q}$ given by $N(x+iy) = x^2 + y^2$ which is $\leq 1/2$ whenever x, y are $\leq 1/2$ in absolute value.

Thus, given any two elements $z_1, z_2 \in \mathbb{Z}[i]$ where $z_2 \neq 0$, the ratio $z_1/z_2 = z_1\overline{z_2}/N(z_2) \in \mathbb{Q}(i)$ is within $\frac{\sqrt{2}}{2}$ of some element $a + bi \in \mathbb{Z}[i]$. So

$$N\left(\frac{z_1}{z_2} - (a+bi)\right) \leq \frac{1}{2}$$

or

$$N\left(z_1 - (a+bi)z_2\right) \leq \frac{1}{2}N(z_2) < N(z_2).$$

So, $\mathbb{Z}[i]$ is Euclidean and is thus a PID.

The units are $\pm 1, \pm i$ since these are the only elements of norm 1.

6.4. Do Ex. 10 on page 115, except replace 10(d) with the problem: Show that $2, 3, 1 \pm \sqrt{-5}$ are irreducible in $\mathbb{Z}[\sqrt{-5}]$.

(c) To show that $R = \mathbb{Z}[\sqrt{-D}]$ has no units except ± 1 when $D \geq 2$, let $N : R \to \mathbb{Z}$ be given by $N(z) = z\overline{z} = a^2 + b^2 D$ when $z = a + b\sqrt{-D}$. Then N is multiplicative (begin a product of two multiplicative maps) and therefore induces a group homomorphism

$$U(R) \to U(\mathbb{Z}) = \{\pm 1\}.$$

But $a^2 + b^2 D = \pm 1$ implies $b = 0$ and $a = \pm 1$.

(d) The points in $\mathbb{Z}[\sqrt{-5}]$ with norm ≤ 9 (absolute value $\leq 3 = \sqrt{9}$) are

$$-3, -2, -1, 0, 1, 2, 3$$

on the real axis and

$$-2 \pm \sqrt{-5}, -1 \pm \sqrt{-5}, \pm\sqrt{-5}, 1 \pm \sqrt{-5}, 2 \pm \sqrt{-5}$$

on the lines $\sqrt{5}$ above and below the real axis. The norms of these numbers are

$$0, 1, 4, 5, 6, 9.$$

No two of these numbers multiply to 4 or 6 or 9. Therefore, 2, $1 \pm \sqrt{-5}$ and 3 are irreducible.