MATH 101A: HOMEWORK

8. Answers to Homework 8

8.1. Take the complex matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

I.e.: $Ae_1 = 0, Ae_2 = Ae_3 = e_1$ and put it into Jordan canonical form: Find an invertible complex matrix B so that $B^{-1}AB$ is in Jordan canonical form. What is the corresponding decomposition of $\mathbb{C}^3$ as a module over $\mathbb{C}[T]$?

The only eigenvalue is 0 and the eigenspace is the two dimensional subspace E_0 given by $x_2 + x_3 = 0$. If we take any vector which is not in this subspace, such as $v_1 = (0,1,0)^t$, we see that $v_2 = Av_1 = (1,0,0)^t$ spans the one dimensional subspace $A\mathbb{C}^3$ of E_0. Complete this to a basis of E_0 by taking $v_3 = (0,-1,1)^t$. Then v_1, v_2, v_3 form a basis for $M = \mathbb{C}^3$ and $Av_1 = v_2, Av_2 = Av_3 = 0$. To get the Jordan canonical form, we switch the order of the basis to v_3, v_2, v_1. Then we get

$$B^{-1}AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The corresponding decomposition of $M = \mathbb{C}^3$ into indecomposable $\mathbb{C}[T]$-modules is

$$M \cong \mathbb{C}[T]/(T^2) \oplus \mathbb{C}[T]/(T).$$

Since students obtained different answers, I considered the extent to which the decomposition is well defined: There are only two invariant subspaces defined by this decomposition: The plane $E_0 = \{v \in \mathbb{C}^3 \mid Av = 0\}$ and the line $L = A\mathbb{C}^3 \subset E_0$. And A gives an isomorphism $\mathbb{C}^3/E_0 \cong L$. The basis given by the columns of B thus consists of

(1) Any generator (nonzero vector) of L. For example, several students took $(2,0,0)^t$.

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(2) Any vector in $\mathbb{C}^3$ which does not lie in L will map to a generator of L. Multiplying by an appropriate scalar gives the chosen generator. Thus, several students took $(0,1,1)^t$.
(3) The last vector can be any element of E_0 which does not lie in L, i.e., any element $(x,y,-y)^t$ where $y \neq 0$.

8.2. Over the real numbers there are irreducible polynomials of degree 2:

$$p(x) = x^2 + bx + c$$

So, $\mathbb{R}[T]/(p(T)^2)$ is a cyclic module over $\mathbb{R}[T]$. Find the corresponding 4×4 matrix.

There are basically two methods. The first method is to choose a basis for the module $M = \mathbb{R}[T]/(p(T)^2)$ and find the matrix with respect to the basis. One basis I which I like is: $p, Tp, 1, T$ (modulo p^2). We see immediately that T sends v_1 to v_2 and v_3 to v_4. Calculation shows that $Tv_2 = -bv_2 - cv_1$ and $Tv_4 = v_1 - bv_4 - cv_3$. So, the matrix of T is:

$$\begin{pmatrix} 0 & -c & 0 & 1 \\ 1 & -b & 0 & 0 \\ 0 & 0 & 0 & -c \\ 0 & 0 & 1 & -b \end{pmatrix}$$

The second method that several students used is to start with a 4×4 real matrix A and prove that the induced $\mathbb{R}[T]$-module structure on $\mathbb{R}^4$ is isomorphic to $\mathbb{R}[T]/(p^2)$.

$$A = \begin{pmatrix} 0 & -c & 1 & 0 \\ 1 & -b & 0 & 1 \\ 0 & 0 & 0 & -c \\ 0 & 0 & 1 & -b \end{pmatrix}$$

Since this matrix has the form $A = \begin{pmatrix} B & I_2 \\ 0 & B \end{pmatrix}$ where B is a 2×2 non-diagonal matrix having the property that $p(B) = 0$, we have:

$$p(A) = \begin{pmatrix} 0 & 2B + bI_2 \\ 0 & 0 \end{pmatrix} \neq 0.$$

But $p(A)^2 = 0$ since any matrix of the form $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ has square zero. Therefore, $p(T)^2$ generates the annihilator of M. This implies that $\mathbb{R}[T]/(p^2)$ is a direct summand of M. Since M is only 4 dimensional, it must be all of $\mathbb{R}[T]/(p^2)$.