

10. LIMITS AS FUNCTORS

I explained the theorem that limits are natural. This means they are functors.

10.1. functors. Given two categories \mathcal{C} and \mathcal{D} a *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ is a mapping which sends objects to objects and morphisms to morphisms and preserves the structure:

- (1) For each $X \in \text{Ob}(\mathcal{C})$ we assign an object $F(X) \in \text{Ob}(\mathcal{D})$.
- (2) For any two objects $X, Y \in \text{Ob}(\mathcal{C})$ we get a mapping of sets:

$$F : \text{Mor}_{\mathcal{C}}(X, Y) \rightarrow \text{Mor}_{\mathcal{D}}(FX, FY).$$

This sends $f : X \rightarrow Y$ to $Ff : FX \rightarrow FY$ so that the next two conditions are satisfied.

- (3) $F(id_X) = id_{FX}$. I.e., F sends identity to identity.
- (4) $F(f \circ g) = Ff \circ Fg$. I.e., F preserves composition.

The first example I gave was the *forgetful functor*

$$F : \mathcal{Gps} \rightarrow \mathcal{Ets}$$

which sends a group to its underlying set and forgets the rest of the structure. Thus

$$F(G, \cdot, e, ()^{-1}) = G.$$

The fact that there is a forgetful functor to the category of sets means that groups are sets with extra structure so that the homomorphisms are the set mappings which preserve this structure. Such categories are called *concrete*. Not all categories are concrete. For example the homotopy category \mathcal{H} whose objects are topological spaces and whose morphisms are homotopy classes of maps is not concrete or equivalent to a concrete category.

10.2. the diagram category. The limit of a diagram in the category of groups is a functor

$$\lim : \text{Fun}(\Gamma, \mathcal{Gps}) \rightarrow \mathcal{Gps}.$$

But what is the domain category?

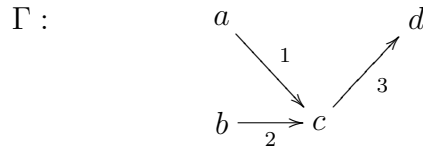
At this point I needed to make precise how diagrams are indexed. I used the following example.

$$D : \begin{array}{ccc} X_a & & X_d \\ & \searrow f_1 & \nearrow f_3 \\ X_b & \xrightarrow{f_2} & X_c \end{array}$$

The objects of the diagram are indexed by the set $\Gamma_0 = \{a, b, c, d\}$. The arrows (morphisms) are indexed by $\Gamma_1 = \{1, 2, 3\}$. These two indexing sets are related by two mappings called *source* and *target*

$$s, t : \Gamma_1 \rightarrow \Gamma_0$$

This structure $(\Gamma_0, \Gamma_1, s, t)$ is called a *directed graph* and is drawn as follows.



For any category \mathcal{C} and any directed graph Γ , the *diagram category* $\text{Fun}(\Gamma, \mathcal{C})$ is defined to be the category whose objects are diagrams consisting of

- (1) an object X_α in \mathcal{C} for every $\alpha \in \Gamma_0$ and
- (2) a morphism $f_j : X_{s(j)} \rightarrow X_{t(j)}$ for every $j \in \Gamma_1$.

A morphism in the category

$$((X_\alpha)_\alpha, (f_j)_j) \rightarrow ((Y_\alpha)_\alpha, (g_j)_j)$$

consists of a morphism $\phi_\alpha : X_\alpha \rightarrow Y_\alpha$ for each $\alpha \in \Gamma_0$ so that the following square commutes for every $j \in \Gamma_1$ with source α and target β .

$$\begin{array}{ccc} X_\alpha & \xrightarrow{f_j} & X_\beta \\ \phi_\alpha \downarrow & & \downarrow \phi_\beta \\ Y_\alpha & \xrightarrow{g_j} & Y_\beta \end{array}$$

10.3. The limit is a functor. When we say that the limit is a functor, we mean that for any map of diagrams we should get a map of limits. For example, suppose that the graph is

$$\Gamma : \quad \cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

And we have two diagrams indexed by Γ :

$$D_1 : \quad \cdots \longrightarrow G_2 \longrightarrow G_1 \longrightarrow G_0$$

$$D_2 : \quad \cdots \longrightarrow H_2 \longrightarrow H_1 \longrightarrow H_0$$

Then, a morphism from D_1 to D_2 in the diagram category is a sequence of vertical arrows $\phi_i : G_i \rightarrow H_i$ making the following diagram commute.

$$\begin{array}{ccccccc} D_1 & & \cdots & \longrightarrow & G_2 & \longrightarrow & G_1 & \longrightarrow & G_0 \\ (\phi_\alpha) \downarrow & & & & \phi_2 \downarrow & & \phi_1 \downarrow & & \phi_0 \downarrow \\ D_2 & & \cdots & \longrightarrow & H_2 & \longrightarrow & H_1 & \longrightarrow & H_0 \end{array}$$

I explained how the universal property of the limit gave the induced map

$$G_\infty = \lim_{\leftarrow} G_i \rightarrow H_\infty = \lim_{\leftarrow} H_i$$

The reason is because of the following commuting diagram.

$$\begin{array}{ccccccc} G_\infty & & & & & & & & \\ & \searrow^{f_1} & & & & & & & \\ & & & & & & & & \\ & \searrow^{f_2} & & & & & & & \\ \cdots & \longrightarrow & G_2 & \longrightarrow & G_1 & \longrightarrow & G_0 & & \\ & & \phi_2 \downarrow & & \phi_1 \downarrow & & \phi_0 \downarrow & & \\ \cdots & \longrightarrow & H_2 & \xrightarrow{h_2} & H_1 & \longrightarrow & H_0 & & \end{array}$$

When we compose the vertical arrows we get the following diagram.

$$\begin{array}{ccccccc} G_\infty & & & & & & & & \\ & \searrow^{\phi_1 f_1} & & & & & & & \\ & \searrow^{\phi_2 f_2} & & & & & & & \\ \cdots & \longrightarrow & H_2 & \xrightarrow{h_2} & H_1 & \longrightarrow & H_0 & & \end{array}$$

The commutativity of the triangles (of which only one is drawn) gives, by the universal property of $\lim_{\leftarrow} H_i$, a uniquely determined homomorphism

$$\phi : G_\infty \rightarrow H_\infty$$

The explicit formula for this homomorphism is

$$\phi(g_0, g_1, g_2, \cdots) = (\phi_0(g_0), \phi_1(g_1), \phi_2(g_2), \cdots)$$

This is the unique mapping which commutes with the projection map to the n th coordinate:

$$\begin{array}{ccc} G_\infty & \xrightarrow{p_n} & G_n \\ \phi \downarrow & & \downarrow \phi_n \\ H_\infty & \xrightarrow{p_n} & H_n \end{array}$$

Theorem 10.1. *For any directed graph Γ the limit gives a functor*

$$\lim : \text{Fun}(\Gamma, \mathcal{Gps}) \rightarrow \mathcal{Gps}$$

from the category of diagrams in \mathcal{Gps} indexed by Γ to the category \mathcal{Gps} . The same holds in the category $\mathcal{E}ns$.

Remark 10.2. *If a category \mathcal{C} has limits of diagrams indexed by Γ we need to assume that the category is small (i.e., the collection of objects is a set) in order for the limit to be an actual functor. The reason is that the limit is only defined up to isomorphism and you need to choose one object from each isomorphism class. Most people don't make such a fuss. (Because if you understand the problem you probably also understand the solution.)*

Another theorem which I mentioned earlier is the following.

Theorem 10.3. *Limit commutes with the forgetful functor. I.e., the following diagram of categories and functors commutes.*

$$\begin{array}{ccc} \text{Fun}(\Gamma, \mathcal{Gps}) & \xrightarrow{\lim} & \mathcal{Gps} \\ F \downarrow & & \downarrow F \\ \text{Fun}(\Gamma, \mathcal{E}ns) & \xrightarrow{\lim} & \mathcal{E}ns. \end{array}$$

This theorem is a fancy way of saying that the formula for the limit is the same in the two categories.

10.4. adjoint functors. Finally, I talked about adjoint functors. This is yet another explanation of the concept of universality. In the definition of universality it says “There exists a unique ϕ .” These words should be reminiscent of the expressions:

$$\begin{aligned} (\forall x)(\exists! y)f(x) &= y, \\ (\forall y)(\exists! x)f(x) &= y. \end{aligned}$$

These say that f is a *bijection* between two sets. If universality is a bijection between two sets then what are those two sets?

One set is $Mor_{\mathcal{C}}(Y, \lim X_{\alpha})$. This contains the morphism $\phi : Y \rightarrow \lim X_{\alpha}$. What data do you need to give you this unique morphism? You need morphisms $\phi_{\alpha} : Y \rightarrow X_{\alpha}$ for all α . These morphisms can be made into a single morphism between two diagrams:

$$\begin{array}{ccccccc} ((Y), (id)) : & \cdots & \longrightarrow & Y & \xrightarrow{id} & Y & \xrightarrow{id} & Y \\ (\phi_{\alpha}) \downarrow & & & \phi_2 \downarrow & & \phi_1 \downarrow & & \phi_0 \downarrow \\ ((X_{\alpha}), (f_i)) & \cdots & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 \end{array}$$

The diagram $((Y), (id))$ is the trivial diagram with Y at each vertex and the identity morphism on Y on each edge. A morphism in the diagram category from this diagram to the diagram $((X_\alpha), (f_i))$ is the same as a family of morphisms $\phi_\alpha : Y \rightarrow X_\alpha$ so that $f_i \circ \phi_{s(i)} = \phi_{t(i)}$:

$$\begin{array}{ccccc}
 & & Y & & \\
 & \swarrow \phi_2 & \downarrow \phi_1 & \searrow \phi_0 & \\
 \cdots & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X_0
 \end{array}$$

I pointed out that this trivial diagram $TY = ((Y), (id))$ is a functor

$$T : \mathcal{C} \rightarrow \text{Fun}(\Gamma, \mathcal{C}).$$

Definition 10.4. Two functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ are called adjoint functors if there is a natural bijection

$$\text{Mor}_{\mathcal{D}}(GX, Y) \cong \text{Mor}_{\mathcal{C}}(X, FY)$$

for all objects X in \mathcal{C} and all objects Y in \mathcal{D} . F is called the right adjoint of G and G is called the left adjoint of F .

Theorem 10.5. The trivial diagram functor and the limit are adjoint functors for the category of groups and for any category that has limits of diagrams indexed by Γ . I.e.,

$$\text{Mor}_{\text{Fun}(\Gamma, \mathcal{G}_{ps})}(((Y), (id)), ((X_\alpha), (f_i))) \cong \text{Mor}_{\mathcal{G}_{ps}}(Y, \lim X_\alpha).$$