11. Categorical limits

(Two lectures by Ivan Horozov. Notes by Andrew Gainer and Roger Lipsett. [Comments by Kiyoshi].)

We first note that what topologists call the "limit", algebraists call the "inverse limit" and denote by $\lim_{\leftarrow}$. Likewise, what topologists call the "colimit", algebraists call the direct limit and denote $\lim_{\rightarrow}$.

Example 11.1. Take the inverse system

$$\cdots \to \mathbb{C}[x]/(x^n) \to \cdots \to \mathbb{C}[x]/(x^3) \to \mathbb{C}[x]/(x^2) \to \mathbb{C}[x]/(x)$$

Let $f \in \lim_{\leftarrow} \mathbb{C}[x]/(x^n) = \mathbb{C}[[x]]$. Then $f = \sum_{n \geq 0} a_n x^n$ is a formal power series in x over $\mathbb{C}$.

We note that $\mathbb{Z}_p = \lim_{\longleftarrow n} \mathbb{Z}/p^n$ [is the inverse limit of]

$$\cdots \to \mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n \to \cdots \to \mathbb{Z}/p$$

11.1. colimits in the category of sets.

Definition 11.2. Let X_α be sets indexed by $\alpha \in I$ and let $f_{\alpha,\beta} : X_\alpha \to X_\beta$ be functions with $\alpha, \beta \in I$. [Only for pairs (α, β) so that $\alpha < \beta$ in some partial ordering of I.] Then $\{f_{\alpha.\beta}, X_\alpha\}_{\alpha \in I}$ is a *directed system of sets* if, for every pair of composable morphisms $f_{\alpha,\beta} : X_\alpha \to X_\beta$, $f_{\beta,\gamma} : X_\beta \to X_\gamma$ [i.e., wherever $\alpha < \beta < \gamma$ in I], the following diagram commutes

$$\begin{tikzcd}
X_\alpha \arrow[r, "f_{\alpha,\beta}"] \arrow[dr, "f_{\alpha,\gamma}"'] & X_\beta \arrow[d, "f_{\beta,\gamma}"] \\
& X_\gamma
\end{tikzcd}$$

and, for every $\alpha, \beta \in I$ there exists a $\gamma \in I$ for which there are maps $f_{\alpha,\gamma}$ and $f_{\beta,\gamma}$ [i.e., $\alpha, \beta \leq \gamma$] as such:

$$\begin{tikzcd}
X_\alpha \arrow[dr, "f_{\alpha,\gamma}"] & \\
& X_\gamma \\
X_\beta \arrow[ur, "f_{\beta,\gamma}"'] &
\end{tikzcd}.$$

One can think of a directed system as a graph with sets as points and arrows as edges.

We can now define the direct limit on directed systems of sets by

$$\lim_{\longrightarrow \alpha \in I} X_\alpha = \coprod_\alpha X_\alpha / \sim$$

where, for each $f_{\alpha,\beta}$ and for all $x \in X_\alpha$, we set $x \sim f_{\alpha,\beta}(x)$. Informally then, the direct limit is the set of equivalence classes induced by all functions $f_{\alpha,\beta}$. That $\sim$ is an equivalence relation follows from the two properties illustrated diagramatically above. [The colimit of any diagram of sets exists. The assumption of being "directed" implies that any two elements of the colimit, represented by say $x \in X_\alpha, y \in X_\beta$, are equivalent to elements of the same set X_γ.]

11.2. **pull-back.** In the following diagram, $*$ is a "pull-back":

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & G \arrow[d, "\alpha"] \\
H \arrow[r, "\beta"'] & K
\end{tikzcd}$$

The pull-back here is a subgroup (or subset) of $G \times H$ given by

$$G \times_K H = \{(g,h) \in G \times H \mid \alpha(g) = \beta(h)\}.$$

11.2.1. *universal property of pull-back.* If G' is a group such that

$$\begin{tikzcd}
G' \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& (g,h) \in G \times_K H \arrow[r] \arrow[d, "h"] & G \arrow[d, "\alpha"] \\
& H \arrow[r, "\beta"'] & K
\end{tikzcd}$$

commutes then there exists a unique map $G' \to G \times_K H$ such that

$$\begin{tikzcd}
G' \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& G \times_K H \arrow[r] \arrow[d] & G \arrow[d] \\
& H \arrow[r] & K
\end{tikzcd}$$

commutes.

11.3. **push-forward [push-out] of groups.**

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & G \arrow[d] \arrow[ddr, bend left] & \\
H \arrow[r] \arrow[drr, bend right] & G *_K H \arrow[dr] & \\
& & G
\end{tikzcd}$$

In this diagram, if $K = \{e,\}$ then $G*H$ is the *amalgamated free product* of G and H given by

$$G * H = \{g_1h_1g_2h_2\cdots g_nh_n \mid g_i \in G, h_i \in H\}.$$

Note that $(g_1hg_2)^{-1} = g_2^{-1}h^{-1}g_1^{-1}$. [So, the set of such products is closed under the operation of taking inverse. So, $G * H$ is a group.]

More generally, $G *_K H$ is the quotient group $G * H/\sim$ where

$$(g\alpha(k)) \cdot \left(\beta(k)^{-1}h\right) \sim gh$$

and

$$hg \sim (h\beta(k) \cdot \left(\alpha(k)^{-1}g\right).$$

Exercise. Compute $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ and explain the computation.

11.4. **direct limit of groups.** In order to take the direct limit of groups, we require a directed system of groups:

Definition 11.3. $\{G_\alpha, f_{\alpha,\beta}\}$ is a *directed system of groups* if

i) $f_{\alpha,\beta} : G_\alpha \to G_\beta, f_{\beta,\gamma} : G_\beta \to G_\gamma$ are homomorphism then [there is a homomorphism $f_{\alpha,\gamma} : G_\alpha \to G_\gamma$ in the system and]

$$\begin{tikzcd}
G_\alpha \arrow[r, "f_{\alpha\beta}"] \arrow[dr, "f_{\alpha\beta}"'] & G_\beta \arrow[d, "f_{\beta\gamma}"] \\
& G_\gamma
\end{tikzcd}$$

[i.e. the diagram commutes.]

ii) For every $\alpha, \beta \in I$ there exists $\gamma \in I$ such that $f_{\alpha,\gamma}, f_{\beta.\gamma}$ are defined and

$$\begin{tikzcd}
G_\alpha \arrow[dr, "f_{\alpha\gamma}"] & \\
& G_\gamma \\
G_\beta \arrow[ur, "f_{\beta\gamma}"'] &
\end{tikzcd}$$

We then define a new object [the weak product] $\prod'_{\alpha\in I} G_\alpha \subseteq \prod_{\alpha\in I} G_\alpha$:

Definition 11.4. For $x = \{x_\alpha\}_\alpha$ we let $x \in \prod'_{\alpha\in I} G_\alpha$ if x has only finitely many x_α coordinates which are not $e_\alpha \in G_\alpha$.

[The direct limit of a directed system of groups is then the same set as the direct limit of sets:

$$\varinjlim_{\alpha\in I} G_\alpha = \coprod_{\alpha\in I} G_\alpha/\sim$$

where, $x_\alpha \in G_\alpha$ is equivalent to $x_\beta \in G_\beta$ if and only if

$$f_{\alpha,\gamma}(x_\alpha) = f_{\beta,\gamma}(x_\beta)$$

for some $\gamma \in I$, with the additional structure that the product of $x_\alpha \in G_\alpha, x_\beta \in G_\beta$ is defined to be the product of their images in G_γ.]

In practice, one works with $\lim_{\to\alpha} G_\alpha$ in the following way: [Here Horozov says that each element of the direct limit is represented by a single element of a single group G_α. I wrote that as the definition.]

11.5. **universal property of the direct limit of groups.** If $\{G_\alpha, f_{\alpha,\beta}\}$ is a directed system of groups and $g_\alpha : G_\alpha \to H$ are homomorphisms such that

$$\begin{tikzcd}
G_\alpha \arrow[r, "g_\alpha"] \arrow[d, "f_{\alpha\beta}"'] & H \\
G_\beta \arrow[ur, "g_\beta"'] &
\end{tikzcd}$$

commutes, then there exists a unique homomorphism $g : \lim_{\to} G_\alpha \to H$ such that

$$\begin{tikzcd}
G_\alpha \arrow[r, "h_\alpha"] \arrow[drr, "g_\alpha"'] & \lim_{\to} G_\alpha \arrow[dr, "\exists ! g"] & \\
& & H
\end{tikzcd}$$

commutes for all $\alpha \in I$.

[Any element $\overline{x}$ of the direct limit is represented by some element of some group $x_\alpha \in G_\alpha$. Then we let $g(\overline{x}) = g_\alpha(x_\alpha)$. If $x_\beta \in G_\beta$ is another representative of the same equivalence class $\overline{x}$ then, by definition, $f_{\alpha,\gamma}(x_\alpha) = f_{\beta,\gamma}(x_\beta) = x_\gamma \in G_\gamma$ for some $\gamma \in I$. But then $g_\alpha(x_\alpha) = g_\gamma(x_\gamma) = g_\beta(x_\beta)$. So, g is well-defined.]

11.6. **free groups.** Let X be a set. The free group on X, $F(X)$, is defined by the following universal property: given any group G and set map $f : X \to G$, there is a unique $g : F(X) \to G$ that is a group homomorphism such that the diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "i"] & & G \\
& F(X) \arrow[ur, "\exists ! g"'] &
\end{tikzcd}$$

commutes.

It remains to define $F(X)$ and say what the map i is. Suppose $X = \{x_1, x_2, \cdots\}$ (note that X need not be countable; we use this subscript notation simply for ease of use). Then the *words in* X, w, are all finite sequences chosen from the set $X \cup X^{-1}$, where $X^{-1} = \{x_1^{-1}, x_2^{-1}, \cdots\}$

(here the $^{-1}$ notation is purely formal). If $W = \{w\}$ is the set of such word then

$$F(X) = W/\sim$$

where $\sim$ is the smallest possible relation so that we get a group: i.e., that for all i, both $x_i x_i^{-1}$ and $x_i^{-1} x_i$ are trivial.

Then each word in $F(X)$ has a unique reduced form, in which no further simplification induced by the above relation are possible, and $F(X)$ thus consists of all the reduced words.

11.7. **an important example.** $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\pm I$. This group acts on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$: $A \in SL_2(\mathbb{Z})$ gives a map

$$z \mapsto \frac{az+b}{cz+d} = \frac{-az-b}{-cz-d}.$$

Thus, for example, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ translates by 1, while $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is inversion.

It turns out that

$$PSL_2(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$$

This has something to do with fixed points in $\mathbb{C}$ under this action. Similarly,

$$SL_2(\mathbb{Z}) = \mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$$

12. More about free products

I decided to explain the example of $PSL_2(\mathbb{Z})$ more thoroughly.

12.1. Amalgamated products again.

The free product formulas for $PSL_2(\mathbb{Z})$ and $SL_2(\mathbb{Z})$ are an example of the following theorem.

Theorem 12.1. *Suppose that G and H have isomorphic normal subgroups $N_1 \trianglelefteq G$, $N_2 \trianglelefteq H$. $N_1 \cong N_2 = N$. Then $N \trianglelefteq G *_N H$ and*

$$\frac{G *_N H}{N} = G/N * H/N.$$

Proof. $G *_N H$ is the push-out (colimit) of the following diagram.

$$\begin{tikzcd}
N \arrow[r] \arrow[d] & G \\
H &
\end{tikzcd}$$

G/N is also given by a universal property: It is the pushout of the diagram

$$\begin{tikzcd}
N \arrow[r] \arrow[d] & G \arrow[d] \\
\{e\} = 1 \arrow[r] & G/N
\end{tikzcd}$$

To show that $\frac{G *_N H}{N} \cong G/N * H/N$ we need to show that: For any group X and homomorphism $G *_N H \to X$ which is trivial on N, there exists a unique homomorphism $G/N * H/N \to X$ making the appropriate diagram commute.

This condition is equivalent to the commuting diagram

$$\begin{tikzcd}
N \arrow[rr] \arrow[dr] \arrow[dd] & & G \arrow[dr] & \\
& H \arrow[rr] & & X \\
1 \arrow[urrr] & & &
\end{tikzcd}$$

However, this commuting diagram includes the diagram

$$\begin{tikzcd}
N \arrow[r] \arrow[d] & G \arrow[d] \\
1 \arrow[r] & X
\end{tikzcd}$$

So, there is an induced map $G/N \to X$ and similarly, there is an induced map $H/N \to X$ as indicated in the following diagram.

$$\begin{tikzcd}
N \arrow[rr] \arrow[dr] \arrow[dd] & & G \arrow[dr] \arrow[dd] & \\
& H \arrow[rr] \arrow[dd] & & X \\
1 \arrow[rr] \arrow[dr] & & G/N \arrow[ur] & \\
& H/N \arrow[uurr] & &
\end{tikzcd}$$

The two morphism $G/N \to X, H/N \to X$ induce a morphism from the free product $G/N * H/N \to X$:

$$\begin{tikzcd}
N \arrow[rr] \arrow[dr] \arrow[dd] & & G \arrow[dr] \arrow[dd] & \\
& H \arrow[rr] \arrow[dd] & & X \\
1 \arrow[rr] \arrow[dr] & & G/N \arrow[dr] & \\
& H/N \arrow[rr] & & G/N * H/N \arrow[uu, dashed]
\end{tikzcd}$$

This proves the theorem. $\square$

I restated the theorem in terms of specific 2×2 matrices A, B

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2$$

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \qquad B^2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \qquad B^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2$$

So, $A^4 = I_2 = B^6$. The statement is

$$SL_2(\mathbb{Z}) \cong \langle A \rangle *_{\langle -I_2 \rangle} \langle B \rangle = \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$$

This means that every element of $SL_2(\mathbb{Z})$ has a unique expansion of one of the following two forms

(1) $A^m B^{n_1} A B^{n_2} A \cdots A B^{n_k}$
(2) $A^m B^{n_1} A B^{n_2} A \cdots A B^{n_k} A$

where, in both cases, each $n_i = 1$ or 2 and $m = 0, 1, 2$ or 3.

Proof. By definition, elements of the amalgamated product $\langle A\rangle *_{\langle -I_2\rangle} \langle B\rangle$ have the form

$$A^{m_1}B^{n_1}A^{m_2}B^{n_2}A^{m_3}\cdots A^{m_k}B^{n_k}$$

subject to the condition that $A^4 = B^6 = I$ and $A^2 = B^3$. This last condition implies that, if any of the powers of A are 2 or more or any of the powers of B are 3 or more, then we can convert it into a power of the other letter and move it to the left. For example,

$$\begin{aligned} ABAB^2AB^4 &= ABAB^2AB(B^3) = ABAB^2(A^2)AB \\ &= ABA(B^3)B^2AB = \cdots = (A^2)ABAB^2AB \end{aligned}$$

The only question is whether the last letter is A or B. □

12.2. **free group as adjoint functor.** If S is a set, let $G(S)$ be the free group generated by S. This is the set of all sequences (reduced words)

$$w = s_1^{n_1}s_2^{n_2}\cdots s_k^{n_k}, \quad k \geq 0$$

where $n_i \in \mathbb{Z}$, $n_i \neq 0$ and $s_i \neq s_{i+1} \in S$. The *length* of w is $\ell(w) = \sum |n_i|$.

Theorem 12.2. *$G : \mathcal{E}ns \to \mathcal{G}ps$ is adjoint to the forgetful functor F. I.e.,*

$$\mathrm{Hom}_{\mathcal{E}ns}(S, FH) \cong \mathrm{Hom}_{\mathcal{G}ps}(GS, H)$$

Proof. The bijection G sends the mapping $f : S \to H$ to the group homomorphism $Gf : GS \to H$ given by

$$Gf(s_1^{n_1}s_2^{n_2}\cdots s_k^{n_k}) = f(s_1)^{n_1}f(s_2)^{n_2}\cdots f(s_k)^{n_k}.$$

The inverse is the restriction map; $G^{-1}(f : GS \to H) = f|S$. □

12.3. **actions and free products.** As Ivan Horozov pointed out, we can tell that $PSL_2(\mathbb{Z})$ is a free product from the way that it acts on upper half-space. The following theorem, which is Exercise 54 on p.81, explains how this works.

Theorem 12.3. *Suppose that $G_1, G_2, \cdots, G_n \leq G$ are subgroups of G which generate G. Suppose that G acts on a set S. Suppose there are subsets $S_1, S_2, \cdots, S_n \subseteq S$ and an element*

$$s \in S\backslash \bigcup S_i$$

in the complement of the sets S_i with the following property. For all $g \in G_i, g_i \neq e$,

1. *$g(S_j) \subseteq S_i$ for all $j \neq i$ and*
2. *$g(s) \in S_i$.*

Then G is the free product of the groups G_i:

$$G = G_1 * G_2 * \cdots * G_n$$

Proof. By the universal property of the free product, there is a homomorphism

$$\phi : G_1 * G_2 * \cdots * G_n \to G$$

which is the inclusion map on each G_i. Since the groups G_i generate G, this homomorphism is onto. Thus, it suffices to show that the kernel of ϕ is trivial. So, suppose that there is an element in the kernel of ϕ. This has the form

$$g_1 g_2 \cdots g_k$$

where $g_j \neq e$ is an element of G_{i_j} and $i_j \neq i_{j+1}$.

Suppose for example that this element is $g_1g_2g_3$ where $g_1 \in G_5, g_2 \in G_9, g_3 \in G_4$. Then

$$g_1g_2g_3(s) \in G_5$$

since $g_3(s) \in S_4, g_2(g_3(s)) \in g_2(S_4) \subseteq S_9, g_1(g_2g_3(s)) \in g_1(S_4) \subseteq S_5$. Therefore, $g_1g_2g_3 \neq e$. And in general, $g_1 \cdots g_k \neq e$ which implies that ϕ has a trivial kernel and is thus an isomorphism. □

Corollary 12.4. $PSL_2(\mathbb{Z}) \cong \mathbb{Z}/2 * \mathbb{Z}/3$.

Proof. We apply the theorem to $G = PSL_2(\mathbb{Z}), G_1 = \langle A \rangle$ where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $G_2 = \langle B \rangle$ where $B = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$. Let $S = \mathbb{H}$ be the upper half plane. Let

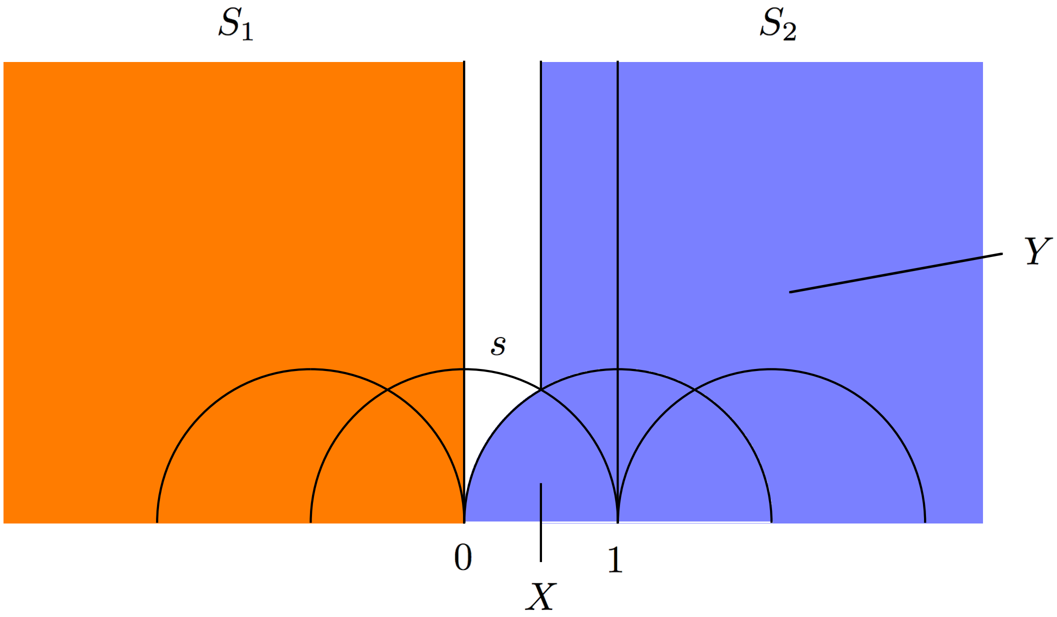
$$S_1 = \{z = x + iy \in \mathbb{H} \,|\, x < 0\}$$

Then $Az = -1/z$. So, A reverses the sign of the real part of z. Therefore, A sends $s = 1//4 + i$ and the set $\{z = x + iy \in \mathbb{H} \,|\, x > 0\}$ into S_1. Let $S_2 = X \cup Y$ where

$$X = \{z \in \mathbb{H} \,|\, |z - 1| < 1 \text{ and } |z| \leq 1\}$$

$$Y = \{z = x + iy \in \mathbb{H} \,|\, x \geq 1/2 \text{ and } |z| > 1\}$$

Then $B(X) = Y$, $Bs \in X$ and $B(S_1) \subseteq X$. Therefore, the conditions of the theorem are satisfied and we conclude that $PSL_2(\mathbb{Z})$ is the free product of the subgroups generated by A and B. □


S_1
S_2
Y
s
0
1
X