Definition 1.10. *If S is a subset of a group G then the* subgroup generated by *S, written $\langle S\rangle$, is defined to be the intersection of all subgroups of G which contain S:*

$$\langle S\rangle := \bigcap_{S\subseteq H\leq G} H.$$

Definition 1.11. *The* commutator subgroup *(also called the* derived subgroup*) $G' = [G,G]$ is the subgroup of G generated by all commutators $[a,b]$.*

In general, if $A, B \leq G$ then $[A,B]$ is defined to be the subgroup of G generated by all commutators $[a,b]$ where $a \in A$ and $b \in B$.

Theorem 1.12. *$G' = [G,G]$ is a normal subgroup of G.*

This follows immediately from the following two facts.

Lemma 1.13.
(1) $x\langle S\rangle x^{-1} = \langle S\rangle$
(2) $x[a,b]x^{-1} = [xax^{-1}, xbx^{-1}]$.

This lemma has a generalization:

Lemma 1.14. *Given a homomorphism $\phi : G \to H, a, b \in G, S \subseteq G$ we have:*
(1) $\phi\langle S\rangle = \langle\phi(S)\rangle$.
(2) $\phi[a,b] = [\phi(a), \phi(b)]$.

Why is this a generalization of the previous lemma?
The main theorem about the commutator subgroup is the following.

Theorem 1.15. *The image $\phi(G)$ of a homomorphism $\phi : G \to H$ is abelian if and only if the kernel of ϕ contains the commutator subgroup*

For this we need the following lemma whose proof is obvious.

Lemma 1.16. *S is a subset of $H \leq G$ iff $\langle S\rangle$ is a subgroup of H.*

Proof of Theorem 1.15. The argument which we did in class is reversible, i.e., "iff" at every step: For any $a, b \in G$ we have

$$[\phi(a), \phi(b)] = \phi[a,b].$$

$\phi(G)$ is abelian iff the LHS is always e. But, the RHS is always equal to e iff $[a,b] \in \ker\phi$ for all $a, b \in G$ which, by Lemma 1.16, is equivalent to saying that $G' \leq \ker\phi$. □

The theorem has the following variation as an obvious corollary:

Corollary 1.17. *Suppose that $N \trianglelefteq G$. Then G/N is abelian iff $G' \leq N$.*

2. Solvable groups

After doing a review of group theory at lightning speed we managed to get to the first topic of the course on the first day: Solvable groups.

2.1. two definitions.

I gave two equivalent definitions of a solvable group. Here is the first definition.

Definition 2.1. *A group G is* solvable *if an iterated derived subgroup $G^{(n)}$ is trivial for some positive integer n. Here $G^{(n)}$ is defined recursively as follows.*

1. $G^{(0)} = G$
2. $G^{(i+1)} = (G^{(i)})' = [G^{(i)}, G^{(i)}]$.

Serge Lang uses normal towers to define solvable groups.

Definition 2.2. *A* normal tower *for a group G is a sequence of subgroups:*

$$G = G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_n = \{e\}$$

so that each subgroup is normal in the previous one: $G_i \trianglelefteq G_{i-1}$. The quotient groups G_{i-1}/G_i are called the subquotients *of the tower.*

In general a *subquotient* of a group G is a quotient of a subgroup of G. (This is more general than a subgroup of a quotient. Why is that?) Here is Lang's definition of a solvable group.

Definition 2.3. *A group is* solvable *if it has a normal tower whose subquotients are all abelian. Lang calls these* abelian towers*.*

We need to show that these definitions are equivalent. One direction is obvious. The first definition implies the second. This is because the derived series:

$$G \trianglerighteq G' \trianglerighteq G'' \trianglerighteq \cdots \trianglerighteq G^{(n)} = \{e\}$$

is a normal tower with abelian subquotients. To prove the converse, we need one more corollary or Theorem 1.15.

Corollary 2.4. *Suppose that $H \leq G, N \trianglelefteq G$ and G/N is abelian. Then $H' \leq H \cap N$.*

Proof. Take the composition

$$\phi : H \hookrightarrow G \twoheadrightarrow G/N.$$

Since $\phi(H) \leq G/N$ is abelian,

$$H' \leq \ker\phi = \{x \in H \mid \underbrace{\phi(x) = e}_{x \in N}\} = H \cap N.$$

□

Coming back to the equivalence of definitions, suppose that G has a normal tower with abelian subquotients. Since G/G_1 is abelian, $G' \leq G_1$. Suppose by induction that $G^{(k)} \leq G_k$. We know that $G_{k+1} \trianglelefteq G_k$ with abelian quotient. The corollary tells us that

$$(G^{(k)})' = G^{(k+1)} \leq G^{(k)} \cap G_{k+1} \leq G_{k+1}.$$

Therefore $G^{(n)} \leq G_k = \{e\}$ making G solvable by the first definition.

2.2. **degree of solvability and examples.**

We say that G is solvable of *degree* n if $G^{(n)} = \{e\}$ and $G^{(n-1)}$ is nontrivial.

1. Abelian groups are solvable of degree 1 (except for the trivial group which is solvable of degree 0).
2. S_3, the symmetric group on 3 letters is solvable of degree 2.
3. $T(n, \mathbb{Z})$, the group of unipotent matrices with coefficients in $\mathbb{Z}$ is solvable, but of what degree?

To prove that S_3 is solvable, take the normal tower:

$$S_3 \trianglerighteq A_3 \trianglerighteq \{e\}.$$

Here $A_3 = \{e, (123), (132)\}$ is the alternating group. This is a cyclic group and thus abelian and $S_3/A_3 \cong \mathbb{Z}/2$ is also abelian. So, S_3 is solvable of degree 2.

As I mentioned in class, *unipotent* means upper triangular with 1's on the diagonal. Any unipotent matrix can be written in the form $I_n + X$ where I_n is the $n \times n$ identity matrix and X is a strictly upper triangular matrix, i.e.,

$$x_{ij} = 0 \text{ unless } j \geq i + 1$$

Let U_k be the set of strictly upper triangular matrices $X = (x_{ij})$ so that

$$x_{ij} = 0 \text{ unless } j \geq i + k$$

Then

$$U_j U_k \subseteq U_{j+k}$$

Therefore, every element of $U_k, k \geq 1$ is nilpotent: $X^n = 0$. This means that

$$(I + X)^{-1} = I - X + X^2 - X^3 + \cdots + (-1)^{n-1} X^{n-1} = I - X(I + X)^{-1}.$$

Let

$$T_k = \{I + X \mid X \in U_k\}.$$

Then $T(n, \mathbb{Z}) = \{I + X \mid X \in U_1\} = T_1$.

Lemma 2.5. *$T_1 \geq T_2 \geq T_3 \geq \cdots \geq T_n = \{I\}$ is a normal tower with abelian subquotients.*

Proof. Take arbitrary elements $I+X, I+Y$ in T_1, T_k resp. Then

$$(I+X)(I+Y)(I+X)^{-1} = (I+X+Y+XY)(I+X)^{-1} = I+(Y+XY)(I+X)^{-1}$$

This is an element of T_k since $(Y+XY)(I+X)^{-1} \in U_k U_0 \subseteq U_k$. Therefore, $T_k \trianglelefteq T_1$.

If $X, Y \in U_k$ then expanding $(I+X)^{-1}$ as $I - X(I+X)^{-1}$ we get:

$$(I+X)(I+Y)(I+X)^{-1} = I + Y + XY - (Y+XY)X(I+X)^{-1}.$$

$$[I+X, I+Y] = I + (XY - (Y+XY)X(I+X)^{-1})(I+Y)^{-1} \in T_{2k}$$

Therefore, T_k/T_{2k} is abelian. ☐

This shows that $T(n, \mathbb{Z})$ is solvable of degree $\leq k$ if $n \leq 2^k$.

2.3. **subgroups and quotient groups.** We want to know that subgroups and quotient groups of solvable groups are solvable.

Theorem 2.6. *Every subgroup of a solvable group is solvable.*

Proof. If $H \leq G$ then $H' \leq G'$. This, in turn, implies that $(H')' = H^{(2)} \leq G^{(2)} = (G')'$. Eventually we get $H^{(n)} \leq G^{(n)} = \{e\}$. So, H is solvable of degree $\leq n$. ☐

Theorem 2.7. *Every quotient group of a solvable group is solvable.*

Proof. I claim that

$$G/N \trianglerighteq G'N/N \trianglerighteq G^{(2)}N/N \trianglerighteq \cdots \trianglerighteq G^{(n)}N/N = \{e\}$$

is a normal tower for G/N with abelian quotients. In fact $G^{(k)}N/N$ is the image of $G^{(k)}$ under the homomorphism

$$G^{(k)} \hookrightarrow G \twoheadrightarrow G/N.$$

Therefore,

$$G^{(k)}N/N = (G/N)^{(k)}.$$

This uses the following lemma. ☐

Lemma 2.8. *If $\phi : G \twoheadrightarrow H$ is an epimorphism (surjective homomorphism) then $\phi(G^{(k)}) = H^{(k)}$ for all $k \geq 0$.*

Theorem 2.9. *Suppose that $N \trianglelefteq G$ and $N, G/N$ are solvable. Then G is solvable.*

Proof. We are given that $N, G/N$ are solvable. So, we have abelian towers

$$N \trianglerighteq N_1 \trianglerighteq N_2 \cdots N_n = \{e\}$$

Since subgroups of G/N all have the form H/N for some $N \leq H \leq G$, the abelian tower for G/N looks like this:

$$G/N \trianglerighteq G_1/N \trianglerighteq G_2/N \cdots G_m/N = \{e\}$$

Where $G_m = N$. By the following lemma the abelian subquotients of this tower are

$$\frac{G^{(k)}/N}{G^{(k+1)}/N} \cong G^{(k)}/G^{(k+1)}$$

Therefore,

$$G \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_m = N \trianglerighteq N_1 \trianglerighteq \cdots \trianglerighteq N_n = \{e\}$$

is a normal tower for G with abelian subquotients proving that G is solvable of degree $\leq n+m$. □

Lemma 2.10. *If* N, H *are normal subgroups of* G *with* $N \leq H$ *then*

$$\frac{G/N}{H/N} \cong \frac{G}{H}.$$

Proof. Let $\phi : G/N \to G/H$ be the homomorphism given by $\phi(aN) = aNH = aH$. Then ϕ is clearly onto and $\ker\phi = \{aN \in G/N \mid aH = H\}$. But $aH = H$ iff $a \in H$. So, $\ker\phi = H/N$. The lemma follows from the equation

$$image(\phi) = \frac{domain(\phi)}{\ker\phi}$$

□