3. Operations of a group on a set

3.1. definition and basic properties.

Definition 3.1. *If X is a set, the* permutation group *of X, $Perm(X)$ is the group of all bijections $f : X \to X$ under composition (i.e., with composition as the group law).*

For example, if $X = \{1, 2, 3\}$ then $Perm(X) = S_3$ is the permutation group on three letters.

Definition 3.2. *An* action *of a group G on a set X is a homomorphism*

$$\pi : G \to Perm(X)$$

sending every $g \in G$ to a permutation π_g of X.

The notation is $\pi_g(x) = gx$.

Definition 3.3. *If $x \in X$, the* orbit *of x is the set*

$$\pi_G(x) := \{\pi_g(x) = gx \mid g \in G\}.$$

The stabilizer *of x in G is the subgroup*

$$G_x := \{g \in G \mid gx = x\}.$$

Theorem 3.4. *(1) X is a disjoint union of orbits: $X = \coprod \pi_G(x_i)$.*
(2) The stabilizer of any two elements of the same orbit are conjugate.

Proof. I only proved the second part. In fact I showed that

$$G_{gx} = gG_xg^{-1}$$

by the following reversible proof:

$$a \in G_{gx} \Leftrightarrow agx = gx \Leftrightarrow g^{-1}agx = x \Leftrightarrow g^{-1}ag \in G_x \Leftrightarrow a \in gG_xg^{-1}.$$

□

In the lecture I realized that this proof assumes the following lemma.

Lemma 3.5. *Two elements $x, y \in X$ lie in the same orbit of an action of G if and only if $y = gx$ for some $g \in G$.*

Proof. Suppose that x, y lie in the orbit $\pi_G(z)$. Then they can be written as $x = hz, y = kz$. So, $z = h^{-1}x$ and $y = kh^{-1}x$. □

3.2. examples.

I believe I gave three examples of group actions.

3.2.1. *example: left multiplication.* Let H be a subgroup of a group G. Then the group H acts on the set G by left multiplication:

$$\lambda_h(x) = hx.$$

What does it mean that this is an action? The definition says that this must be a homomorphism

$$\lambda : H \to Perm(G)$$

and this is just one equation:

$$\lambda_g \pi_h = \lambda_{gh}$$

Verification is trivial:

$$\lambda_g \lambda_h(x) = \lambda_g(hx) = ghx = \lambda_{gh}(x).$$

I also pointed out that right multiplication $\rho_h(x) = xh$ is not an action.

Questions:

(1) What are the orbits of this action?
(2) What is the stabilizer subgroup of $g \in G$?

Answers:

(1) The orbits are the right cosets of H.
(2) The stabilizers are all trivial.

3.2.2. *example: conjugation of elements.* Let G be any group and take the action of G on G by conjugation:

$$\gamma_g(x) = gxg^{-1}.$$

The orbits of this action are the *conjugacy classes* of elements of G:

$$\gamma_G(x) = \{gxg^{-1} \mid g \in G\}.$$

I write this as $C(x)$.

The stabilizers of the action also have a name. They are called the *centralizers* of the elements of G:

$$Z_G(x) := \{g \in G \mid gx = xg\}.$$

For example, if $G = S_3$ then

$$\begin{array}{rlrl} \gamma_G(e) = & \{e\} & G_e = & S_3 \\ \gamma_G(12) = & \{(12), (23), (13)\} & G_{(12)} = & \{e, (12)\} \\ \gamma_G(123) = & \{(123), (132)\} & G_{(123)} = & A_3 \end{array}$$

3.2.3. *example: conjugation of subgroups.* Let G be any group and let S be the set of all subgroups of G. Then G acts on S by conjugation:

$$\gamma_g(H) = gHg^{-1}.$$

What can we say about the orbits and stabilizers of this action?

The only things I could think of were the following.

(1) A subgroup $N \leq G$ is normal if and only if its orbit under the conjugation action is a singleton: $\gamma_G(N) = \{N\}$. (A *singleton* is a set with one element.)
(2) The stabilizer subgroup of $H \leq G$ is the *normalizer*
$$N_G(H) := \{g \in G \mid gHg^{-1} = H\}.$$
(3) The stabilizer of a normal subgroup is the whole group: $N_G(N) = N$.

For example, if $G = S_3$ then S has six elements:

$$\{e\}, \langle(12)\rangle, \langle(13)\rangle, \langle(23)\rangle, A_3, S_3$$

These form four orbits: $\{e\}, A_3, S_3$ are normal. So, they are singleton orbits. The other three form one orbit. The stabilizers of the normal subgroups are the whole group. The stabilizer of $\langle(ab)\rangle$ is itself. (These subgroups are *self-normalizing*.)

3.3. **morphisms of G-sets.**

Definition 3.6. *If X, Y are G-sets (sets with G-action), a* morphism *of G-sets is a mapping*

$$f : X \to Y$$

which commutes with the action of G. This means that

$$f \circ \pi_g = \pi_g \circ f$$

for all $g \in G$. Equivalently, $f(gx) = gf(x)$ for all $g \in G, x \in X$. You can also write this as a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \pi_g}\downarrow & & \downarrow{\scriptstyle \pi_g} \\ X & \xrightarrow{f} & Y \end{array}$$

One example of a morphism of G-sets is given by right multiplication. Take $X = Y = G$ with group H acting by left multiplication. Then for a fixed $g \in G$ right multiplication by g is a morphism of H-sets since

$$\lambda_h \circ \rho_g = \rho_g \circ \lambda_h.$$