7. Category theory and products

I want to prove the theorem that a finite group is nilpotent if and only if it is the product of its Sylow subgroups. For this we first have to go over the product of groups. And this looks like a good time to introduce category theory.

7.1. categories.

I gave the definition of a category and two examples to illustrate the definition: $\mathcal{G}ps$ is the category of groups and $\mathcal{E}ns$ is the category of sets.

Definition 7.1. *A* category $\mathcal{C}$ *consists of four things:* $\mathcal{C} = (Ob(\mathcal{C}), Mor, \circ, id)$ *where*

1. $Ob(\mathcal{C})$ *is a collection of* objects. *This collection is usually not a set. For example,* $Ob(\mathcal{G}ps)$ *is the collection of all groups and* $Ob(\mathcal{E}ns)$ *is the collection of all sets.*
2. *For any two objects* $X, Y \in Ob(\mathcal{C})$ *there is a set of* morphisms
$$Mor_{\mathcal{C}}(X, Y)$$
which are written $f : X \to Y$. *For example, in* $Mor_{\mathcal{G}ps}(G, H)$ *is the set of homomorphisms* $\phi : G \to H$ *and* $Mor_{\mathcal{E}ns}(S, T)$ *is the set of all mappings* $f : S \to T$.
3. *For any three objects* X, Y, Z, *we have a composition law:*
$$Mor_{\mathcal{C}}(Y, Z) \times Mor_{\mathcal{C}}(X, Y) \to Mor_{\mathcal{C}}(X, Z)$$
sending (g, f) *to* $g \circ f$. *Composition must be associative.*
4. *Every object* $X \in Ob(\mathcal{C})$ *has an* identity $id_X \in Mor_{\mathcal{C}}(X, X)$ *so that* $id_Y \circ f = f = f \circ id_X$ *for any* $f : X \to Y$.

Note that there are only two assumptions about the structure. Namely, associativity of composition and the existence of units.

The idea of category theory is to extract elementary concepts out of difficult mathematics. We look only at composition of morphisms and forget the rest of the structure. Then we can ask: What are the properties that can be expressed only in terms of composition of morphisms? One of these is the product.

7.2. product of groups.

Definition 7.2. *If* G, H *are groups, then the* product $G \times H$ *is defined to be the cartesian product of sets*
$$G \times H = \{(g, h) \mid g \in G, h \in H\}$$

with the group law given coordinate-wise by

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2).$$

These are several things to notice about this definition. The first is that $G \times H$ contains a copy of G, H which commute. By this I mean that there are monomorphisms ($1-1$ homomorphisms):

$$j_1 : G \to G \times H, \qquad j_2 : H \to G \times H$$

given by $j_1(g) = (g, e), j_2(h) = (e, h)$. These inclusion maps have commuting images since

$$j_1(g)j_2(h) = (g, e)(e, h) = (g, h) = (e, h)(g, e) = j_2(h)j_1(g)$$

7.2.1. *internal direct product.*

Lemma 7.3. *If $\phi : G \to K$, $\psi : H \to K$ are homomorphisms with commuting images then there is a unique homomorphism $f : G \times H \to K$ so that $f \circ j_1 = \phi$ and $f \circ j_2 = \psi$.*

Proof. This is obvious. f must be given by $f(g, h) = j_1(g)j_2(h)$. This is a homomorphism since $[j_1(G), j_2(H)] = \{e\}$. □

Theorem 7.4. *Suppose that G contains normal subgroups H, K so that $H \cap K = \{e\}$, $[H, K] = \{e\}$ and $HK = G$. Then the homomorphism $f : H \times K \to G$ given by the inclusion maps $H \hookrightarrow G, K \hookrightarrow G$ is an isomorphism.*

We say that $G = H \times K$ is the *internal direct product* in this case.

Proof. The map is given by $f(h, k) = hk$. This is surjective since $HK = G$. It is $1-1$ since $H \cap K = \{e\}$. It is a homomorphism since $[H, K] = \{e\}$. □

7.2.2. *universal property.* The product $G \times H$ has two other projection homomorphisms

$$p_1 : G \times H \to G, \qquad p_2 : G \times H \to H$$

given by $p_1(g, h) = g, p_2(g, h) = h$. These satisfy the following "universal" property which is obvious (obviously true) and which I also explained in categorical terms.

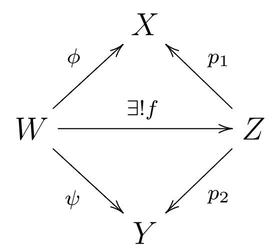
Theorem 7.5. *Suppose that G, H, K are groups and $\phi : K \to G, \psi : K \to H$ are homomorphisms. Then there exists a unique homomorphism $f : K \to G \times H$ so that $p_1 \circ f = \phi$ and $p_2 \circ f = \psi$.*

The unique homomorphism is $f(x) = (\phi(x), \psi(x))$ and it is written $f = \phi \times \psi$.

7.3. **categorical product.** The last theorem is categorical since it involves only composition of homomorphism. It says that $G \times H$ is a categorical product.

Definition 7.6. *Suppose that X, Y are objects of a category $\mathcal{C}$. Then $Z \in \mathcal{C}$ is the* product *of X and Y if there are morphisms $p_1 : Z \to X, p_2 : Z \to Y$ so that for any other object W and any morphisms $\phi : W \to X, \psi : W \to Y$ there is a unique morphism $f : W \to Z$ so that $p_1 \circ f = \phi$ and $p_2 \circ f = \psi$.*

The condition can be written as a commuting diagram:



We say that Z is the *product* of X, Y in the category $\mathcal{C}$ and we write $Z = X \times Y$. We also call Z the *categorical product* of X and Y. Theorem 7.5 was written in such a way that it is obvious that the product of groups is the categorical product.

The next point I made was that the definition of a product defines $Z = X \times Y$ uniquely up to isomorphism.

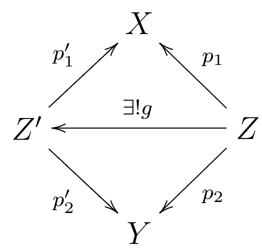
The concept of an isomorphism is categorical:

Definition 7.7. *Two objects X, Y in any category $\mathcal{C}$ are* isomorphic *and we write $X \cong Y$ if there are morphisms $f : X \to Y$ and $g : Y \to X$ so that $f \circ g = id_Y$ and $g \circ f = id_X$.*

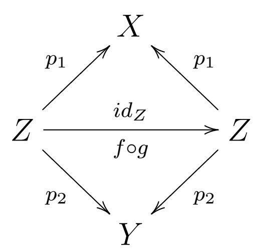
The definition of product is by a "universal condition" which forces the object Z to be unique up to isomorphism if it exists. (If the product does not exist, it suggests that the category is not large enough and perhaps we should add more objects.)

Theorem 7.8. *The product $Z = X \times Y$ is unique up to isomorphism assuming it exists.*

Proof. Suppose that Z' is another product. This means what we have morphisms $p_1' : Z \to X, p_2' : Z \to Y$ so that for any W, such as $W = Z$, any morphisms $W \to X, W \to Y$ (such as p_1, p_2) there is a unique morphism g so that $p_i' \circ g = p_i$ for $i = 1, 2$. In other words, the following diagram commutes.



Similarly, since Z is the product, there is a unique morphism $f : Z' \to Z$ so that $p_i \circ f = p_i'$ for $i = 1, 2$. Now, take Z and Z. We have two morphisms $Z \to Z$ making the following diagram commute:



By the uniqueness clause in the definition of the product, we must have

$$f \circ g = id_Z.$$

Similarly, $g \circ f = id_{Z'}$. So, $Z \cong Z'$. □

7.4. **products of nilpotent groups.** Finally, I proved the following theorem which we need. I used three lemmas without proof. But, I am giving the proofs here after the proof of the theorem (and retroactively in Corollary 5.3).

Theorem 7.9. *If G, H are nilpotent groups of nilpotency class c_1, c_2 resp. then $G \times H$ is nilpotent of class $c = \max(c_1, c_2)$.*

The proof is by induction on c using the following lemma.

Lemma 7.10. *The center of $G \times H$ is $Z(G) \times Z(H)$.*

Proof. This is obvious. An element $(x, y) \in G \times H$ is central if

$$(x, y)(g, h) = (xg, yh) = (gx, hy) = (g, h)(x, y)$$

for all $g \in G, h \in H$. But this is the same as saying that $x \in Z(G)$ and $y \in Z(H)$. □

Lemma 7.11. *If $A \trianglelefteq G, B \trianglelefteq H$ then*

$$\frac{G \times H}{A \times B} \cong G/A \times H/B.$$

Proof. It suffices to find an epimorphism $\phi : G\times H \to G/A\times H/B$ with kernel $A\times B$. Such an epimorphism is given by $\phi(g, h) = (gA, hB)$. □

Proof of Theorem 7.9. By induction on c. If $c = 1$ then G, H are both abelian. Then $G \times H$ is also abelian, which is the same as being nilpotent of class 1.

Now suppose that the theorem is true for $c-1$. Since $G/Z(G), H/Z(H)$ are nilpotent of class $c_1 - 1, c_2 - 1$, their product

$$G/Z(G) \times H/Z(H)$$

is nilpotent of class $\max(c_1 - 1, c_2 - 1) = c - 1$. By the lemmas above, this is isomorphic to $G \times H$ modulo its center. Therefore, by Corollary 5.3, $G \times H$ is nilpotent of class c. □