8. PRODUCTS WITH MANY FACTORS

I am giving a long explanation about why finite nilpotent groups are products of Sylow subgroups:

$$G = P_1 \times P_2 \times \cdots \times P_n$$

So, I should explain first about products with many factors. There is a group theoretic definition and a categorical definition and they agree.

8.1. product of many groups.

A product of finitely many groups is given by:

$$G_1 \times \cdots \times G_n = \{(g_1, \cdots, g_n) \,|\, g_i \in G_i \text{ for } i = 1, 2, \cdots, n\}$$

If we have an infinite family of groups G_α indexed by $\alpha \in I$ then the *product* of these groups is the cartesian product

$$\prod_{\alpha \in I} G_\alpha = \{g : I \to \cup G_\alpha \,|\, g(\alpha) \in G_\alpha \ \forall \alpha \in I\}.$$

We write $g(\alpha) = g_\alpha$ and call it the α coordinate of g. Multiplication is defined coordinatewise by

$$(fg)_\alpha = (f_\alpha)(g_\alpha)$$

This can be written as $p_\alpha(fg) = p_\alpha(f)p_\alpha(g)$. I.e., the projection maps

$$p_\alpha : \prod G_\alpha \to G_\alpha$$

are homomorphisms.

Elements of an infinite product may have infinitely many nontrivial coordinates. This sometimes causes trouble. However, it is required by category theory. The product as we just defined it satisfies the following categorical condition for obvious reasons.

Theorem 8.1. *Suppose that $G_\alpha, \alpha \in I$ is a family of groups and H is another group. Let $f_\alpha : H \to G_\alpha$ be arbitrary homomorphisms. Then there is a unique homomorphism $\phi : H \to \prod G_\alpha$ so that $p_\alpha \circ \phi = f_\alpha$ for all $\alpha \in I$.*

This means that $\prod G_\alpha$ is the product of the objects G_α of the category of groups.

Definition 8.2. *If $X_\alpha, \alpha \in I$ is a family of objects in a category $\mathcal{C}$ indexed by the set I then $Z \in Ob(\mathcal{C})$ is called the* product *of the X_α if there are morphisms $p_\alpha : Z \to X_\alpha$ for all $\alpha \in I$ so that for any other object W of $\mathcal{C}$ and morphisms $f_\alpha : W \to X_\alpha$ there exists a unique morphism $\phi : W \to Z$ so that $p_\alpha \circ \phi = f_\alpha$ for all $\alpha \in I$.*

The product of the X_α, if it exists, is unique up to isomorphism in the category $\mathcal{C}$ because it is given by a universal condition. I will explain later a rigorous definition of "universal condition" which makes this statement obvious. What is not obvious is the definitions which become more and more complicated.

8.2. **weak product.** When the index set I is infinite we often consider the *weak product*

$$\prod_{\alpha\in I}{}' \, G_\alpha = \{(g_\alpha)_{\alpha\in I} \mid g_\alpha = e \text{ for all but a finite number of } \alpha\}.$$

When the groups G_α are abelian, this is called the *sum* and written $\bigoplus G_\alpha$. The weak product contains a copy of each group G_α via the monomorphism

$$j_\alpha : G_\alpha \to \prod_{\alpha\in I}{}' \, G_\alpha$$

which sends $x \in G_\alpha$ to the element $(e, \cdots, e, x, e, \cdot, e)$ with x in the α coordinate and e everywhere else. These elements (for different α) commute and weak product is universal with this property. I.e.:

Theorem 8.3. *Suppose that G_α are groups indexed by I and H is another group. Let $f_\alpha : G_\alpha \to H$ be homomorphisms whose images commute, i.e.,*

$$f_\alpha(x)f_\beta(y) = f_\beta(y)f_\alpha(x)$$

whenever $x \in G_\alpha, y \in G_\beta$ and $\alpha \neq \beta$. Then there exists a unique homomorphism

$$\phi : \prod_{\alpha\in I}{}' \, G_\alpha \to H$$

so that $\phi \circ j_\alpha = f_\alpha$ for all $\alpha \in I$.

Proof. An element of the infinite product (g_α) has only finitely many nontrivial coordinates $g_{\alpha_1}, g_{\alpha_2}, \cdots, g_{\alpha_n}$. $\phi(g)$ must be equal to the product

$$\phi(g) = j_{\alpha_1}(g_{\alpha_1}) j_{\alpha_2}(g_{\alpha_2}) \cdots j_{\alpha_n}(g_{\alpha_n})$$

where the factors can be multiplied in any order since they commute. □

8.3. **recognizing weak products.** I decided to take a finite product. But the same statement is true for an infinite weak product.

Suppose that $N_1, N_2, \cdots, N_n$ are normal subgroups of a group G so that

1. $N_i \cap N_j = \{e\}$ for all $i \neq j$ and
2. $N_1 N_2 \cdots N_n = G$.

Then does this imply that G is isomorphic to the product $N_1 \times \cdots \times N_n$? The following example says not.

Take the group $G = \mathbb{Z}/2 \times \mathbb{Z}/2$. This has four elements e, a, b, ab. Each of the three nontrivial elements generates a (normal) subgroup of order 2: $N_1 = \langle a \rangle, N_2 = \langle b \rangle, N_3 = \langle ab \rangle$. These normal subgroups satisfy (1) and (2) but $N_1 \times N_2 \times N_3$ is not isomorphic to G since it has 8 elements. However, condition (1) implies that the elements of N_i and N_j commute for $i \neq j$. Therefore, Theorem 8.3 implies that there is a homomorphism

$$\phi : N_1 \times N_2 \times \cdots N_n \to G$$

which is the inclusion on each N_i. Condition (2) implies that ϕ is onto.

Theorem 8.4. *Suppose that $N_1, N_2, \cdots, N_n$ are normal subgroups of G satisfying conditions (1),(2) above. Then the homomorphism ϕ above is an isomorphism if and only if*

$$N_j \cap N_1 N_2 \cdots \widehat{N_j} \cdots N_n = \{e\}$$

for all j.

Proof. This new condition is certainly necessary is G is to be the product of the N_i. To show that it is sufficient, suppose it is true. Then ϕ is a monomorphism. Otherwise, there are elements $x_i \in N_i$ not all trivial so that

$$\phi(x_1, x_2, \cdots, x_n) = x_1 x_2 \cdots x_n = e$$

If $x_j \neq e$ then it is the inverse of the product of the other x_i's contradicting the new condition. □

8.4. **products of Sylow subgroups.** Since p-groups are nilpotent, Theorem 7.9 implies that a product of p-groups is also nilpotent. Using Theorem 8.4 this implies the following.

Corollary 8.5. *If G is a finite group whose Sylow subgroups are normal then G is a product of its Sylow subgroups and therefore nilpotent.*

Proof. Suppose that $P_1, P_2, \cdots, P_n$ are the Sylow subgroups of G. If $P_i \trianglelefteq G$ and $|P_i| = p_i^{k_i}$ then P_i is the unique Sylow p_i-subgroup of G. Therefore the primes p_i are all distinct and $P_i \cap P_j = \{e\}$. This is condition (1). Also we must have

$$P_j \cap P_1 \cdots \widehat{P}_j \cdots P_n = \{e\}$$

since any element of P_j has order a power of p_j and elements of $P_1 \cdots \widehat{P_j} \cdots P_n$ have order prime to p_j. (Use the fact that the order of ab is least common multiple of the orders of a, b when a, b commute.) Therefore,

$$\phi : P_1 \times P_2 \times \cdots \times P_n \to G$$

is a monomorphism. But the two groups have the same number of elements. So, ϕ must be an isomorphism. □

We need two more lemmas and HW 3.1.

Lemma 8.6. *Let $N \trianglelefteq G$ and $N \leq H \leq G$. Then*

$$N_{G/N}(H/N) = N_G(H)/N.$$

Proof. Let $N_{G/N}(H/N) = K/N$. Then $H/N \trianglelefteq K/N$ and we have a quotient map $K/N \to (K/N)/(H/N)$. The composite homomorphism:

$$K \to K/N \to \frac{K/N}{H/N}$$

has kernel H. Therefore, $H \trianglelefteq K$ and $K \leq N_G(H)$. Similarly, $H \trianglelefteq N_G(H)$ implies that $H/N \trianglelefteq N_G(H)/N$. Therefore, $N_G(H)/N$ is contained in the normalizer K/N of H/N. So $N_G(H) \leq K$. So, they must be equal. □

Lemma 8.7. *Suppose that G is nilpotent and H is a proper subgroup of G. Then $H \neq N_G(H)$.*

Proof. We have a central series

$$G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = \{e\}.$$

I.e., $G_i \trianglelefteq G$ and $G_{i-1}/G_i \leq Z(G/G_i)$. Let i be minimal so that $G_i \leq H$. Then $G_{i-1} \nleq H$. But

$$G_{i-1}/G_i \leq Z(G/G_i) \leq N_{G/G_i}(H/G_i) = N_G(H)/G_i$$

by the previous lemma. Therefore, $G_{i-1} \leq N_G(H)$ which shows that $H \neq N_G(H)$. □

Theorem 8.8. *A finite group G is nilpotent if and only if it is a product of its Sylow subgroups.*

Proof. Since p-groups are nilpotent and a finite product of nilpotent groups is nilpotent, a group which is the product of its Sylow subgroups must be nilpotent. Conversely, suppose that G is nilpotent. Then we claim that every Sylow subgroup must be normal and thus G is a product of its Sylow subgroups by Corollary 8.5. To prove this claim suppose not. Then there is a Sylow subgroup P which is not normal in G. Then $N_G(P) \neq G$. But HW 3.1 says that $H = N_G(P)$ is self-normalizing. This is impossible for nilpotent groups by Lemma 8.7. □