

## 9. UNIVERSAL OBJECTS AND LIMITS

I explained how universal constructions were all examples of initial or terminal objects in some category. I also explained that these are equivalent if we reverse arrows.

## 9.1. initial and terminal objects.

**Definition 9.1.** *Suppose that  $\mathcal{C}$  is a category. Then an object  $X_0$  of  $\mathcal{C}$  is called initial if for any object  $Y$  of  $\mathcal{C}$  there is a unique morphism  $f : X_0 \rightarrow Y$ . Similarly,  $X_\infty \in \text{Ob}(\mathcal{C})$  is called terminal if for any object  $W$  of  $\mathcal{C}$  there is a unique morphism  $f : W \rightarrow X_\infty$ . If  $X_0$  is both initial and terminal it is called a zero object.*

**Example 9.2.** *In  $\mathcal{E}ns$ , the category of sets, the empty set  $\emptyset$  is initial and any one point set  $\{*\}$  is terminal. In the category of groups  $\{e\}$  is both initial and terminal. So, the trivial group is the zero object.*

**Theorem 9.3.** *Initial and terminal objects are unique up to isomorphism if they exist.*

This is trivial but we went through it carefully because, as we will see later, it implies the uniqueness of any universal object.

*Proof.* Suppose that there are two initial objects  $X_0, X_1$ . Then

$$\begin{aligned} X_0 \text{ initial} &\Rightarrow \exists! f : X_0 \rightarrow X_1 \\ X_1 \text{ initial} &\Rightarrow \exists! g : X_1 \rightarrow X_0 \\ X_0 \text{ initial} &\Rightarrow \text{Any two morphisms } X_0 \rightarrow X_0 \text{ are equal.} \end{aligned}$$

Therefore,  $g \circ f = id_{X_0}$ . Similarly,  $f \circ g = id_{X_1}$ . Therefore,  $X_0 \cong X_1$ . The uniqueness of terminal objects is similar (and also follows from the next theorem).  $\square$

**Definition 9.4.** *If  $\mathcal{C}$  is any category, its opposite category  $\mathcal{C}^{op}$  is “the same thing with arrows reversed.” By this I mean that*

- (1)  $\text{Ob}(\mathcal{C}^{op}) = \text{Ob}(\mathcal{C})$ . *The opposite category has the same objects. However, we put a little “op” as a superscript to indicate that we are considering the object as being in  $\mathcal{C}^{op}$ . So, if  $X \in \text{Ob}(\mathcal{C})$  then  $X^{op}$  is  $X$  considered as an object of  $\mathcal{C}^{op}$ .*
- (2)  $\text{Mor}_{\mathcal{C}^{op}}(X^{op}, Y^{op}) = \text{Mor}_{\mathcal{C}}(Y, X)$ . *The morphism sets are equal. But the morphism  $f : Y \rightarrow X$  in  $\mathcal{C}$  is written*

$$f^{op} : X^{op} \rightarrow Y^{op}$$

*in  $\mathcal{C}^{op}$ .*

- (3)  $id_{X^{op}} = (id_X)^{op}$ . *(Identities are the same.)*
- (4)  $f^{op} \circ g^{op} = (g \circ f)^{op}$ . *(Composition is reversed.)*

For example, when we say that

$$f^{op} : G^{op} \rightarrow H^{op}$$

is a morphism in  $\mathcal{Gps}^{op}$ , we do not mean that we have created new objects called  $G^{op}$  and  $H^{op}$ . All this means is that we have an ordinary group homomorphism

$$f : H \rightarrow G.$$

The purpose is to change the description of the objects.

**Theorem 9.5.**  *$X$  is an initial (resp. terminal) object of  $\mathcal{C}$  if and only if  $X^{op}$  is a terminal (resp. initial) object of  $\mathcal{C}^{op}$ .*

**9.2. products as terminal objects.** If  $X_\alpha, \alpha \in I$  is a family of objects in  $\mathcal{C}$ , I created a new category  $\mathcal{B}$  so that a terminal object of  $\mathcal{B}$  is the same as the product of the objects  $X_\alpha$  in  $\mathcal{C}$ .

The *objects* of the new category  $\mathcal{B}$  consist of

- (1) an object  $Y$  of  $\mathcal{C}$  and
- (2) morphisms  $f_\alpha : Y \rightarrow X_\alpha$  for all  $\alpha \in I$ .

I wrote the element as:  $(Y, (f_\alpha)_{\alpha \in I})$ . If  $(Z, (g_\alpha))$  is another object in  $\mathcal{B}$  then a morphism

$$(Y, (f_\alpha)) \rightarrow (Z, (g_\alpha))$$

is defined to be a morphism  $\phi : Y \rightarrow Z$  in  $\mathcal{C}$  so that  $f_\alpha = g_\alpha \circ \phi$  for all  $\alpha \in I$ . In other words, the following diagram commutes for each  $\alpha$ .

$$\begin{array}{ccc} & & X_\alpha \\ & \nearrow f_\alpha & \uparrow g_\alpha \\ Y & \xrightarrow{\phi} & Z \end{array}$$

**Proposition 9.6.** *If  $(Z, (g_\alpha))$  is terminal in  $\mathcal{B}$  then  $Z \cong \prod X_\alpha$ .*

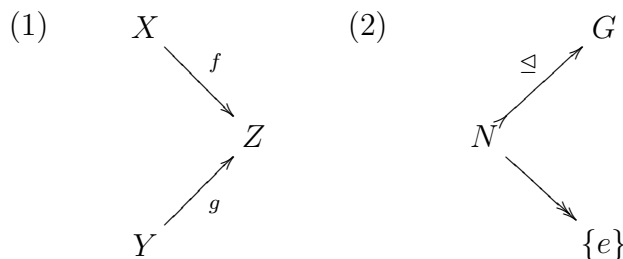
*Proof.*  $(Z, (g_\alpha))$  is terminal implies  $\phi : Y \rightarrow Z$  is unique which implies that  $Z = \prod X_\alpha$ .  $\square$

**9.3. general limits.** A categorical product is the limit of a diagram with no arrows. We need to generalize this construction to create more general limits. We will do this now in an arbitrary category and next week we will look in the category of groups and sets and specialize to particular diagrams.

**Definition 9.7.** *A diagram  $D$  in a category  $\mathcal{C}$  is a set of objects  $X_\alpha, \alpha \in I$  and a set of morphisms between these objects.*

I couldn't think of a good way to index the arrows in general. It depends on the diagram.

**Example 9.8.** Here are some important examples of diagrams.



(3)  $D = \{X_1, X_2, X_3\}$  (no arrows).

(4)

$$\dots \xrightarrow{f_3} G_2 \xrightarrow{f_2} G_1 \xrightarrow{f_1} G_0$$

(5)  $X, Y$  with two morphisms  $f, g : X \rightarrow Y$ .

**Definition 9.9.** If  $D$  is a diagram in a category  $\mathcal{C}$  then the category of objects over  $D$  which we write as  $\mathcal{C}/D$  has objects consisting of

- (1) one object  $Y$  of  $\mathcal{C}$  and
- (2) morphisms  $g_\alpha : Y \rightarrow X_\alpha$  going from  $Y$  to each object in the diagram  $D$  so that, for any morphism  $f : X_\alpha \rightarrow X_\beta$  in  $\mathcal{C}$ ,  $f \circ g_\alpha = g_\beta$ .

If  $(Y, (g_\alpha))$  is a terminal object in  $\mathcal{C}/D$ , then  $Y$  is called the limit of the diagram  $D$ .

Being a terminal object, the limit of a diagram is unique up to isomorphism if it exists. For the special case when the diagram has no arrows, the limit is the product.