9.4. **existence of limits.** I showed that limits exist in the category of sets and in the category of groups. I also showed that the category of sets has colimits. Colimits in the category of groups are complicated and will be discussed next week.

Theorem 9.10. *The category of sets has arbitrary limits.*

Proof. This means that, for any diagram of sets, with objects X_α and morphisms $f : X_\alpha \to X_\beta$ (which I did not index the first day), there exists a limit of the diagram. This is given by letting $\lim X_\alpha$ be the set of all (x_α) in the Cartesian product $\prod X_\alpha$ so that $f(x_\alpha) = x_\beta$ for all morphisms $f : X_\alpha \to X_\beta$ in the diagram. □

Theorem 9.11. *The category of sets has arbitrary colimits.*

Proof. Given a diagram D of sets with objects X_α and morphisms $f : X_\alpha \to X_\beta$, the *colimit* or *direct limit* is the set

$$\operatorname{colim} D = \coprod X_\alpha / \sim$$

which is the disjoint union of the sets X_α modulo the equivalence relation given by $x \in X_\alpha \sim f(x) \in X_\beta$.

The *disjoint union* is the union made disjoint. Formally this is the set

$$\coprod_{\alpha \in I} X_\alpha = \{(\alpha, x) \in I \times \bigcup X_\alpha \,|\, x \in X_\alpha\}.$$

When we say we "mod out the equivalence relation" we mean: Take the set of equivalence classes. The fact that this is the colimit is obvious. Given a mappings $g_\alpha : X_\alpha \to Y$, we get a mapping on the disjoint union. Since we require $g_\beta \circ f = g_\alpha$, the identified elements $x \in X_\alpha$ and $f(x) \in X_\beta$ map to the same element of Y, so we get an induced map of the quotient $\coprod X_\alpha / \sim$ into Y. □

Theorem 9.12. *Arbitrary limits exist in the category of groups.*

Proof. This is easy. It is the same thing as in the category of sets. Given a diagram with groups G_α connected my morphisms $f : G_\alpha \to G_\beta$, the limit of the diagram is

$$\{(g_\alpha) \in \prod G_\alpha \,|\, f(g_\alpha) = g_\beta \text{ for all } f : G_\alpha \to G_\beta \text{ in the diagram}\}$$

which is the same as in the category of sets. The proof of universality is also the same. The only thing we need to check (or at least realize needs to be checked), is that this actually defines a subgroup of the product. Sometimes definitions have a hidden commutativity assumption.

So, suppose that $(g_\alpha), (h_\alpha)$ are in the inverse limit set. This means $f(g_\alpha) = g_\beta$ and $f(h_\alpha) = h_\beta$. We need to check that $(g_\alpha h_\alpha)$ is in the set:

$$f(g_\alpha h_\alpha) = f(g_\alpha) f(h_\alpha) = g_\beta h_\beta.$$

So, it works. Also, it is trivial. So, the proper thing would have been to say: "It is easy to see that this is a subgroup of the product." $\square$

At this point I introduced the "forgetful functor" to explain what it means that the formula for limits is the same in the two categories. But in these notes this explanation comes later.

9.5. **examples.** I explained several examples to illustrate how this works.

9.5.1. *double arrow.* The first example (Example 9.8.5) was in answer to the question: What happens if there is more than one morphism between the same two objects.

Consider the diagram in $\mathcal{G}ps$ with two homomorphism $f_1, f_2 : G \to H$. The limit L has the property that it has maps $p_1 : L \to G$, $p_2 : L \to H$ so that $p_2 = f_i \circ p_1$ for $i = 1$ and $i = 2$. This implies that $f_1 \circ p_1 = f_2 \circ p_1$. In other words, $p_1 : L \to G$ has image in the subgroup

$$K = \{g \in G \mid f_1(g) = f_2(g)\}$$

Since K is a subset of G, the mapping $L \to K$ is unique. This means that K is the universal object, i.e., the limit of the diagram. K is called the *equalizer* of the two arrows.

9.5.2. *pull-back.* The second example I gave was the more standard construction of the pull-back which is the limit of the following diagram.

$$\begin{array}{ccc} & & G \\ & & \downarrow{\scriptstyle f_1} \\ H & \xrightarrow[f_2]{} & K \end{array}$$

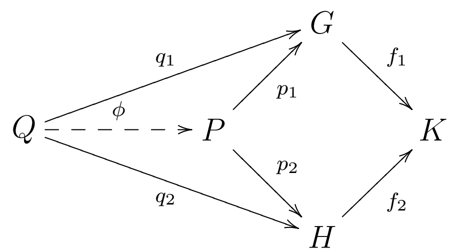
Definition 9.13. *Suppose that $f_1 : G \to K, f_2 : H \to K$ are homomorphisms. Then the* pull-back *P is defined to be the subgroups of the product $G \times H$ given by*

$$P = \{(g, h) \in G \times H \mid f_1(g) = f_2(g)\}$$

together with the projection homomorphisms $p_1 : P \to G, p_2 : P \to H$.

Proposition 9.14. *The pull-back is the limit of the diagram above.*

Proof. Suppose that there is another group Q with homomorphisms $q_1 : Q \to G, q_2 : Q \to H$ making the following diagram commute.



Then $f_1 q_1(x) = f_2 q_2(x)$ for every $x \in Q$. This is the same as saying that $\phi(x) = (q_1(x), q_2(x)) \in P$. So, ϕ in the diagram above exists and is unique. □

According to the formula, the limit is given by

$$\lim(G \to K \leftarrow H) = \{(g, k, h) \in G \times K \times H \mid f_1(g) = k = f_2(h)\}.$$

This is isomorphic to the pull-back because the coordinate $k \in K$ is redundant. An isomorphism $P \to \lim(G \to K \leftarrow H)$ is given by

$$(g, h) \mapsto (g, f_1(g), h).$$

9.5.3. *inverse limit.* The third example I gave is the diagram that most people think of as giving the "inverse limit."

$$\cdots \xrightarrow{f_4} G_3 \xrightarrow{f_3} G_2 \xrightarrow{f_2} G_1 \xrightarrow{f_1} G_0$$

We use the formula which says that the limit of this diagram is

$$\lim D = \varprojlim G_i = \{(g_0, g_1, g_2, \cdots) \in \prod G_i \mid f_i(g_i) = g_{i-1} \forall i\}.$$

An important special case is the inverse system:

$$\xrightarrow{f_4} \mathbb{Z}/n^3 \xrightarrow{f_3} \mathbb{Z}/n^2 \xrightarrow{f_2} \mathbb{Z}/n^1$$

When $n = 10$, the elements are sequences $(n_1, n_2, n_3, \cdots)$ so that each n_j is a j digit decimal whose last $j - 1$ digits give n_{j-1}. For example $(4, 14, 014, 9014, 19014, \cdots) \in \lim \mathbb{Z}/10^i$. This can be viewed as an infinite decimal going to the left:

$$\cdots 19014.$$

There are two important differences between infinite decimals going to the left and those going to the right.

(1) The expression is unique. For the usual decimals, two expressions can be equal. For example,

$$0.999999\cdots = 1.000000\cdots$$

But every element in the inverse limit is given by a unique expression.

(2) Negative quantities are included. For decimal going to the right, we need to put a negative sign in front to get the additive inverse. For these decimals going to the left, negatives are included. For example,

$$\cdots 99999. = -1$$

since, if we add 1 we get $\cdots 00000$.

Definition 9.15. *For any prime p the ring of p*-adic numbers $\mathbb{Z}_p$ *is given by*

$$\mathbb{Z}_p := \lim_{\leftarrow} \mathbb{Z}/p^i\mathbb{Z}.$$

But, we constructed it as a limit of additive groups. I will go over the ring structure later.

We figured out what the formula for the inverse is in $\mathbb{Z}_2$. An element is an arbitrary infinite sequence of 0's and 1's:

$$\cdots 01101000110.$$

To get the negative, you change all 0's to 1's and all 1's to 0's and then add 1.

$$-(\cdots 01000110) = \cdots 10111001 + 1 = \cdots 10111010.$$