2. Examples

Today, I gave some examples of rings and functors associated to rings.

2.1. $\mathbb{Z}$, the integers.

The first example is $\mathbb{Z} = (\mathbb{Z}, +, \cdot, 0, 1)$. This is a ring which has the property of being the initial object in the category of rings. In other words, for any ring R there is a unique ring homomorphism

$$\phi : \mathbb{Z} \to R.$$

This is given by $\phi(1) = 1, \phi(2) = 1 + 1, \cdots$ and $\phi(-n) = -\phi(n)$. The uniqueness follows from the fact that $\mathbb{Z}$ is generated as an additive group by 1. So, ϕ is determined by what it does to 1 which is $\phi(1) = 1$ by definition of a ring homomorphism.

2.2. $U(R)$, the group of units.

Definition 2.1. A *unit* in a ring R is any invertible element. I.e., $x \in R$ is a unit if $xy = yx = 1$ for some $y \in R$. The units of a ring form a group called the *group of units* $U(R)$.

Proposition 2.2. *U is a functor from the category of rings to the category of groups.*

Proof. Any ring homomorphism $\phi : R \to S$ takes units to units since $xy = yx = 1$ implies

$$\phi(xy) = \phi(x)\phi(y) = \phi(yx) = \phi(y)\phi(x) = \phi(1) = 1.$$

□

In the next example, I constructed the adjoint of this functor.

2.3. group rings.

Suppose that G is a multiplicative group and R is a commutative ring. Then the *group ring* RG is defined to be the set of all finite linear combinations

$$\sum r_i g_i$$

where $r_i \in R, g_i \in G$ where we are allowed to simplify (i.e., there is an equivalence relation) using the relation

$$rg + sg = (r + s)g$$

whenever the g_i are not distinct.

Addition is defined by just putting the terms together. For example,

$$(ag_1 + bg_2) + (ch_1 + dh_2) = ag_1 + bg_2 + ch_1 + dh_2.$$

In Σ notation this is

$$\sum_{i=1}^{n} r_i g_i + \sum_{j=1}^{m} s_j h_j = \sum_{i=1}^{n+m} r_i g_i$$

where we use the s's and h's when the index i becomes too large. I.e., $r_{n+j} = s_j, g_{n+j} = h_j$.

Multiplication is given by

$$\left(\sum r_i g_j\right)\left(\sum s_j h_j\right) = \sum_{i,j} r_i s_j g_i h_j$$

Example 2.3. Suppose that $G = \langle t \mid t^3 \rangle$. This is the group generated by t modulo the relation $t^3 = 1$. So, $G = \{1, t, t^2\}$ is the multiplicative group which is isomorphic to the additive group $\mathbb{Z}/3\mathbb{Z}$. The group ring is

$$RG = \{a + bt + ct^2 \mid a, b, c \in R\}$$

Example 2.4. Let G be the infinite cyclic group $Z = \langle t \rangle = \{t^n \mid n \in \mathbb{Z}\}$. This is the multiplicative version of the additive group $\mathbb{Z}$. Then the group ring RZ is the ring of all *Laurent polynomials*

$$a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0 + a_{-1} t^{-1} + \cdots + a_{-m} t^{-m}$$

where $a_i \in R$.

The usual notation for the Laurent polynomial ring is

$$R[t, t^{-1}] = RZ$$

Here, the square brackets indicate that this is the ring generated by R, t, t^{-1}. In general, if R is a subring of a ring S and $x_1, \cdots, x_n \in S$ then $R[x_1, \cdots, x_n]$ is defined to be the smallest subring of S containing R and the elements $x_1, \cdots, x_n$. (I.e., you just take the intersection of all such subrings.) This is in contrast to the round bracket notation which indicates a field. For example $\mathbb{Q}(t)$ is the smallest field containing $\mathbb{Q}$ and t.

Recall that a *field* is a commutative ring so that every nonzero element x has an inverse y ($xy = yx = 1$).

Proposition 2.5. *There is a bijection*

$$U(R) \cong \mathrm{Hom}_{Rings}(\mathbb{Z}[t, t^{-1}], R)$$

given by sending $x \in U(R)$ to the homomorphism $\phi : \mathbb{Z}[t, t^{-1}] \to R$ which sends $\sum n_i t^i$ to $\sum n_i x^i$. (This is called the evaluation map *since it sends a Laurent polynomial to its value at x.)*

Theorem 2.6. *The adjoint of the unit functor U is given by the integer group ring functor $\mathcal{G}ps \to \mathcal{R}ings$ which send G to $\mathbb{Z}G$. I.e.,*

$$\mathrm{Hom}_{\mathcal{G}ps}(G, U(R)) \cong \mathrm{Hom}_{\mathcal{R}ings}(\mathbb{Z}G, R)$$

Proof. Suppose that $f : G \to U(R)$ is a group homomorphism. I.e., $f(gh) = f(g)f(h)$. Then we get a ring homomorphism $\phi : \mathbb{Z}G \to R$ by

$$\phi(\sum n_i g_i) = \sum n_i f(g_i)$$

This is additive by definition and multiplicative since f is multiplicative. Also $\phi(1) = f(1) = 1$. Conversely, G is contained in the group of units of $\mathbb{Z}G$. So, any ring homomorphism $\phi : \mathbb{Z}G \to R$ induces a group homomorphism

$$G \hookrightarrow U(\mathbb{Z}G) \xrightarrow{U\phi} U(R).$$

And it is easy to see that these constructions are inverse to each other and therefore give a bijection between $\mathrm{Hom}_{\mathcal{G}ps}(G, U(R))$ and $\mathrm{Hom}_{\mathcal{R}ings}(\mathbb{Z}G, R)$. □

Corollary 2.7. *If there is an inverse system of rings, the units of the inverse limit are the inverse limit of the units:*

$$U(\lim R_i) \cong \lim U(R_i)$$

What follows is definitely not the best proof of this fact. However, it illustrates the use of category theory.

Proof. I used the theorem which gives a bijection. But the natural mapping is a homomorphism for the following reason. Given any diagram of rings R_i we get a diagram of groups $U(R_i)$ since U is a functor. The homomorphisms $\lim R_i \to R_i$ induce group homomorphisms $U(\lim R_i) \to U(R_i)$ which induces a group homomorphism

$$U(\lim R_i) \to \lim U(R_i).$$

I want to show that this is a bijection.

Now, use the natural bijections:

$$U(R_i) \cong \mathrm{Hom}_{\mathcal{G}ps}(Z, U(R_i)) \cong \mathrm{Hom}_{\mathcal{R}ings}(\mathbb{Z}[t, t^{-1}], R_i)$$

Then the universality of the inverse limit can be stated as:

$$\mathrm{Hom}_{\mathcal{R}ings}(X, \lim R_i) \cong \lim \mathrm{Hom}_{\mathcal{R}ings}(X, R_i)$$

for all rings X. Apply this to $X = \mathbb{Z}[t, t^{-1}]$ gives

$$U(\lim R_i) \cong \lim U(R_i).$$

□

2.4. **matrix rings over fields.** Suppose that K is a field. Consider the set of $n \times n$ matrices with coefficients in K. We write $A = (a_{ij})$. Using the usual addition and multiplication of matrices:

$$(A + B)_{ij} = a_{ij} + b_{ij}$$

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

we get a ring $\mathcal{M}_n(K)$ called the $n \times n$ *matrix ring* over K.

I used this example to illustrate the concept of equivalence between rings and additive categories. When we take $n \times n$ matrices over K, there are some special matrices which have special significance. These are the idempotents, e.g.,

$$e_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

The definition is: $e \in R$ is an *idempotent* if $e^2 = e$. For matrices, the "primitive" idempotents are e_i the diagonal matrix with 1 in the ii position and 0 everywhere else. Two idempotents e_1, e_2 are called *orthogonal* if $e_1 e_2 = e_2 e_1 = 0$. (A *primitive* idempotent is one which cannot be written as a sum of orthogonal idempotents.) We can write unity I_n as a sum of n orthogonal idempotents:

$$1 = e_1 + e_2 + \cdots + e_n.$$

This is a maximal such decomposition. This is significant because of the following general construction.

2.5. **idempotents give a category.** Now suppose that R is any ring and

$$1 = e_1 + e_2 + \cdots + e_n$$

is a decomposition of unity into orthogonal idempotents. Then we can construct a category $\mathcal{C}$ as follows. The objects of $\mathcal{C}$ are the idempotents $e_1, \cdots, e_n$ and the morphism sets are the sets

$$\mathrm{Hom}_{\mathcal{C}}(e_i, e_j) := e_j R e_i.$$

Note that this is a additive group. In fact, aRb is an additive group for any $a, b \in R$ since it is closed under addition:

$$arb + asb = a(r + s)b \in aRb$$

and negation:

$$-arb = a(-r)b \in aRb$$

and contains $0 = a0b$.

Composition is defined by multiplication:

$$(e_k s e_j) \circ (e_j r e_i) := (e_k s e_j)(e_j r e_i) = e_k s e_j r e_i$$

Note that, since multiplication is biadditive, composition gives a biadditive mapping

$$\mathrm{Hom}_{\mathcal{C}}(e_j, e_k) \times \mathrm{Hom}_{\mathcal{C}}(e_i, e_j) \to \mathrm{Hom}_{\mathcal{C}}(e_i, e_k)$$

The identity morphism of e_i is

$$e_i = e_i 1 e_i \in \mathrm{Hom}_{\mathcal{C}}(e_i, e_i) = e_i R e_i$$

It is easy to check that this is the identity morphism:

$$(e_j r e_i)e_i = e_j r e_i = e_j(e_j r e_i)$$

So, we have a category. This category has additional structure since the Hom sets are additive groups and composition is biadditive. A category with this kind of structure is called a *pre-additive category*. (It would be an *additive category* if it also had a zero object and was closed under finite products and/or finite coproducts. (They are equivalent in a pre-additive category.)