4. Localization

Today I talked about localization. I explained the basic definitions, gave examples and I also tried to explain what the word "local" refers to. All rings are commutative here.

4.1. basic definitions.

Definition 4.1. A subset $S \subseteq R$ is called a *multiplicative set* if S is closed under multiplication, $0 \notin S$ and $1 \in S$.

Some people leave out the assumption that $1 \in S$. It is not necessary since you can always add 1 to the set (take the union with $\{1\}$).

Definition 4.2. If S is a multiplicative set in a ring R then $S^{-1}R$ is defined to be the set of equivalence classes of symbols $\frac{x}{a}$ where $x \in R, a \in S$ and

$$\frac{x}{a} \sim \frac{y}{b} \text{ if } (\exists t \in S)\ xbt = ayt$$

This is a clumsy equivalence relation. So, I reformulated it as follows. This equivalence relation is the transitive relation generated by the symmetric relation given by

$$\frac{x}{a} \approx \frac{tx}{ta} \quad (\forall t \in S)$$

The reason is that:

$$\frac{x}{a} \approx \frac{xbt}{abt} = \frac{ayt}{abt} \approx \frac{y}{b}$$

Proposition 4.3. *$S^{-1}R$ is a ring with addition and multiplication given by*

$$\frac{x}{a} + \frac{y}{b} = \frac{xb + ay}{ab}$$
$$\frac{x}{a} \cdot \frac{y}{b} = \frac{xy}{ab}$$

and $j(x) = \frac{x}{1}$ gives a ring homomorphism $j : R \to S^{-1}R$.

Proof. We need to show that addition and multiplication are well-defined. The rest is straightforward. To show that addition is well-defined, we can use the stronger relation $\approx$:

$$\frac{sx}{sa} + \frac{ty}{tb} = \frac{sxtb + saty}{satb} = \frac{(st)(xb + ay)}{(st)ab} \approx \frac{xb + ay}{ab}.$$

Similarly,

$$\frac{sx}{sa} \cdot \frac{ty}{tb} = \frac{sxty}{satb} = \frac{(st)(xy)}{(st)ab} \approx \frac{xy}{ab}.$$

So, multiplication is also well-defined. $\square$

4.2. **examples.** I gave a bunch of examples but I delayed the explanations for the end of the lecture.

Example 1. $R = \mathbb{Z}$ with multiplicative set $S = \{n \in \mathbb{Z} \,|\, n \neq 0\}$. Then $S^{-1}\mathbb{Z} = \mathbb{Q}$.

This is a special case of the more general example:

Example 2. Suppose that R is any domain and $S = R \backslash \{0\}$. Then

$$Q(R) = S^{-1}R$$

is the *quotient field* or *field of fractions* of R. This is a field since the nonzero elements of R are invertible:

$$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

This example required a discussion about which elements are zero and which are 1.

Proposition 4.4. *An element $\frac{x}{s} \in S^{-1}R$ is equal to zero if and only if there is an element $t \in S$ so that $xt = 0$.*

Proof. If such a t exists then

$$\frac{x}{s} \approx \frac{xt}{st} = \frac{0}{st} \sim \frac{0}{1} = j(0)$$

since $0 \cdot 1 = 0 = st \cdot 0$. Conversely, if

$$\frac{x}{s} \sim \frac{0}{1} = j(0)$$

then there is a $t \in S$ so that $xt = x1t = st0 = 0$. □

I forgot to do the same discussion with 1.

Proposition 4.5. *An element $\frac{x}{s} \in S^{-1}R$ is equal to $1 = \frac{1}{1} = j(1)$ if and only if there exists $t \in S$ so that $tx = ts$.*

Proof. This is just the definition of the equivalence relation. □

Example 3. Let $R = C^0(\mathbb{R})$. This is the ring of continuous functions $f : \mathbb{R} \to \mathbb{R}$ with the usual pointwise addition and multiplication. Let S be the set of all functions f which are nonzero at 1:

$$S = \{f \,|\, f(1) \neq 0\}$$

Then S is clearly a multiplicative set. I claim that $S^{-1}R$ is a "local ring" which means that the nonunits form an ideal. A fraction $f(x)/g(x)$ is an element of $S^{-1}R$ if $g(1) \neq 0$. It is invertible if $f(1) \neq 0$. So the nonunits are given by the equation $f(1) = 0$ which clearly defines an ideal. In fact it is the kernel of the evaluation map

$$ev_1 : S^{-1}R \to \mathbb{R}.$$

This is a special case of the following more general example.

Example 4. If $P \subseteq R$ is a prime ideal then its complement $S = R \backslash P$ is a multiplicative set. (In fact these conditions are equivalent.) So, we can form the ring $S^{-1}R$ which is called the *localization* of R at P and written R_P.

Another special case of this is the following.

Example 5. Suppose that $p \in \mathbb{Z}$ is irreducible (i.e. a prime number). Then (p) is a prime ideal and we can form the localization:

$$\mathbb{Z}_{(p)} = S^{-1}\mathbb{Z} = \left\{ \frac{a}{b} \in \mathbb{Q} \,|\, p \not| \, b \right\}$$

This is a subring of $\mathbb{Q}$.

What is the relationship between $\mathbb{Z}_{(p)}$ and $\mathbb{Z}_p$?

4.3. **universal property.** In your homework you proved that if any integer n which is not divisible by p is uniquely invertible in $\mathbb{Z}_p$. Therefore, by the following universal property of the localization, there is a ring homomorphism

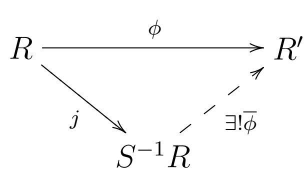
$$\mathbb{Z}_{(p)} \to \mathbb{Z}_p.$$

Is it onto? Is it a monomorphism?

Theorem 4.6. *If $S \subset R$ is a multiplicative set then $S^{-1}R$ has the following universal property: Given any ring homomorphism*

$$\phi : R \to R'$$

so that $\phi(S) \subseteq U(R')$ then there exists a unique ring homomorphism $\overline{\phi} : S^{-1}R \to R'$ so that $\overline{\phi} \circ j = \phi$. I.e., the following diagram commutes.



Proof. (Existence) The ring homomorphism $\overline{\phi}$ is given by $\overline{\phi}(x/s) = \phi(x)\phi(s)^{-1}$. You actually need to show that this is a homomorphism. This follows from the fact that the definition of addition and multiplication in $S^{-1}R$ used the rules which hold for arbitrary elements of the form xs^{-1}.

(Uniqueness) Suppose that $\psi : S^{-1}R \to R'$ is a homomorphism which extends j. Then the equation

$$\frac{x}{s} = \frac{x}{1} \cdot \frac{1}{s}$$

gives the equation

$$\psi\left(\frac{x}{s}\right) = \psi\left(\frac{x}{1}\right)\psi\left(\frac{1}{s}\right) = \phi(x)\phi(s)^{-1}$$

where $\psi(1/s) = \psi(s)^{-1} = \phi(s)^{-1}$ since ψ induces a group homomorphism

$$U(\psi) : U(R) \to U(R').$$

□

4.4. **local rings.** Localization is used to produce local rings.

Definition 4.7. A *local ring* is a ring R with a unique maximal ideal $\mathfrak{m}$.

Proposition 4.8. *A ring R is local if and only if the complement $R \backslash U(R)$ of the set of units is an ideal.*

The proof used the followings two obvious properties of $U(R)$.

Lemma 4.9. *An ideal in R cannot contain any units.*

This implies that any ideal is contained in $R \backslash U(R)$. So, if this is an ideal, it must be maximal.

Lemma 4.10. *Any element $a \in R$ which is not a unit generates an ideal (a).*

So, if there is only one maximal ideal, it contains all such a. So, $\mathfrak{m} = R \backslash U(R)$.

The term "localization" is justified by the following theorem which I forgot to prove. So, you can do it for homework.

Theorem 4.11. *If P is a prime ideal in R then R_P is a local ring.*

What I did explain is what the "local" means, at least topologically.

4.5. **germs of functions.**

Definition 4.12. Suppose that X, Y are topological spaces and $f : X \to Y$ is a mapping. Let $x_0 \in X$. Then the *germ* of f at x_0 is defined to be the equivalence class of f under the equivalence relation $f \sim g$ if there exists an open neighborhood U of x_0 in X so that

$$f|_U = g|_U$$

This is an equivalence relation since, if $g|_V = h|_V$, then

$$f|_{U\cap V} = g|_{U\cap V} = h|_{U\cap V}$$

and $U \cap V$ is an open neighborhood of x_0. Note that f need only be defined in a neighborhood of x_0.

When the target Y is a ring, the set of map germs forms a ring. If we put some restriction on the functions f (e.g., continuous, differentiable, polynomial), we get a subring. For example, we have a ring of germs at 1 of continuous functions $\mathbb{R} \to \mathbb{R}$. The notation is with a comma:

$$gr_1(f) : \mathbb{R}, 1 \to \mathbb{R}$$

In Example 3, we localized the ring of continuous functions $\mathbb{R} \to \mathbb{R}$ at the maximal ideal $S = \{f \mid f(1) \neq 0\}$ (making it into the unique maximal ideal). Two functions f, g give the same map germ at 1 if there exists an $\epsilon > 0$ so that

$$f(x) = g(x) \quad \forall x \in (1-\epsilon, 1+\epsilon)$$

In other words, the functions f, g agree in the ϵ neighborhood of 1 and the germs is just the function restricted to this arbitrarily small interval about 1.

The following theorem represents the intuitive concept that "localization" refers to restricting attention to a neighborhood of a point.

Theorem 4.13. *If $S = \{f \mid f(1) \neq 0\}$ then $S^{-1}C^0(\mathbb{R})$ is isomorphic to the ring of germs at 1 of continuous functions $\mathbb{R}, 1 \to \mathbb{R}$.*

Proof. Let G be the ring of germs at 1 of continuous functions $\mathbb{R}, 1 \to \mathbb{R}$. By definition, we have an epimorphism of rings

$$gr_1 : C^0(\mathbb{R}) \to G$$

For any $f \in S$, there is an $\epsilon > 0$ so that $f(x) \neq 0$ for all $x \in (1-\epsilon, 1+\epsilon)$. Let $g : \mathbb{R} \to \mathbb{R}$ be given by

$$g(x) = \begin{cases} 1/f(x) & \text{if } |x-1| \leq \epsilon/2 \\ 1/f(1-\epsilon/2) & \text{if } x \leq 1-\epsilon/2 \\ 1/f(1+\epsilon/2) & \text{if } x \geq 1+\epsilon/2 \end{cases}$$

This is a continuous function so that the product $f \cdot g$ is equal to 1 in a neighborhood of 1. Therefore, $gr_1(f)$ is invertible in the ring of germs. So, by the universal property we have a ring homomorphism

$$\overline{gr}_1 : S^{-1}C^0(\mathbb{R}) \to G$$

which is surjective. So, we just have to show that the kernel is zero.

Suppose that $f/g \in \ker \overline{gr}_1$. Then, for some $\epsilon > 0$,

$$f(x)/g(x) = 0 \quad \forall x \in (1-\epsilon, 1+\epsilon).$$

But this implies that $f(x) = 0$ in that interval. Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function so that $h(1) \neq 0$ but $h(x) = 0$ whenever $|x-1| \geq \epsilon$. (For example, $h(x) = \max(0, \epsilon - |x-1|)$.) Then the product $f \cdot h = 0$ and in $S^{-1}C^0(\mathbb{R})$ we have

$$\frac{f}{g} = \frac{fh}{gh} = \frac{0}{gh} = 0.$$

Therefore, $\overline{gr}_1$ is an isomorphism of rings. □

In algebraic geometry, we should take $\mathbb{C}$ instead of $\mathbb{R}$ and we should take the Zariski topology. The open subsets of $\mathbb{C}$ are then the complements of finite sets. We should take polynomial functions

$$\mathbb{C} \to \mathbb{C}$$

and invert the ones which are not zero at some point, say a. This is the complement of the maximal ideal $(X-a) \subset \mathbb{C}[X]$. Then we get the local ring

$$\mathbb{C}[X]_{(X-a)}$$

which is supposed to restrict attention to the behavior of polynomials at the point a.